Hamiltonian and 1-tough properties in Cartesian product graphs

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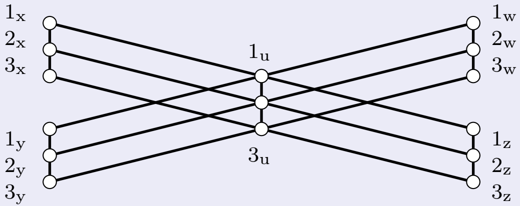
Overview

Cartesian Product graph $G_1 \square G_2$

$$
\begin{aligned}
V(G_1 \square G_2) =& \{v_u : v \in V(G_1), u \in V(G_2)\},\\
E(G_1 \square G_2) =& \{v_u v_w : v \in V(G_1), uw \in E(G_2)\}\\
& \cup \{v_u w_u : u \in V(G_2), vw \in E(G_1)\}.
\end{aligned}
$$

Example

With $V(P_3) = \{1, 2, 3\}$ and $V(K_{1,4}) = \{x, y, z, w, u\}$), the graph $P_3 \square K_{1,4}$ is as follows:

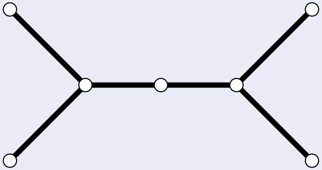


Graph Terminologies

- A graph is Hamiltonian if it contains a spanning cycle.
- A graph is traceable if it contains a spanning path.
- A path factor of a graph is its spanning subgraph such that each component is isomorphic to a path with order at least two.
- If each component in a path factor is isomorphic to P_2, the path factor is called a perfect matching.

Example

A graph without a path factor



Main Theorem

Let $\Delta(G)$ denote the maximum degree of graph G and $c(G)$ denote the number of connected components in G.

Main Theorem

Let G_1 be a traceable graph and G_2 a connected graph. Assume one of the following (a),(b) holds:

(a) G_2 has a perfect matching and $|V(G_1)| \geq \Delta(G_2)$, or

(b) G_2 has a path factor and $|V(G_1)|$ is an even integer with $|V(G_1)| \geq 4\Delta(G_2) - 2$.

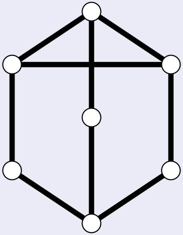
Then $G_1 \square G_2$ has a Hamiltonian cycle.

Toughness

- A graph G is t-tough if t is a rational number such that $|S| \geq t \cdot c(G - S)$ for any cut set S of G.
- If G is not complete, the largest t makes G to be t-tough is called the toughness of G, denoted by t(G).
- Hamiltonian $\Rightarrow$ 1-tough $(|S| \geq c(G - S))$.
- 1-tough $\not\Rightarrow$ Hamiltonian.

Example

A 1-tough graph which is not Hamiltonian.



Toughness and Hamiltonicity

- V. Chvátal (1973) :
  - Introduce the idea of graph toughness
  - Conjecture : There exists a real number t_0 such that all t_0-tough graphs are Hamiltonian.
- J. Harant, P.J. Owens (1995) :
  - Construct non-Hamiltonian graphs with toughness greater than 1.25.
- D. Bauer, H.H. Broersma, H.H. Veldman (2000) :
  - Construct non-Hamiltonian graphs with toughness greater than 2.
- A.Kabela, T.Kaiser (2017) :
  - 10-tough chordal graphs are hamiltonian.

Application of Main Theorem (a)

Corollary A

Let T be a tree with a perfect matching and n be a positive integer. Then the following three statements are equivalent:

1. $P_n \square T$ is Hamiltonian.
2. $P_n \square T$ is 1-tough.
3. $n \geq \Delta(T)$.

Proof.

3⇒1 : This is Theorem 1 (a)

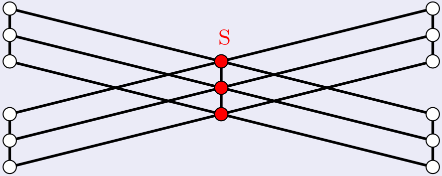
1⇒2 : This is clear

2⇒3 : ?

Proof of Corollary A

Proof.

❷⇒❸ : If $n < \Delta(T)$ then $c(T - S) > |S|$ for some $S \subseteq V(P_n \Box T)$ as shown below:



The graph $P_3 \Box K_{1,4}$ with $|V(P_3)| = 3 < 4 = |\Delta(K_{1,4})|$.

Application of Main Theorem (b)

Corollary B

Let H be a connected bipartite graph. Assume n is an even integer and $n \geq 4\Delta(H) - 2$. Then the following three statements are equivalent:

1. $P_n \square H$ is Hamiltonian.
2. $P_n \square H$ is 1-tough.
3. H has a path factor.

Proof.

3 ⇒ 1 : This is Theorem 1 (a)

1 ⇒ 2 : This is clear

2 ⇒ 3 : ?

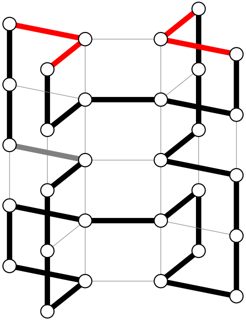
To prove

$$\text{1-tough } P_n \square H \quad \Rightarrow \quad H \text{ has a path factor,}$$

we need a little more works, but one can easily realize

$$\text{Hamiltonian } P_n \square H \quad \Rightarrow \quad H \text{ has a path factor,}$$

from the following graph:



Path factor of a bipartite graph

To complete the proof of Corollary B, we need to proof the following theorem.

Lemma B

If $P_n \square H$ is a 1-tough bipartite graph then H has a path factor.

Let i(G) denote the number of isolated vertices in G. The following result proved by J. Akiyama, D. Avis and H. Era (1980) is useful.

Proposition

Let G be a graph. Then G has a path factor if and only if $i(G - S) \leq 2|S|$ for all $S \subseteq V(G)$.

Path factor of a bipartite graph

Restricted to bipartite graphs, the following proposition is a supplementary.

Proposition

If H is a bipartite graph that does not contain a path factor, then there exists a vertex subset S that belongs to a single partite set of H with $i(H - S) > 2|S|$.

Proof of Lemma B

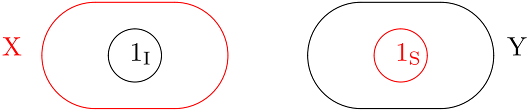
Lemma B

If $P_n \Box H$ is a 1-tough bipartite graph then H has a path factor.

Proof.

Assume H has no path factor and choose a vertex subset S in a single partite set of H such that $|I| = i(H - S) > 2|S|$, where $I := \{u \ : \ u \text{ is isolated in } H - S\}$. Let $V(P_n \Box H) = X \cup Y$ be a bipartition with $|X| \leq |Y|$; and if $|X| = |Y|$, let $1_S := \{1_s : s \in S\} \subseteq Y$, so $1_I \subseteq X$. Hence $2|1_S| = 2|S| < i(G - S) = |I|$.

If $|X| < |Y|$, set $X' = X$, $Y' = Y$; if $|X| = |Y|$, set $X' = (X \cup 1_S) - 1_I$, $Y' = (Y \cup 1_I) - 1_S$. Then $t(G) \leq |X'|/c(G - X') < 1$. □

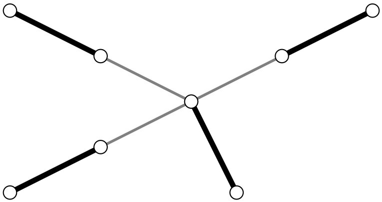


A tree with a path factor

Lemma

For a tree T with a $\{P_2, P_3\}$-factor F, there exists a component c of F such that $T - c$ is a tree with $\{P_2, P_3\}$-factor $F - \{c\}$.

A tree T with a perfect matching

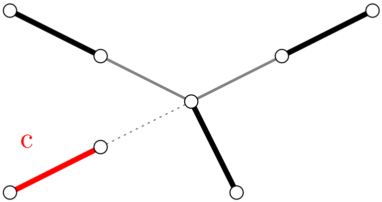


A tree with a path factor

Lemma

For a tree T with a $\{P_2, P_3\}$-factor F, there exists a component c of F such that $T - c$ is a tree with $\{P_2, P_3\}$-factor $F - \{c\}$.

A tree $T - c$ with a perfect matching



Theorem A is a stronger version of Main Theorem (a)

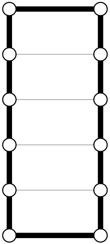
Theorem A

Let T be a tree with a perfect matching and $n \geq \Delta(T)$. Then there exists a Hamiltonian cycle of $P_n \square T$ which contains exactly $n - \mathsf{deg}_T(v)$ of the edges: $\{i_v(i+1)_v : i = 1, 2, \cdots, n-1\}$ for any vertex $v \in V(T)$.

Base Case:

The standard Hamiltonian cycle

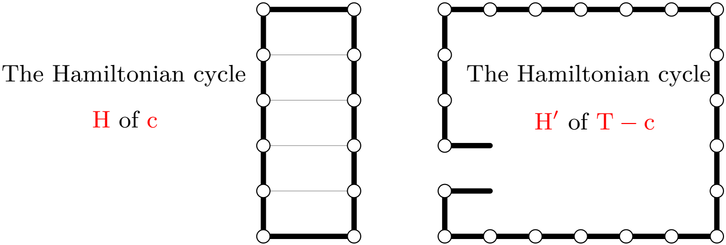
for $P_n \square P_2$



Proof of Theorem A

Theorem A

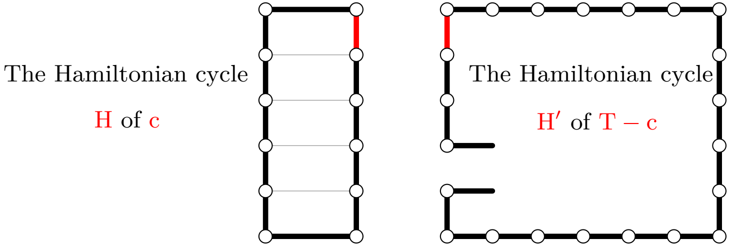
Let T be a tree with a perfect matching and $n \geq \Delta(T)$. Then there exists a Hamiltonian cycle of $P_n \square T$ which contains exactly $n - \deg_T(v)$ of the edges: $\{i_v(i+1)_v : i = 1, 2, \cdots, n-1\}$ for any vertex $v \in V(T)$.



Proof of Theorem A

Theorem A

Let T be a tree with a perfect matching and $n \geq \Delta(T)$. Then there exists a Hamiltonian cycle of $P_n \Box T$ which contains exactly $n - \mathsf{deg}_T(v)$ of the edges: $\{i_v(i+1)_v : i = 1, 2, \cdots, n-1\}$ for any vertex $v \in V(T)$.



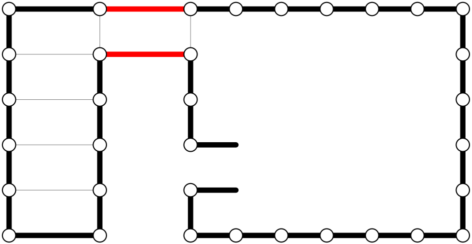
Proof of Theorem A

Theorem A

Let T be a tree with a perfect matching and $n \geq \Delta(T)$. Then there exists a Hamiltonian cycle of $P_n \Box T$ which contains exactly $n - \mathsf{deg}_T(v)$ of the edges: $\{i_v(i+1)_v : i = 1, 2, \cdots, n-1\}$ for any vertex $v \in V(T)$.

The Hamiltonian cycle

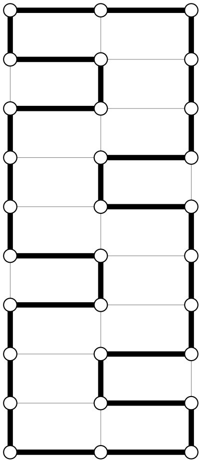
H'' of T



□

Standard Hamiltonian cycle H for $P_n \square P_3$

$L_v = \{i_v(i+1)_v : i \equiv 0, 1, 3 \pmod 4\}$
$C_v = \{i_v(i+1)_v : i \equiv 0, 2 \pmod 4\}$
$R_v = \{i_v(i+1)_v : i \equiv 1, 2, 3 \pmod 4\}$



$|H \cap X_v| \geq \lceil n/4 \rceil - \deg(v)$
$(\forall X \in \{L, C, R\})$

Theorem B is a stronger version of Theorem 1 (b)

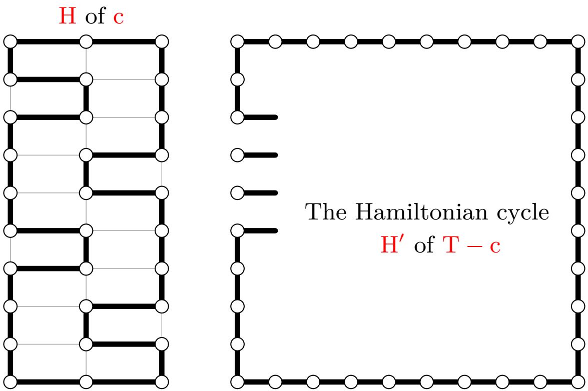
Theorem B

Let G be a connected graph with a path factor and n be an even integer with $n \geq 4\Delta(G) - 2$. Then $P_n \square G$ contains a Hamiltonian cycle H such that for any vertex $a \in V(G)$, the number of edges in the intersection of H and any of the three edge sets L_a, C_a, R_a is at least $\lceil n/4 \rceil - \mathsf{deg}_G(a)$.

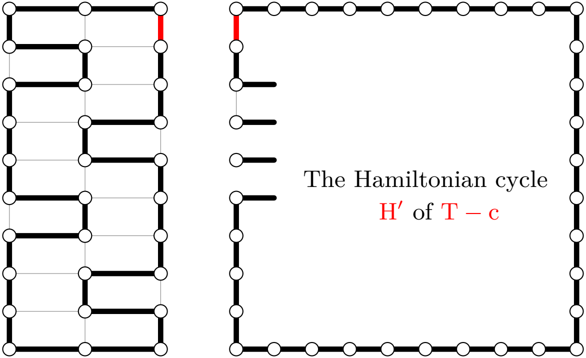
Base cases:

The standard Hamiltonian cycle H_1 for $P_n \square P_2$,
The standard Hamiltonian cycle H_2 for $P_n \square P_3$.

Proof of Theorem B

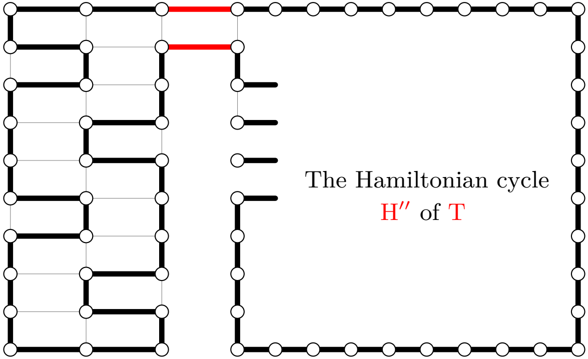


Proof of Theorem B-edges matching



The assumption $n \geq 4\Delta(G) - 2$ ensures a pair of edges matches in a vertical position.

Proof of Theorem 4 - Combine the Hamiltonian cycle



□

The lower bound of n

The condition $n \geq \Delta(T)$ is an equivalent statement to the Hamiltonicity of $P_n \square T$ when T has a perfect matching in Corollary A.

Can the assumption $n \geq 4\Delta(H) - 2$ in Corollary B be replaced by $n \geq \Delta(H)$?

Corollary B

Let H be a connected bipartite graph. Assume n is an even integer and $n \geq 4\Delta(H) - 2$. Then the following three statements are equivalent:

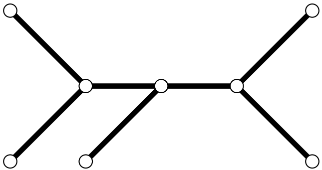
1. $P_n \square H$ is Hamiltonian.
2. $P_n \square H$ is 1-tough.
3. H has a path factor.

The lower bound of n

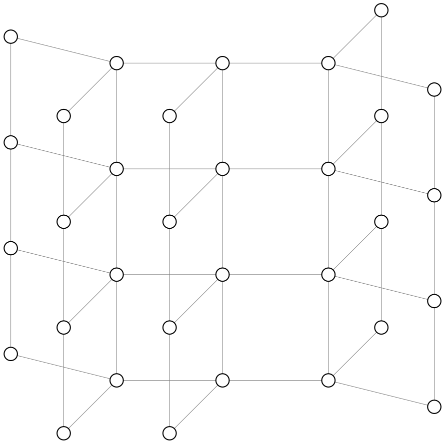
The answer is NO!

Here we provide a 1-tough non-Hamiltonian graph $P_n \square T$ such that T is a tree with a path factor and $n = \Delta(T) + 1$.

T :



$P_4 \square T$ is not Hamiltonian



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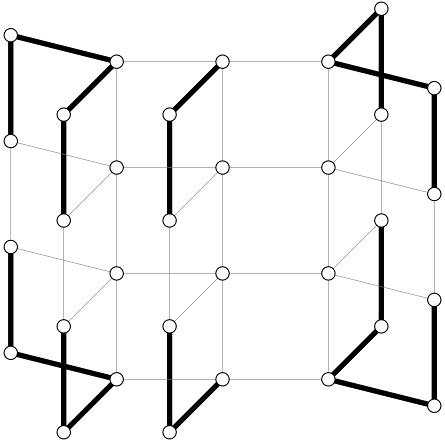

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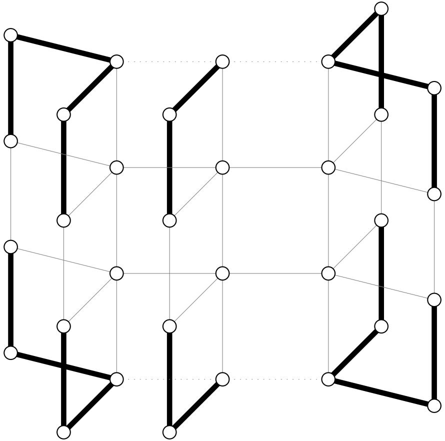
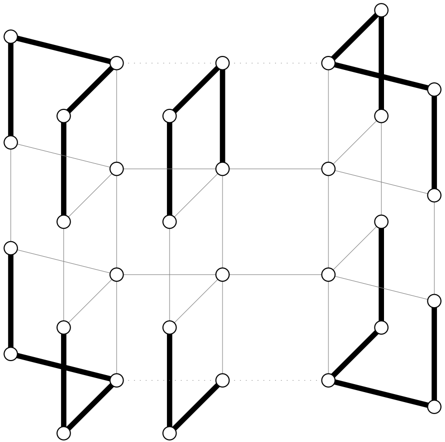
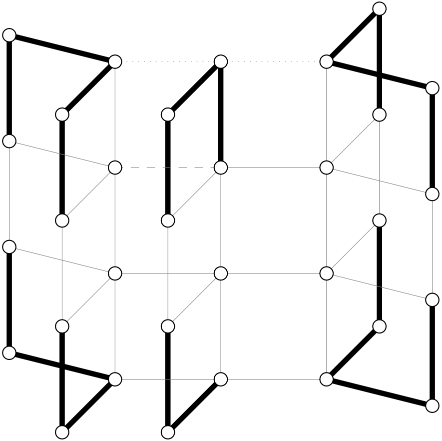


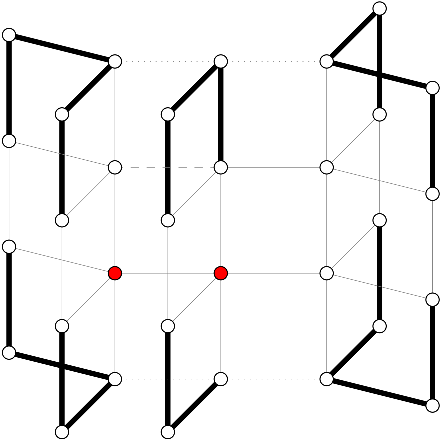
$P_4\square T$ is not Hamiltonian



$P_4 \square T$ is not Hamiltonian

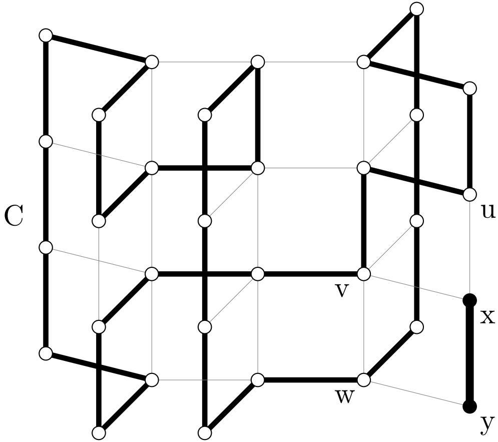


$P_4 \square T$ is not Hamiltonian



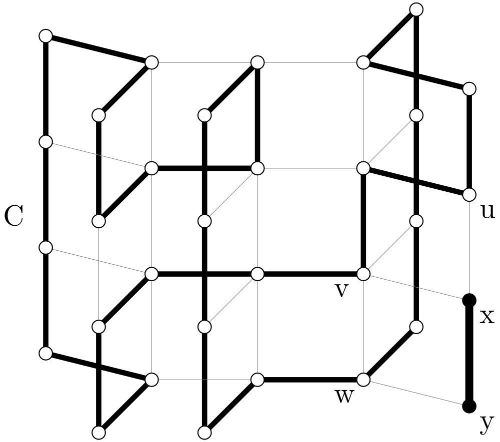
□

$G = P_4 \square T$ is 1-tough



Consider three cases: (i) $S \cap \{x, y\} \neq \emptyset$, (ii) $S \cap \{x, y\} = \emptyset$ and $\{u, v, w\} \subseteq S$, (iii) $S \cap \{x, y\} = \emptyset$ and $\{u, v, w\} \not\subseteq S$.

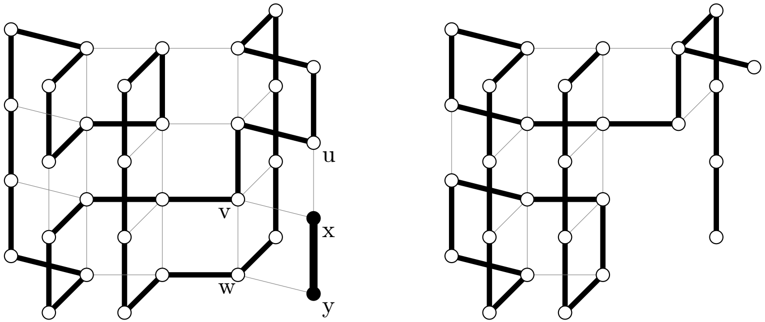
$S \cap \{x, y\} \neq \emptyset$



$$c(G - S) \leq c(C - (S \cap C)) + c(\{x, y\} - (S \cap \{x, y\}))$$
$$\leq |S \cap C| + |S \cap \{x, y\}| = |S|$$

$S \cap \{x, y\} = \emptyset$ and $\{u, v, w\} \subseteq S$

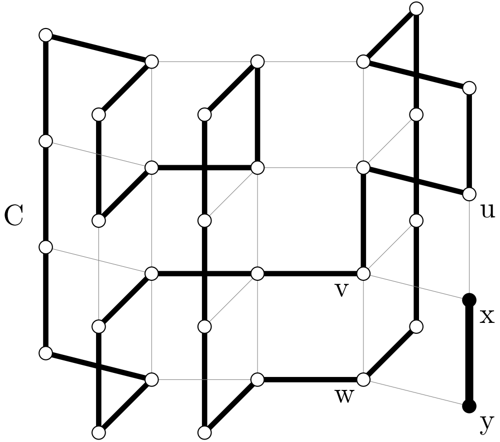
Let $S' = S - \{u, v, w\}$, and T' be spanning tree of $G - \{x, y, u, v, w\}$. Note that T' is obtained by connected a vertex to a path.



$$c(G - S) \leq 1 + c(T' - S') \leq |S'| + 3 = |S|.$$

$S \cap \{x, y\} = \emptyset$ and $\{u, v, w\} \not\subseteq S$

Say $u \notin S$. Then x, y are connected to the component containing u in $C - S$.



Conjecture 1

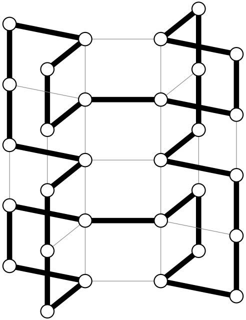
When T has a path factor and $\Delta(T) < n < 4\Delta(T) - 2$, it seems difficult to to find a Hamiltonian cycle in $P_n \square T$ systematically. In general, we have the following conjecture.

Conjecture

For $n \geq 8$ a multiple of 4, there is a connected graph G with a path factor satisfying $n = 4\Delta(G) - 4$ such that $P_n \square G$ is not Hamiltonian.

The case n is odd

For n being an odd integer and G a bipartite graph, the graph $P_n \square G$ is also possible to be Hamiltonian if it is balanced. For instance, let $V(G_2) = \{1, 2, 3, 4, 5, 6\}$ and $E(G_2) = \{12, 23, 34, 25, 36\}$. Then $P_5 \square G_2$ is Hamiltonian.



Conjecture 2

Note that our construction of Hamiltonian cycles of $P_n \Box G$ always contains the edge $1_x 1_y$, where xy is an edge in a component of a $\{P_2, P_3\}$-factor of the graph G. Use these edges as bridges, we can construct a Hamiltonian cycle for $P_{2k+5} \Box G$ from those of $P_5 \Box G$ and $P_{2k} \Box G$.

Conjecture

Let G be a graph with path factor and $n \geq 4\Delta(G) - 2$ such that $P_n \Box G$ is balanced bipartite. Then $P_n \Box G$ is Hamiltonian.

Thank you for your attention.