The spectral excess theorem for general graphs

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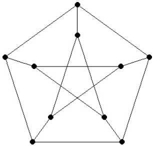
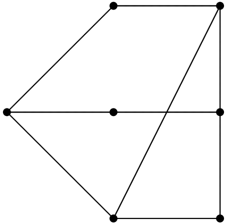
Overview

Let D denote the diameter of a graph and d denote the spectral diameter of a graph. ($d+1$ is the number of distinct eigenvalues of G)



Petersen graph has odd-girth

$$5 = 2D+1 = 2d+1.$$



A graph with odd-girth

$$5 = 2D+1 < 2d+1.$$

Overview

The theory we derive here will show that a graph with odd-girth 2d+1 must be regular, and then indeed distance-regular by a recent result of E.R. van Dam and W.H. Haemers.

(E.R. van Dam and W.H. Haemers, An odd characterization of the generalized odd graphs, *J. Combin. Theory Ser. B* (2011), doi:10.1016/j.jctb.2011.03.001).

Outline

- Orthogonal polynomials associated with a graph
- Hoffman polynomial
- The generalized spectral excess theorem
- $D = d$
- Graphs with odd-girth $2d+1$

- Orthogonal polynomials associated with a graph

Hoffman polynomials

The generalized spectral excess theorem

$D = d$

Graphs with odd-girth $2d+1$

Notations

Let $G = (VG, EG)$ be a connected graph on n vertices, with diameter D, adjacency matrix A, and distance function ∂. Assume that A has $d+1$ distinct eigenvalues $\lambda_0 > \lambda_1 > \ldots > \lambda_d$ with corresponding multiplicities $1 = m_0$, m_1, ..., m_d. The spectrum of G will be denoted by the multi-set

$$\text{sp } G = \{\lambda_0^{m_0}, \lambda_1^{m_1}, \ldots, \lambda_d^{m_d}\}.$$

The parameter d is called the spectral diameter of G. It is well known that $D \leq d$ and

$$Z(x) := \prod_{i=0}^{d} (x - \lambda_i)$$

is the minimal polynomial of A.

Inner product polynomial space

Consider the $(d+1)$-dimensional vector space $\mathbb{R}_d[x] \cong \mathbb{R}[x]/\langle Z(x)\rangle$ with inner product defined by

$$\langle p(x), q(x)\rangle_{\triangle} := \sum_{i=0}^{d} \frac{m_i}{n} p(\lambda_i) q(\lambda_i) = \mathrm{tr}(p(A)q(A))/n,$$

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and norm defined by

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for $p(x), q(x) \in \mathbb{R}_d[x]$.

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for $p(x), q(x) \in \mathbb{R}_d[x]$.

Note that

(1) $1, x, \ldots, x^d$ is a basis of $\mathbb{R}_d[x]$.

(2) $\langle p(x), p(x)\rangle_\triangle = 0$ iff $p(x) = 0$.

(3) $\langle xp(x), q(x)\rangle_\triangle = \langle p(x), xq(x)\rangle_\triangle$.

Gram-Schmidt process

The projection of $q(x)$ into $p(x)$ is defined by

$$\mathrm{Proj}_{p(x)}(q(x)) := \frac{\langle p(x), q(x)\rangle_{\triangle}}{\|p(x)\|^2_{\triangle}} p(x).$$

Set $p'_0(x) = 1$ and

$$p'_{i+1}(x) = x^{i+1} - \sum_{k=0}^{i} \mathrm{Proj}_{p'_k(x)}(x^{i+1}) \tag{1}$$

for $0 \le i \le d-1$ recursively. Then $p'_0(x), p'_1(x), \ldots, p'_d(x)$ is an orthogonal basis of $\mathbb{R}_d[x]$ such that $p'_i(x)$ has degree i and leading coefficient 1.

Lemma

$p_i'(\lambda_0) > 0$ for $0 \leq i \leq d$.

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Proof.

Let θ_1, θ_2, ..., θ_h be zeros of $p'_i(x)$ in (λ_d, λ_0) for which $p'_i(x)$ takes opposite signs in $(\theta_j - \varepsilon, \theta_j)$ and in $(\theta_j, \theta_j + \varepsilon)$ for all $1 \leq j \leq h$ and for some $\varepsilon > 0$.

Set $q(x) = \prod_{j=1}^{h} (x - \theta_j)$.

Then $q(x)p'_i(x) \geq 0$ for all $x \in [\lambda_d, \lambda_0]$ or $q(x)p'_i(x) \leq 0$ for all $x \in [\lambda_d, \lambda_0]$.

Since $h \leq i < d+1$, there exists an eigenvalue λ_j such that $q(\lambda_j)p'_i(\lambda_j) \neq 0$.

Hence $\langle q(x), p'_i(x) \rangle_\triangle \neq 0$ for all $x \in [\lambda_d, \lambda_0]$.

As $q(x)$ can be written as a linear combination of $p'_0(x), p'_1(x), \ldots, p'_h(x)$, $h = i$ and all zeros of $p'_i(x)$ appear in (λ_d, λ_0).

Thus $q(x) = p'_i(x)$ and hence $p'_i(\lambda_0) = q(\lambda_0) > 0$. □

The predistance polynomials

Set

$$p_i(x) = \frac{p_i'(\lambda_0)}{\|p_i'(x)\|_\triangle^2} p_i'(x). \tag{2}$$

Then $p_0(x), p_1(x), \ldots, p_d(x)$ is the unique system of orthogonal polynomials in $\mathbb{R}_d[x]$ satisfying

$$\deg\ p_i(x) = i$$

and

$$\|p_i(x)\|_\triangle^2 = p_i(\lambda_0).$$

for $0 \leq i \leq d$. The $p_i(x)$ is referred as the i-th predistance polynomial of G. Note that $p_d(\lambda_0) > 0$.

Three-term relations

Lemma

$$xp_i(x) = c_{i+1}p_{i+1}(x) + a_ip_i(x) + b_{i-1}p_{i-1}(x) \qquad 0 \le i \le d \tag{3}$$

for some scalars c_{i+1}, a_i, $b_{i-1} \in \mathbb{R}$ with $b_{-1} = c_{d+1} := 0$.

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for some scalars c_{i+1}, a_i, $b_{i-1} \in \mathbb{R}$ with $b_{-1} = c_{d+1} := 0$.

Proof.

Since $xp_i(x)$ has degree $i+1$, write $xp_i(x) = \sum_{j=0}^{i+1} \alpha_{ij} p_j(x)$ for some $\alpha_{ij} \in \mathbb{R}$.

Then

$$\begin{aligned} \alpha_{ij}\langle p_j(x), p_j(x)\rangle_\triangle &= \langle \sum_{k=0}^{i+1} \alpha_{ik} p_k(x), p_j(x)\rangle_\triangle = \langle xp_i(x), p_j(x)\rangle_\triangle \\ &= \langle p_i(x), xp_j(x)\rangle_\triangle = 0 \end{aligned}$$

if $\deg(xp_j(x)) = j+1 < i$. □

Note that

$$c_{i+1} = \frac{\langle xp_i(x), p_{i+1}(x)\rangle_\triangle}{\|p_{i+1}(x)\|_\triangle^2} \neq 0$$

and

$$b_i = \frac{\langle xp_{i+1}(x), p_i(x)\rangle_\triangle}{\|p_i(x)\|_\triangle^2} = \frac{\langle p_{i+1}(x), xp_i(x)\rangle_\triangle}{\|p_i(x)\|_\triangle^2} \neq 0$$

for $0 \leq i \leq d-1$.

The number

$$\overline{k}_d := |\{(u,v) \,|\, u,v \in VG, \partial(u,v) = d\}|/n$$

is called the the average excess of G, and the number $p_d(\lambda_0)$ is called the spectral excess of G.

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The spectral excess theorem states that

$$\overline{k}_d \leq p_d(\lambda_0) \tag{4}$$

if G is regular, and the equality holds iff G is distance-regular.

(M.A. Fiol and E. Garriga, From local adjacency polynomials to local pseudo-distance-regular graphs, *J. Combin. Theory Ser. B* 71 (1997), 162-183).

The following example shows that the regularity assumption of G in the spectrum excess theorem is necessary.

Example

Let G be a path of three vertices. Then $\mathrm{sp}(G) = \{\sqrt{2}, 0, -\sqrt{2}\}$.
By (1), $p'_0(x) = 1$, $p'_1(x) = x$, $p'_2(x) = x^2 - 4/3$.
By (2), $p_0(x) = 1$, $p_1(x) = 3\sqrt{2}x/4$, $p_2(x) = 3(x^2 - 4/3)/4$.
Note that $\overline{k}_2 = 2/3$ and $p_2(\lambda_0) = 1/2$. This shows that (4) does not hold.

We will generalize the spectrum excess theorem to the non-regular graphs.

Inner product matrix space

Definition

For two $n \times n$ symmetric matrices M, N over $\mathbb{R}$, define the inner product

$$\langle M, N \rangle := \frac{1}{n}\mathrm{tr}(MN) = \frac{1}{n}\sum_{i,j} M_{ij}N_{ij} = \frac{1}{n}\sum_{i,j}(M \circ N)_{ij}, \tag{5}$$

and the norm

$$\|M\| = \sqrt{\langle M, M \rangle},$$

where " $\circ$ " is the entrywise or Hadamard product of matrices.

Thus $\langle p(A), q(A) \rangle = \langle p(x), q(x) \rangle_{\triangle}$ for $p(x), q(x) \in \mathbb{R}_d[x]$.

Orthogonal polynomials associated with a graph

- Hoffman polynomials

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$$H(x) := n\prod_{i=1}^{d}\frac{x-\lambda_i}{\lambda_0-\lambda_i}$$

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Let $q_i(x) = \sum_{j=0}^{i} p_j(x)$. Then $q_i(x)$ has degree i and $q_0(x)$, $q_1(x)$, …, $q_d(x)$ is a basis of $\mathbb{R}_d[x]$. Note that

$$\|q_i(x)\|_{\triangle}^2 = \sum_{j=0}^{i}\|p_j(x)\|_{\triangle}^2 = \sum_{j=0}^{i} p_j(\lambda_0) = q_i(\lambda_0).$$

An optimization problem

Lemma

For $p(x) \in \mathbb{R}_d[x]$ with degree at most i and $\|p(x)\|_\triangle = \|q_i(x)\|_\triangle$, $p(\lambda_0)^2 \leq q_i(\lambda_0)^2$ with equality iff $p(x) = \pm q_i(x)$.

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Proof.

Let $p(x) = \sum_{j=0}^{i} \alpha_j p_j(x)$ for some $\alpha_j \in \mathbb{R}$.

As $q_i(\lambda_0) = \|q_i(x)\|_\triangle^2 = \|p(x)\|_\triangle^2 = \sum_{j=0}^{i} \alpha_j^2 p_j(\lambda_0)$, by Cauchy's inequality,

$$p(\lambda_0)^2 = \left[\sum_{j=0}^{i} \alpha_j p_j(\lambda_0)\right]^2 \le \left[\sum_{j=0}^{i} \alpha_j^2 p_j(\lambda_0)\right] \left[\sum_{j=0}^{i} p_j(\lambda_0)\right] = q_i(\lambda_0)^2,$$

with equality iff all α_j are equal; indeed $\alpha_j = \pm 1$. □

The dual problem

Lemma

For $p(x) \in \mathbb{R}_d[x]$ with degree at most i and $\|p(x)\|_\triangle = \|q_i(x)\|_\triangle$, $\sum_{j=1}^{d} m_j q_i(\lambda_j)^2 \leq \sum_{j=1}^{d} m_j p(\lambda_j)^2$ with equality iff $p(x) = \pm q_i(x)$.

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Proof.

This follows from the previous lemma and

$$\frac{1}{n}(p(\lambda_0)^2 + \sum_{j=1}^{d} m_j p(\lambda_j)^2) = \|p(x)\|_\triangle^2 = \|q_i(x)\|_\triangle^2 = \frac{1}{n}(q_i(\lambda_0)^2 + \sum_{j=1}^{d} m_j q_i(\lambda_j)^2).$$

□

Lemma

For any graph, the sum of all the predistance polynomials gives the Hoffman polynomial, i.e.,

$$H(x) = q_d(x) = p_0(x) + p_1(x) + \cdots + p_d(x). \qquad (6)$$

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Proof.

Let $p(x) = c\prod_{i=1}^{d}\frac{x-\lambda_i}{\lambda_0-\lambda_i}$ for some $c \in \mathbb{R}$ such that $\|p(x)\|_\triangle = \|q_d(x)\|_\triangle$.

By dual problem lemma, $\sum_{j=1}^{d} m_j q_d(\lambda_j)^2 \leq \sum_{j=1}^{d} m_j p(\lambda_j)^2 = 0$.

Then $\sum_{j=1}^{d} m_j q_d(\lambda_j)^2 = 0$ and thus $q_d(x) = \pm p(x)$.

Hence $q_d(\lambda_0) = \|q_d(x)\|_\triangle^2 = (q_d(\lambda_0)^2 + \sum_{j=1}^{d} m_j q_d(\lambda_j)^2)/n = q_d(\lambda_0)^2/n$.

Therefore, $q_d(\lambda_0) = n$, and $q_d(x) = n\prod_{i=1}^{d}\frac{x-\lambda_i}{\lambda_0-\lambda_i} = H(x)$.

Let α be the eigenvector of A corresponding to λ_0 such that $\alpha^t\alpha = n$ and all entries are positive. Note that $\alpha = (1,1,\ldots,1)^t$ iff G is regular.

Lemma

For the graph G,

$$H(A) = \frac{n\alpha\alpha^t}{\alpha^t\alpha} = \alpha\alpha^t.$$

Moreover, G is regular iff $H(A) = J$, the all 1's matrix.

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Moreover, G is regular iff $H(A) = J$, the all 1's matrix.

Proof.

The first equality follows since the matrix in the middle of the equation acts as $H(A)$ on the right eigenvectors of A. The second equality follows from the assumption $\alpha^t\alpha = n$. The remaining is clear.

□

Orthogonal polynomials associated with a graph

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Graphs with odd-girth $2d+1$

For $u \in VG$, let α_u be the entry corresponding to u in the eigenvector α. Let A_i be the i-th distance matrix, i.e., an $n \times n$ matrix with rows and columns indexed by the vertex set VG such that

$$(A_i)_{uv} = \begin{cases} 1, & \text{if } \partial(u,v) = i; \\ 0, & \text{else.} \end{cases}$$

In particular, $A_0 = I$ and $A_1 = A$.

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Define

$$\begin{aligned} p_{\geq D}(x) &:= p_D(x) + p_{D+1}(x) + \cdots + p_d(x), \\ \widetilde{A}_i &:= A_i \circ H(A), \\ \delta_i &:= \|\widetilde{A}_i\|^2. \end{aligned}$$

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Define

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More precisely, $\widetilde{A}_i$ is regarded as a “weighted” version of A_i as follows:

$$(\widetilde{A}_i)_{uv} = \begin{cases} \alpha_u \alpha_v, & \text{if } \partial(u,v) = i; \\ 0, & \text{else.} \end{cases} \tag{7}$$

Note that $\delta_d = 0$ iff $d > D$. The number δ_D is referred as average weighted excess and $p_{\geq D}(\lambda_0)$ is as generalized spectral excess of G.

Note that if $D=d$ we have $p_{\geq D}(x)=p_D(x)$.
By the above definitions, we have

$$\|p_{\geq D}(A)\|^2=\|p_{\geq D}(x)\|_{\triangle}^2=p_{\geq D}(\lambda_0) \tag{8}$$

and

$$\widetilde{A}_0+\widetilde{A}_1+\cdots+\widetilde{A}_D=H(A). \tag{9}$$

It is well-known that $(A^i)_{uv}$ counts the number of walks of length i in G from u to v. In particular, if there exists a cycle of length i in G then $\mathrm{tr}(A^i)\neq 0$. Although $\widetilde{A}_i$ might be different to A_i, they are similar as for $j<i$,

$$\langle A_i, p_j(A)\rangle=0=\langle \widetilde{A}_i, p_j(A)\rangle \tag{10}$$

from (5).

Lemma

The projection of $\widetilde{A}_D$ into $p_{\geq D}(A)$ is

$$\mathrm{Proj}_{p_{\geq D}(A)} \widetilde{A}_D = \frac{\delta_D}{p_{\geq D}(\lambda_0)} \; p_{\geq D}(A).$$

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Proof.

By (5), (8), (9), and (10),

$$\begin{aligned}\mathrm{Proj}_{p_{\geq D}(A)}\widetilde{A}_D &= \frac{\langle \widetilde{A}_D, p_{\geq D}(A)\rangle}{\|p_{\geq D}(A)\|^2}\ p_{\geq D}(A) \\ &= \frac{\langle \widetilde{A}_D, H(A)\rangle}{p_{\geq D}(\lambda_0)}\ p_{\geq D}(A) \\ &= \frac{\delta_D}{p_{\geq D}(\lambda_0)}\ p_{\geq D}(A).\end{aligned}$$

□

Generalized spectral excess theorem

Theorem

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Proof.

By Lemma 3.1,

$$0 \leq \|\widetilde{A}_D\|^2 - \|\mathrm{Proj}_{p_{\geq D}(A)}\widetilde{A}_D\|^2 = \delta_D - \frac{\delta_D^2}{p_{\geq D}(\lambda_0)}.$$

The equality is attained iff $\widetilde{A}_D = \mathrm{Proj}_{p_{\geq D}(A)}\widetilde{A}_D = p_{\geq D}(A)$. □

Revisiting the case that G is a path of three vertices in Example 1.3, $d = D = 2$ and thus $p_{\geq D}(\lambda_0) = p_2(\lambda_0) = 1/2$. Note that $\alpha = (\sqrt{3}/2, \sqrt{6}/2, \sqrt{3}/2)^t$. By (7), we have

$$\widetilde{A}_D = \begin{pmatrix} 0 & 0 & 3/4 \\ 0 & 0 & 0 \\ 3/4 & 0 & 0 \end{pmatrix}.$$

Hence $\delta_D = 3/8 \leq 1/2 = p_{\geq D}(\lambda_0)$ satisfies the inequality in Theorem 3.2.

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Hence $\delta_D = 3/8 \leq 1/2 = p_{\geq D}(\lambda_0)$ satisfies the inequality in Theorem 3.2.

Remark

If G is regular with diameter $D = 2$, then the equality in Theorem 3.2 holds. Indeed $\widetilde{A}_2 = A_2 = J - I - A = H(A) - I - A = p_{\geq 2}(A)$.

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Proof.

From (7), $(\widetilde{A}_0)_{uu} = \alpha_u^2$ for $u \in VG$. Since α has positive entries and $\alpha^t \alpha = n$, $\alpha^t = (1, 1, \ldots, 1)$ iff G is regular. □

Theorem

Let G be a connected graph of diameter D equal to spectral diameter d.
Then $\widetilde{A}_D = p_D(A)$ iff $\widetilde{A}_i = p_i(A)$ for $0 \leq i \leq D-1$.
Moreover, if $\widetilde{A}_D = p_D(A)$ then G is distance-regular.

Theorem

Let G be a connected graph of diameter D equal to spectral diameter d.
Then $\widetilde{A}_D = p_D(A)$ iff $\widetilde{A}_i = p_i(A)$ for $0 \le i \le D-1$.
Moreover, if $\widetilde{A}_D = p_D(A)$ then G is distance-regular.

Proof.

The sufficiency follows from deleting $\widetilde{A}_i = p_i(A)$ for $0 \le i \le D-1$ in both sides of

$$\widetilde{A}_0 + \widetilde{A}_1 + \cdots + \widetilde{A}_D = H(A) = p_0(A) + p_1(A) + \cdots + p_D(A). \tag{11}$$

The necessity follows by (backward) induction on $0 \le i \le D$. The base case is the assumption that $\widetilde{A}_D = p_D(A)$. Suppose now that $p_k(A) = \widetilde{A}_k$ for $D \ge k \ge i$. Then deleting these common terms from both sides of (11), we have

$$\widetilde{A}_0 + \widetilde{A}_1 + \cdots + \widetilde{A}_{i-1} = p_0(A) + p_1(A) + \cdots + p_{i-1}(A), \tag{12}$$

Proof.

and by induction hypothesis to the three-term recurrence in (3),

$$\begin{aligned} A\widetilde{A}_i &= c_{i+1}p_{i+1}(A)+a_ip_i(A)+b_{i-1}p_{i-1}(A) \\ &= c_{i+1}\widetilde{A}_{i+1}+a_i\widetilde{A}_i+b_{i-1}p_{i-1}(A). \end{aligned} \tag{13}$$

It remains to show that $p_{i-1}(A)=\widetilde{A}_{i-1}$. To this end, consider the following two cases:

(*i*) For $\partial(u,v)\geq i-1$, $(p_{i-1}(A))_{uv}=(\widetilde{A}_{i-1})_{uv}$ by (12).

(*ii*) For $\partial(u,v)<i-1$, $(A\widetilde{A}_i)_{uv} = \sum_{w\in G(u)}(\widetilde{A}_i)_{wv} = 0$, where the last equality follows since $\partial(w,v)\leq 1+ \partial(u,v)<i$.
Then $(p_{i-1}(A))_{uv}=0$ by (13) and since $b_{i-1}\neq 0$.

This proves the necessity. Suppose $\widetilde{A}_D=p_D(A)$. Then G is regular by applying the necessary condition in the case $i=0$ to Lemma 4.1. Thus G is distance-regular by the spectral excess thoerem.

□

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Graph with odd girth $2d+1$

From now on, assume that G has odd-girth $2d+1$, i.e., the shortest odd cycle has length $2d+1$. As an application of Theorem 4.2, we will show that G has diameter $D=d$ and G must be distance-regular.

Graph with odd girth $2d+1$

From now on, assume that G has odd-girth $2d+1$, i.e., the shortest odd cycle has length $2d+1$. As an application of Theorem 4.2, we will show that G has diameter $D=d$ and G must be distance-regular.
For a vertex u, let $G_d(u)$ be the set of vertices at distance d from u.
If $D<d$ then $G_d(u)=\emptyset$.

Let $c=n/\prod_{i=1}^{d}(\lambda_0-\lambda_i)$ and note that c is the leading coefficient of the Hoffman polynomial $H(x)$. For two vertices $u,v\in VG$ with $\partial(u,v)=d$,

$$(A^d)_{uv}=H(A)_{uv}/c \tag{14}$$

and

$$(A^{d+1})_{uv}=Z(A)_{uv}+(\sum_{i=0}^{d}\lambda_i)(A^d)_{uv}=(\sum_{i=0}^{d}\lambda_i)H(A)_{uv}/c. \tag{15}$$

Lemma

The average weighted excess δ_D of G equals $c^2\mathrm{tr}(A^{2d+1})/(n\sum_{i=0}^{d}\lambda_i)$.
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Lemma

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Proof.

For vertices $u, v \in VG$ with $\partial(u,v) < d$, $(A^d)_{uv} = 0$ or $(A^{d+1})_{vu} = 0$ since no odd cycle has length less than $2d+1$. By (5), (9), (14), (15),

$$\begin{aligned} n(\textstyle\sum_{i=0}^d \lambda_i)\delta_d &= (\textstyle\sum_{i=0}^d \lambda_i) \displaystyle\sum_{u,v \in VG} [(\widetilde{A}_d)_{uv}]^2 \\ &= (\textstyle\sum_{i=0}^d \lambda_i) \displaystyle\sum_{u \in VG} \sum_{v \in G_d(u)} [H(A)_{uv}]^2 \\ &= c^2 \sum_{u \in VG} \sum_{v \in VG} (A^d)_{uv} (A^{d+1})_{uv} = c^2 \text{tr}(A^{2d+1}). \end{aligned}$$

As $\text{tr}(A^{2d+1}) \neq 0$, we have $\sum_{i=0}^d \lambda_i \neq 0$ and $\delta_d = c^2\text{tr}(A^{2d+1})/(n\sum_{i=0}^d \lambda_i) > 0$. This also implies $D = d$. □

Lemma

Referring the notations of three-term recurrence in (3),

(*i*) $a_{j-1} = 0$ for $1 \leq j \leq d$;

(*ii*) $p_j(x)$ is an even or odd polynomial depending on whether j is even or odd for $0 \leq j \leq d$.

Moreover, the generalized spectral excess $p_d(\lambda_0)$ is $c^2 \mathrm{tr}(A^{2d+1})/(n\sum_{i=0}^{d} \lambda_i)$.

Proof.

Clearly, $p_0(x) = 1$ is even. We prove (i)-(ii) by induction on $j \geq 1$.
By (2), $p_1(x) = n\lambda_0 x / \sum_{i=0}^{d} m_i \lambda_i^2$ is odd. Setting $i = 0$ in (3), $a_0 = 0$.
Hence we have (i)-(ii) in the base case $j = 1$. By (3),

$$a_k p_k(\lambda_0) = \langle a_k p_k(x), p_k(x) \rangle_\triangle = \langle x p_k(x), p_k(x) \rangle_\triangle = \mathrm{tr}(A p_k^2(A))/n \qquad (16)$$

for $0 \leq k \leq d$. Now suppose (i)-(ii) for $j = k < d$. Since $x p_k^2(x)$ is an odd polynomial of degree $2k+1 < 2d+1$, the last term in (16) is zero.
Hence $a_k = 0$ and (i) holds for $j = k+1$. From (i) and setting $i = k$ in (3), the polynomial $p_{k+1}(x)$ satisfies (ii). This proves (i)-(ii) in any j.

Proof.

For the remaining, since the last term in (16) with $k=d$ equals $c^2\text{tr}(A^{2d+1})/n$, it suffices to show $a_d=\sum_{i=0}^{d}\lambda_i$. Choose two vertices u and v at distance d. Then by (3), (6), (15),

$$a_dH(A)_{uv}=a_dp_d(A)_{uv}=(Ap_d(A))_{uv}=c(A^{d+1})_{uv}=(\sum_{i=0}^{d}\lambda_i)H(A)_{uv},$$

where the third equality follows because $xp_d(x)$ has no term of degree d. Dividing both sides by $H(A)_{uv}$, we have $a_d=\sum_{i=0}^{d}\lambda_i$.

□

Proof.

For the remaining, since the last term in (16) with $k = d$ equals $c^2\mathrm{tr}(A^{2d+1})/n$, it suffices to show $a_d = \sum_{i=0}^{d} \lambda_i$. Choose two vertices u and v at distance d. Then by (3), (6), (15),

$$a_d H(A)_{uv} = a_d p_d(A)_{uv} = (Ap_d(A))_{uv} = c(A^{d+1})_{uv} = (\sum_{i=0}^{d} \lambda_i) H(A)_{uv},$$

where the third equality follows because $xp_d(x)$ has no term of degree d. Dividing both sides by $H(A)_{uv}$, we have $a_d = \sum_{i=0}^{d} \lambda_i$. □

From Lemma 5.1-5.2, and Theorem 4.2, we immediately have the following theorem.

Theorem

Any connected graph with $d+1$ distinct eigenvalues and odd-girth $2d+1$ must be distance-regular.

Reference

- N. Biggs, *Algebraic Graph Theory*, Cambridge University Press, Cambridge, 1993.
- A.E. Brouwer, A.M. Cohen and A. Neumaier, *Distance-Regular Graphs*, Springer-Verlag, Berlin, 1989.
- C. Dalfó, E.R. van Dam, M.A. Fiol and E. Garriga, Dual concept of almost distance-regularity and the spectral excess theorem, preprint.
- C. Dalfó, E.R. van Dam, M.A. Fiol, E. Garriga and B.L. Gorissen, On almost distance-regular graphs, *J. Combin. Theory Ser. A* 118 (2011), 1094-1113.
- E.R. van Dam, The spectral excess theorem for distance-regular graphs: a global (over)view, *Electron. J. Combin.* 15 (1) (2008), R129.
- E.R. van Dam and W.H. Haemers, An odd characterization of the generalized odd graphs, *J. Combin. Theory Ser. B* (2011), doi:10.1016/j.jctb.2011.03.001

- M.A. Fiol and E. Garriga, From local adjacency polynomials to local pseudo-distance-regular graphs, *J. Combin. Theory Ser. B* 71 (1997), 162-183.
- M.A. Fiol, S. Gago and E. Garriga, A simple proof of the spectral excess theorem for distance-regular graphs, *Linear Algebra Appl.* 432 (2010), 2418-2422.
- A.J. Hoffman, On the polynomial of a graph, *Amer. Math. Monthly* 70 (1963), 30-36.
- T. Huang, Spectral characterization of odd graphs O_k, $k \leq 6$, *Graphs Combin.* 10 (1994), 235-240.
- T. Huang and C. Liu, Spectral characterization of some generalized odd graphs, *Graphs Combin.* 15 (1999), 195-209.
- P. Terwilliger, The subconstituent algebra of an association scheme I, *J. Alg. Combin.* 1 (1992), 363-388.
- P.M. Weichsel, On distance-regularity in graphs, *J. Combin. Theory Ser. B* 32 (1982), 156-161.