## Bidiagonal triples and the quantum group $U_q(sl_2)$

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The Basis of 
$$sl_2(\mathbb{K})$$

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

h, e, f satisfy

$$[h, e] = 2e,$$
  $[h, f] = -2f,$   $[e, f] = h.$ 

## Another Basis of $sl_2(\mathbb{K})$

$$A = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, C = \begin{pmatrix} -1 & 0 \\ -2 & 1 \end{pmatrix}$$

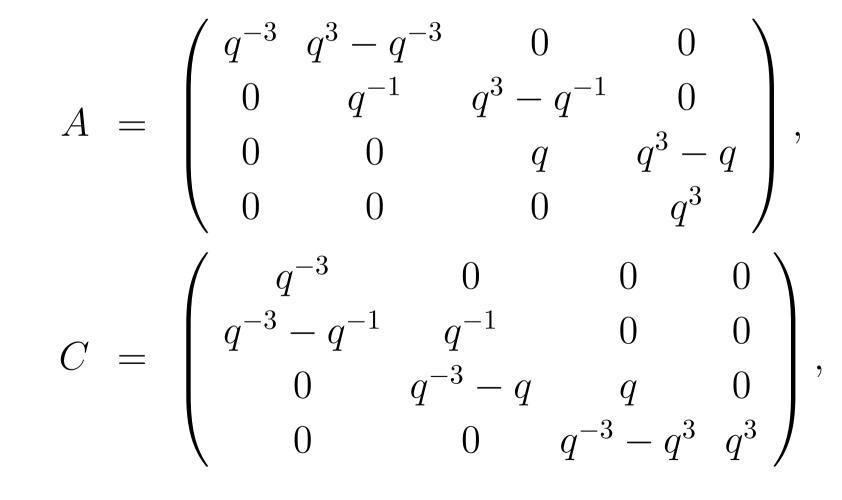
A, B, C satisfy

$$[A, B] = -2A - 2B, [B, C] = -2B - 2C,$$
$$[C, A] = -2C - 2A.$$

### **Bidiagonal Matrices**

Let X denote a square matrix. We say X is upper bidiagonal whenever both (i) each nonzero entry of X is on the diagonal or superdiagonal; (ii) each entry on the superdiagonal of X is nonzero. We say X is lower bidiagonal whenever the transpose of X is upper bidiagonal.

### Examples



where  $q^2 \neq 1, q^4 \neq 1, q^6 \neq 1$ .

### **Bidiagonal Triple**

Let  $\mathbb{K}$  be an algebraically closed field with characteristic 0. Let  $\mathbb{V}$  denote a vector space over  $\mathbb{K}$  with finite positive dimension. By a **bidiagonal triple** on  $\mathbb{V}$  we mean a sequence of linear transformations  $A, B, C : \mathbb{V} \to \mathbb{V}$  that satisfy the following three conditions:

### **Bidiagonal Triple**

- (i) There exists a basis for  $\mathbb{V}$  with respect to which the matrices representing A, B, C are upper bidiagonal, diagonal, and lower bidiagonal, respectively.
- (ii) There exists a basis for  $\mathbb{V}$  with respect to which the matrices representing B, C, A are upper bidiagonal, diagonal, and lower bidiagonal, respectively.
- (iii) There exists a basis for  $\mathbb{V}$  with respect to which the matrices representing C, A, B are upper bidiagonal, diagonal, and lower bidiagonal, respectively.

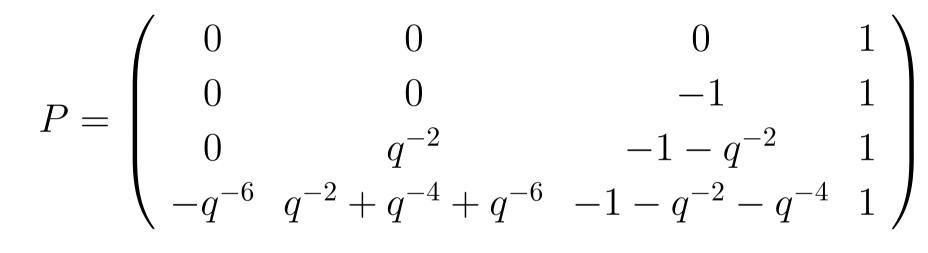


A, C as before and

$$B = \begin{pmatrix} q^3 & 0 & 0 & 0 \\ 0 & q & 0 & 0 \\ 0 & 0 & q^{-1} & 0 \\ 0 & 0 & 0 & q^{-3} \end{pmatrix}$$

Then A, B, C is a bidiagonal triple.

#### Proof



Then

 $P^{-1}BP = A, P^{-1}CP = B, P^{-1}AP = C.$ 

#### **More Examples**

(i) A is upper bidiagonal with entries  $A_{ii} = q^{2i-n}$ for  $0 \le i \le n$  and  $A_{i,i+1} = q^n - q^{2i-n}$  for  $0 \le i \le n-1$ .

(ii) B is diagonal with  $B_{ii} = q^{n-2i}$  for  $0 \le i \le n$ .

(iii) *C* is lower bidiagonal with entries  $C_{ii} = q^{2i-n}$ for  $0 \le i \le n$  and  $C_{i,i-1} = q^{-n} - q^{2i-n}$  for  $1 \le i \le n$ .

Then the sequence A, B, C is a bidiagonal triple on  $\mathbb{K}^{n+1}$  (with base q).

### **More Examples**

(i) A is upper bidiagonal with entries  $A_{ii} = 2i - n$ for  $0 \le i \le n$  and  $A_{i,i+1} = 2n - 2i$  for  $0 \le i \le n - 1$ .

(ii) B is diagonal with  $B_{ii} = n - 2i$  for  $0 \le i \le n$ .

(iii) *C* is lower bidiagonal with entries  $C_{ii} = 2i - n$ for  $0 \le i \le n$  and  $C_{i,i-1} = -2i$  for  $1 \le i \le n$ .

Then the sequence A, B, C is a bidiagonal triple on  $\mathbb{K}^{n+1}$  (with base q = 1).

#### Normalized Bidiagonal Triples

We refer all of the above mentioned bidiagonal triples as normalized bidisgonal triples with base q.

#### Lemma

Let A, B, C denote a bidiagonal triple on  $\mathbb{V}$ . Let  $\alpha^{\pm}, \beta^{\pm}, \gamma^{\pm}$  denote scalars in  $\mathbb{K}$  with  $\alpha^{+}, \beta^{+}, \gamma^{+}$  nonzero. Then the sequence

 $\alpha^+ A + \alpha^- I, \quad \beta^+ B + \beta^- I, \quad \gamma^+ C + \gamma^- I$ 

is a bidiagonal triple on  $\mathbb{V}$ .

### Affine Equivalence

Let A, B, C and A', B', C' denote two bidiagonal triples on  $\mathbb{V}$ . We say these two sequences are affine equivalent whenever

 $A' = \alpha^{+}A + \alpha^{-}I, B' = \beta^{+}B + \beta^{-}I, C' = \gamma^{+}C + \gamma^{-}I$ 

for some scalars  $\alpha^{\pm}, \beta^{\pm}, \gamma^{\pm}$  K with  $\alpha^{+}, \beta^{+}, \gamma^{+}$  nonzero.

#### Main Theorem

Each bidiagonal triple is affine equivalent to a normalized bidiagonal triple with base q.

### Lie Algebra $sl_2(\mathbb{K})$

This algebra has a basis e, f, h satisfying

[h, e] = 2e, [h, f] = -2f, [e, f] = h,

where [,] denotes the Lie bracket.

## $\begin{array}{c} \mbox{Irreducible} \\ sl_2\mbox{-Modules} \end{array}$

There exists a family

$$\mathbb{V}_n \qquad n = 0, 1, 2, \dots \tag{1}$$

of finite dimensional irreducible  $sl_2$ -modules with the following properties. The module  $\mathbb{V}_n$  has a basis  $v_0, v_1, \ldots, v_n$  satisfying  $hv_i = (n - 2i)v_i$  for  $0 \le i \le n, fv_i = (i + 1)v_{i+1}$  for  $0 \le i \le n - 1,$  $fv_d = 0, ev_i = (n - i + 1)v_{i-1}$  for  $1 \le i \le n, ev_0 = 0.$ 

## $\begin{array}{c} \text{Irreducible} \\ sl_2 \text{-Modules} \end{array}$

Every irreducible  $sl_2$ -module of dimension n+1 is isomorphic to the  $\mathbb{V}_n$  in previous slide.

## **Alternative Basis for** $sl_2$

Set x = -h + 2e, y = h, z = -h - 2f in  $sl_2$ . Then x, y, z is another basis of  $sl_2$  satisfying

[x, y] = -2x - 2y, [y, z] = -2y - 2z, [z, x] = -2z - 2x.

## Bidiagonal Triples and $sl_2$

The alternate basis x, y, z of  $sl_2$  act on  $\mathbb{V}_n$  as a bidiagonal triple.

 $U_q(sl_2)$ 

Quantum algebra  $U_q(sl_2)$  is the unital associative K-algebra with generators  $e, f, k, k^{-1}$  and the following relations:

$$kk^{-1} = k^{-1}k = 1,$$
  

$$kek^{-1} = q^{2}e, kfk^{-1} = q^{-2}f,$$
  

$$ef - fe = \frac{k - k^{-1}}{q - q^{-1}},$$

where  $q \in \mathbb{K}$  is not a root of unity.

#### Alternative Presentation

The quantum algebra  $U_q(sl_2)$  is isomorphic to the unital associative  $\mathbb{K}$ -algebra with generators  $x, y, z, z^{-1}$  and the following relations:

$$\begin{aligned} yy^{-1} &= y^{-1}y = 1, \\ \frac{qxy - q^{-1}yx}{q - q^{-1}} &= 1, \\ \frac{qyz - q^{-1}zy}{q - q^{-1}zy} &= 1, \\ \frac{qzx - q^{-1}xz}{q - q^{-1}xz} &= 1. \end{aligned}$$

#### Proof

An isomorphism is given by:

$$y^{\pm 1} \rightarrow k^{\pm 1},$$
  
 $z \rightarrow k^{-1} + f,$   
 $x \rightarrow k^{-1} - q(q - q^{-1})^2 k^{-1} e.$ 

The inverse of this isomorphism is given by:

$$\begin{array}{rcl} k^{\pm 1} & \rightarrow & y^{\pm 1}, \\ f & \rightarrow & z - y^{-1}, \\ e & \rightarrow & \frac{1 - yx}{q(q - q^{-1})^2}. \end{array}$$

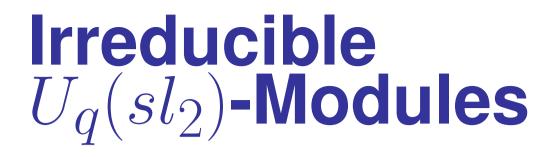
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## Irreducible $U_q(sl_2)$ -Modules

#### There exists a family

$$\mathbb{V}_{\varepsilon,n}$$
  $\varepsilon \in \{1, -1\},$   $n = 0, 1, 2...$ 

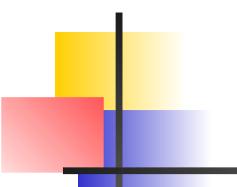
of finite dimensional irreducible  $U_q(sl_2)$ -modules with the following properties. The module  $\mathbb{V}_{\varepsilon,n}$  has a basis  $u_0, u_1, \ldots, u_n$  such that  $ku_i = \varepsilon q^{n-2i}u_i$  for  $0 \le i \le n$ ,  $fu_i = [i+1]_q u_{i+1}$  for  $0 \le i \le n-1$ ,  $fu_n = 0$ ,  $eu_i = \varepsilon [n-i+1]_q u_{i-1}$  for  $1 \le i \le n$ ,  $eu_0 = 0$ .



Every irreducible  $U_q(sl_2)$ -module of dimension n+1 is isomorphic to  $\mathbb{V}_{-1,n}$  or  $\mathbb{V}_{1,n}$ .

# Bidiagonal Triples and $U_q(sl_2)$

Let  $\mathbb{V}_{\varepsilon,n}$  denote the finite dimensional irreducible  $U_q(sl_2)$ -module. Then the alternate generators  $\varepsilon x, \varepsilon y, \varepsilon z$  act on  $\mathbb{V}_{\varepsilon,n}$  as a bidiagonal triple.



#### **Thank You**