# Polynomials with Real Zeros and Pólya Frequency Sequences

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**Abstract**

Let $f(x)$ and $g(x)$ be two real polynomials whose leading coefficients have the same sign. Suppose that $f(x)$ and $g(x)$ have only real zeros and that $g$ interlaces $f$ or $g$ alternates left of $f$. We show that if $ad \geq bc$ then the polynomial

$$(bx+a)f(x)+(dx+c)g(x)$$

has only real zeros. Applications are related to certain results of F.Brenti (Mem. Amer. Math. Soc. 413 (1989)) and transformations of Pólya frequency sequences. More specifically, suppose that $A(n,k)$ are nonnegative numbers which satisfy the recurrence

$$A(n,k)=(rn+sk+t)A(n-1,k-1)+(an+bk+c)A(n-1,k)$$

for $n \geq 1$ and $0 \leq k \leq n$, where $A(n,k)=0$ unless $0 \leq k \leq n$. We show that if $rb \geq as$ and $(r+s+t)b \geq (a+c)s$, then for each $n \geq 0$, $A(n,0), A(n,1), \ldots, A(n,n)$ is a Pólya frequency sequence. This gives a unified proof of the PF property of many well-known sequences including the binomial coefficients, the Stirling numbers of two kinds and the Eulerian numbers.

**Keywords** Unimodality; Log-concavity; Pólya frequency sequences

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# 1 Introduction

Let $a_0, a_1, a_2, \ldots$ be a sequence of nonnegative real numbers. It is *unimodal* if $a_0 \le a_1 \le \cdots \le a_{k-1} \le a_k \ge a_{k+1} \ge \cdots$ for some $k$. It is *log-concave*(LC) if $a_{i-1}a_{i+1} \le a_i^2$ for all $i > 0$. It is said to *have no internal zeros* if there are no three indices $i < j < k$ such that $a_i, a_k \ne 0$ and $a_j = 0$. Clearly, a log-concave sequence with no internal zeros is unimodal. Unimodal and log-concave sequences occur naturally in combinatorics, algebra, analysis, geometry, computer science, probability and statistics. We refer the reader to the survey papers by Stanley[27] and Brenti[7] for various results on unimodality and log-concavity.

A classical approach for attacking unimodality and log-concavity problems of finite sequences is to use the following result originally due to Newton[14, p.104].

**Newton's Inequality** *Given a finite sequence $a_0, a_1, \ldots, a_n$ of nonnegative numbers. Suppose that its generating function $\sum_{i=0}^{n} a_i x^i$ has only real zeros. Then*

$$a_i^2 \ge a_{i-1}a_{i+1}\left(\frac{i+1}{i}\right)\left(\frac{n-i+1}{n-i}\right), \qquad i = 1, 2, \ldots, n-1,$$

*and in particular, the sequence is log-concave(with no internal zeros).*

It is natural to look at those sequences whose associated polynomial has only real zeros. A characterization for such sequences comes from the theory of total positivity. Let $A = (a_{ij})_{i,j\ge 0}$ be an infinite matrix. We say that $A$ is *totally positive* (or *TP*, for short) if all minors of $A$ have nonnegative determinants. An infinite sequence $a_0, a_1, a_2, \ldots$ of nonnegative numbers is called a *Pólya frequency sequence*(or a *PF* sequence) if the matrix $(a_{i-j})_{i,j\ge 0}$ is a TP matrix (where $a_k = 0$ if $k < 0$). A finite sequence $a_0, a_1, \ldots, a_n$ is *PF* if the infinite sequence $a_0, a_1, \ldots, a_n, 0, 0, \ldots$ is PF. By definition, a PF sequence is necessarily log-concave(with no internal zeros). A deeper result is the following theorem which provides the basic link between finite PF sequences and polynomials having only real zeros. For more information about TP matrices and PF sequences, see [17].

**Aissen-Schoenberg-Whitney Theorem**[3] *A finite sequence $a_0, \ldots, a_n$ of nonnegative numbers is PF if and only if its generating function $\sum_{i=0}^{n} a_i x^i$ has only real zeros.*

Brenti[6] was the first one who applied total positivity techniques to study systematically unimodality and log-concavity problems. He noted that PF sequences have much better behavior than unimodal and log-concave sequences. It may often be more convenient to prove that a sequence is PF even if we are actually interested only in the unimodality or log-concavity. On the other hand, many of unimodal and log-concave sequences arising in combinatorics turn out to be PF sequences. So it is natural to pay more attention to PF sequences, as well as polynomials having only real zeros. Indeed, there are a number of open problems in combinatorics concerning whether certain polynomials have only real zeros (see [7, 27, 28] for instance). The present paper is devoted to studying certain transformations of polynomials(resp. sequences) that preserve the reality of zeros(resp. PF property).

Let **RZ** be the set of real polynomials having only real zeros and **PF** the set of polynomials in **RZ** whose coefficients are nonnegative. In other words, **PF** is the set of polynomials whose coefficients form a PF sequence. It is clear that all zeros of any polynomial in **PF** are nonpositive. For convenience let

**RZ** contain all real constants and **PF** contain all nonnegative numbers. Let $\mathbf{RZ}_n$ and $\mathbf{PF}_n$ be the sets of the corresponding polynomials of degree $n$ in **RZ** and **PF** respectively.

Suppose that $f(x) \in \mathbf{RZ}_n$ and $g(x) \in \mathbf{RZ}_m$ where $n \geq m \geq 1$. Let $r_n \leq \cdots \leq r_1$ and $s_m \leq \cdots \leq s_1$ be the zeros of $f$ and $g$ respectively. Following [34], we say that $g$ *interlaces* $f$ if $m = n - 1$ and

$$r_n \leq s_{n-1} \leq r_{n-1} \leq \cdots \leq r_2 \leq s_1 \leq r_1,$$

and that $g$ *alternates left of* $f$ if $m = n$ and

$$s_n \leq r_n \leq s_{n-1} \leq r_{n-1} \leq \cdots \leq r_2 \leq s_1 \leq r_1.$$

By $g \rightsquigarrow f$ we denote "either $g$ interlaces $f$ or $g$ alternates left of $f$". For notational convenience, let $a \rightsquigarrow bx + c$ for all real constants $a, b, c$. Clearly, $g \rightsquigarrow f$ yields $ag \rightsquigarrow bf$ for any $ab \neq 0$.

Our first main result concerns certain transformations on polynomials having only real zeros.

**Theorem 1** *Let $f(x)$ and $g(x)$ be two real polynomials whose leading coefficients have the same sign and let $F(x) = (bx+a)f(x)+(dx+c)g(x)$. Suppose that $f, g \in \mathbf{RZ}$ and $g \rightsquigarrow f$. Then, if $ad \geq bc$, $F(x) \in \mathbf{RZ}$.*

To get a glimpse of the preceding theorem, readers are suggested to consider the simplest possible example with $\deg f = 1$ and $\deg g \leq 1$. Theorem 1 will prove to be an extremely useful tool. We will give certain applications in several aspects, especially in transformations of PF sequences.

Let $\{x(n,k)\}_{n \geq k \geq 0}$ be a triangular array of nonnegative numbers satisfying a two-term recursion

$$x(n,k) = b(n,k)x(n-1,k-1) + a(n,k)x(n-1,k)$$

for $n \geq 1, 0 \leq k \leq n$, where $x(n,k) = 0$ unless $0 \leq k \leq n$, and the coefficients $a(n,k), b(n,k)$ are nonnegative. For convenience we take $x(0,0) = 1$. Such a triangular array is said to be *unimodal*(resp. *LC*, *PF*) if for each $n \geq 0$, the sequence $x(n,0), x(n,1), \ldots, x(n,n)$ has the corresponding property. Canfield[8] considered the unimodality problem. Kurtz[19] and Sagan[23] provided certain sufficient conditions on coefficients such that triangular arrays are LC respectively. What conditions will insure that triangular arrays are PF? This question is, in general, very difficult to answer. Since our interest in this matter stems from combinatorial motivations, we consider only triangular arrays $\{A(n,k)\}$ which satisfy a recurrence of "bilinear" form

$$A(n,k) = (rn + sk + t)A(n-1,k-1) + (an + bk + c)A(n-1,k) \tag{1}$$

for $n \geq 1, 0 \leq k \leq n$, where $A(n,k) = 0$ unless $0 \leq k \leq n$, and $r, s, t, a, b, c$ are real numbers. Many important triangular arrays arising in combinatorics satisfy such a recurrence. It is easy to see that such triangular arrays are necessarily LC[19].

Our second main result provides sufficient conditions for which $\{A(n,k)\}$ is PF. This yields a unified proof of the PF property of many well-known sequences, including the binomial coefficients, the Stirling numbers of both kinds, and the Eulerian numbers.

**Theorem 2** *Let $\{A(n,k)\}$ be a triangular array defined by (1). Suppose that $rb \geq as$ and $(r+s+t)b \geq (a+c)s$. Then the triangular array $\{A(n,k)\}$ is PF.*

This paper is organized as follows. The next section is devoted to the proof of Theorem 1. In §3 we present several applications of Theorem 1 related to certain results of Brenti[6] and transformations of PF sequences, the latter of which induces a proof of Theorem 2. We also provide some examples of triangular arrays that can be verified to be PF by Theorem 2.

## 2 Proof of Theorem 1

Let *sgn* denote the sign function defined by

$$\mathrm{sgn}(x) = \begin{cases} +1 & \text{for } x > 0 \\ 0 & \text{for } x = 0 \\ -1 & \text{for } x < 0. \end{cases}$$

Let $f(x)$ be a real function. If $f(x) > 0$(resp. $f(x) < 0$) for sufficiently large $x$, then we denote $\mathrm{sgn} f(+\infty) = +1$(resp. $-1$). The meaning of $\mathrm{sgn} f(-\infty)$ is similar.

Before showing Theorem 1, we provide four lemmas to deal with certain special cases of the theorem. The first one is a fundamental and known result (see Section 3 of [34] for instance). For completeness we include a proof of it.

**Lemma 1** *Suppose that $f, g \in \mathbf{RZ}$ and $g \rightsquigarrow f$. Then $f + g \in \mathbf{RZ}$. Furthermore, if the leading coefficients of $f$ and $g$ have the same sign, then $g \rightsquigarrow f + g \rightsquigarrow f$.*

**Proof** Let $g \rightsquigarrow f$. Clearly, if $r$ is an $m$-fold zero of $f(x)$ and a $k$-fold zero of $g(x)$ respectively, then $|m - k| \leq 1$. Hence we may assume, without loss of generality, that $f$ and $g$ have no zeros in common (which implies that they have only simple zeros).

The result is obvious if $g$ is a constant. So, we let $\deg g \geq 1$. We may assume that $f(x) = \alpha \prod_{i=1}^{n} (x - r_i)$ and $g(x) = \prod_{i=1}^{m} (x - s_i)$ where

$$s_n < r_n < s_{n-1} < r_{n-1} < \cdots < r_2 < s_1 < r_1$$

(set $s_n = -\infty$ if $m = n-1$). Denote $h(x) = f(x)+g(x)$. For $j = 1, 2, \ldots, n-1$, we have

$$\mathrm{sgn} h(s_j) = \mathrm{sgn}\left(\alpha \prod_{i=1}^{n} (s_j - r_i)\right) = (-1)^j \mathrm{sgn}(\alpha)$$

and

$$\mathrm{sgn} h(r_j) = \mathrm{sgn}\left(\prod_{i=1}^{n-1} (r_j - s_i)\right) = (-1)^{j-1}.$$

Also, $\mathrm{sgn} h(-\infty) = (-1)^n \mathrm{sgn}(\alpha)$ and $\mathrm{sgn} h(+\infty) = \mathrm{sgn}(\alpha)$.

If $\alpha > 0$, then $h(s_j)h(r_j) < 0$ for $j = 1, 2, \ldots, n$. By the intermediate-value theorem, there exists (exactly) one zero of $h(x)$ in each interval $(s_j, r_j)$. Thus $h(x) \in \mathbf{RZ}_n$ and $g \rightsquigarrow h \rightsquigarrow f$.

Assume now that $\alpha < 0$. Then $h(r_j)h(s_{j-1}) < 0$ for $j = 2, \ldots, n$. It follows that $h(x)$ has at least $n-1$ real zeros. However, $\deg h \leq n$. Hence $h \in \mathbf{RZ}$, and thus the proof is complete. $\square$

Let $P(x)$ is a real polynomial of degree $n$. Define its *reciprocal* polynomial by

$$P^*(x) = x^n P(1/x).$$

The following facts are elementary but very useful in the sequel:

(i) If $P(0) \neq 0$ then $\deg P^* = \deg P$ and $(P^*)^* = P$.

(ii) $P \in \mathbf{RZ}$ if and only if $P^* \in \mathbf{RZ}$.

(iii) Suppose that all zeros of $f$ and $g$ are negative and $g \rightsquigarrow f$. If $g$ interlaces $f$, then $g^*$ interlaces $f^*$. If $g$ alternates left of $f$, then $f^*$ alternates left of $g^*$.

In the following three lemmas we assume that $f$ and $g$ are two monic polynomials with only simple negative zeros and that $g$ interlaces $f$. More precisely, let $f(x) = \prod_{i=1}^{n}(x - r_i)$ and $g(x) = \prod_{i=1}^{n-1}(x - s_i)$ where

$$r_n < s_{n-1} < r_n < \cdots < s_2 < r_2 < s_1 < r_1 < 0$$

($g(x) = 1$ provided $n = 1$).

**Lemma 2** *Suppose that $ad > 0$. Then*

(i) $af + dxg \in \mathbf{RZ}$ *and* $f, g \rightsquigarrow af + dxg$*;*

(ii) $af + (dx + c)g \in \mathbf{RZ}$ *for any* $c$*;*

(iii) $(bx + a)f + dxg \in \mathbf{RZ}$ *for any* $b$*.*

**Proof** (i) Let $F = af + dxg$. Then $F^* = af^* + dg^*$. By the assumption, $g$ interlaces $f$, so $g^*$ interlaces $f^*$. From Lemma 1 it follows that $F^* \in \mathbf{RZ}$, $g^*$ interlaces $F^*$ and $F^*$ alternates left of $f^*$. Thus $F \in \mathbf{RZ}$, $g$ interlaces $F$ and $f$ alternates left of $F$.

(ii) By (i), $F \in \mathbf{RZ}$ and $g \rightsquigarrow F$. Thus $F + cg \in \mathbf{RZ}$ for any $c$ by Lemma 1, i.e., $af + (dx + c)g \in \mathbf{RZ}$.

(iii) By (i), $f \rightsquigarrow F$. Note that all zeros of the polynomial $F$ are negative since its coefficients has the same sign. Hence $F \rightsquigarrow xf$. Thus $bxf + F \in \mathbf{RZ}$ for any $b$ by Lemma 1, i.e., $(bx + a)f + dxg \in \mathbf{RZ}$. $\square$

**Remark 1** The condition $ad > 0$ in Lemma 2 is not necessary. Actually, $g \rightsquigarrow f$ implies $f \rightsquigarrow xg$ since all zeros of $f$ are negative, and thus $af + dxg \in \mathbf{RZ}$ for any $a, d$ by Lemma 1. We can also show that $ac > 0$ implies $af + (dx + c)g \in \mathbf{RZ}$ for any $d$, which is in a sense "dual" to Lemma 2(ii) since $[af + (dx + c)g]^* = af^* + (cx + d)g^*$. Similarly, as the dual version of Lemma 2(iii), we have $(bx + a)f + dxg \in \mathbf{RZ}$ when $bd > 0$.

**Lemma 3** *Suppose that $bc < 0$. Then*

(i) $bxf + cg \in \mathbf{RZ}$ *and* $f, xg \rightsquigarrow bxf + cg$;

(ii) $bxf + (dx + c)g \in \mathbf{RZ}$;

(iii) $(bx + a)f + cg \in \mathbf{RZ}$ *and* $f \rightsquigarrow (bx + a)f + cg$ *for any* $a$.

**Proof** Let $F = bxf + cg$. Without loss of generality, let $b > 0$ and $c < 0$. We have

$$\mathrm{sgn} F(r_j) = \mathrm{sgn}\left(c\prod_{i=1}^{n-1}(r_j - s_i)\right) = (-1)^j, \quad j = 1, 2\cdots, n$$

and

$$\mathrm{sgn} F(s_j) = \mathrm{sgn}\left(bs_j\prod_{i=1}^{n}(s_j - r_i)\right) = (-1)^{j+1}, \quad j = 1, 2\cdots, n-1.$$

Also, $\mathrm{sgn}F(0) = -1, \mathrm{sgn}F(-\infty) = (-1)^{n+1}$ and $\mathrm{sgn}F(+\infty) = 1$. We make a list of signs of $F$ as follows:

| $x:$ | $-\infty$ | $r_n$ | $s_{n-1}$ | $r_{n-1}$ | $\cdots$ | $s_2$ | $r_2$ | $s_1$ | $r_1$ | $0$ | $+\infty$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $b > 0, c < 0$ | $(-1)^{n+1}$ | $(-1)^n$ | $(-1)^n$ | $(-1)^{n-1}$ | $\cdots$ | $-$ | $+$ | $+$ | $-$ | $-$ | $+$ |

By the intermediate-value theorem, $F$ has $n+1$ real zeros $t_1, \ldots, t_{n+1}$ satisfying

$$t_{n+1} < r_n < s_{n-1} < t_n < r_{n-1} \cdots < s_2 < t_3 < r_2 < s_1 < t_2 < r_1 < 0 < t_1.$$

Thus $F \in \mathbf{RZ}$ and $f, xg \rightsquigarrow F$. This proves (i).

Write $bxf + (dx + c)g = F + d(xg)$ and$(bx + a)f + cg = F + af$. Then both (ii) and (iii) follow from (i) and Lemma 1. □

**Lemma 4** *Suppose that $ad \geq bc$. Then $F = (bx + a)f + (dx + c)g \in \mathbf{RZ}$.*

**Proof** If $abcd = 0$, then the statement follows from the previous three lemmas. If $ad = bc$, then the statement follows from Lemma 1 since $F$ has the factor $bx + a$. So let $abcd \neq 0$ and $ad > bc$. Without loss of generality, we assume that $a > 0$. There are four cases to consider.

Case 1. $c > 0$. Then $g \rightsquigarrow af + cg$ by Lemma 1. Note that

$$aF = (bx + a)(af + cg) + (ad - bc)xg.$$

Hence $aF \in \mathbf{RZ}$ by Lemma 2(iii), and so $F \in \mathbf{RZ}$.

Case 2. $b, c < 0$. Then $d > 0$ and $-a/b > -c/d > 0$. Thus $(dx + c)g$ interlaces $(bx + a)f$, and so $F = (bx + a)f + (dx + c)g \in \mathbf{RZ}$ by Lemma 1.

Case 3. $b, d > 0$. Then $g \rightsquigarrow bf + dg$ by Lemma 1. Note that

$$bF = (bx + a)(bf + dg) + (bc - ad)g.$$

Hence $bF \in \mathbf{RZ}$ by Lemma 3(iii), and so $F \in \mathbf{RZ}$.

Case 4. $a, b > 0, c, d < 0$.

Assume that $r_k < -c/d < r_{k-1}$ for some $k$. Then $dr_j + c < 0$ for $1 \leq j \leq k - 1$ and $dr_j + c > 0$ for $k \leq j \leq n$. Thus

$$\mathrm{sgn} F(r_j) = \mathrm{sgn}\left((dr_j + c)\prod_{i=1}^{n-1}(r_j - s_i)\right) = \begin{cases} (-1)^j & \text{if } 1 \leq j \leq k-1, \\ (-1)^{j+1} & \text{if } k \leq j \leq n. \end{cases}$$

Moreover,

$$\mathrm{sgn} F\left(-\frac{c}{d}\right) = \mathrm{sgn}\left(\frac{ad - bc}{d}\prod_{i=1}^{n}\left(-\frac{c}{d} - r_i\right)\right) = (-1)^k$$

and $\mathrm{sgn} F(+\infty) = 1$. Make a list of signs of $F(x)$ as follows:

| $x:$ | $r_n$ | $\cdots$ | $r_k$ | $-c/d$ | $r_{k-1}$ | $\cdots$ | $r_1$ | $+\infty$ |
|---|---|---|---|---|---|---|---|---|
| $r_k < -c/d < r_{k-1}$ | $(-1)^{n+1}$ | $\cdots$ | $(-1)^{k+1}$ | $(-1)^k$ | $(-1)^{k-1}$ | $\cdots$ | $-$ | $+$ |

From this we see that $F(x)$ has $n + 2$ changes of sign. Hence $F(x)$ has $n + 1$ real zeros, and so $F \in \mathbf{RZ}$.

Similarly, we may prove $F \in \mathbf{RZ}$ provided $r_1 < -c/d$ or $r_n > -c/d$.

Assume next that $r_k = -c/d$ for some $k$. We consider only the case $1 < k < n$ since the proof for the case $k = 1$ or $k = n$ being similar. We have

$$\mathrm{sgn} F(r_j) = \mathrm{sgn}\left((dr_j + c)\prod_{i=1}^{n-1}(r_j - s_i)\right) = \begin{cases} (-1)^j & \text{if } 1 \le j \le k-1, \\ 0 & \text{if } j = k, \\ (-1)^{j+1} & \text{if } k+1 \le j \le n \end{cases}$$

and $\mathrm{sgn} F(+\infty) = 1$. Also,

$$\mathrm{sgn} F(s_k) = \mathrm{sgn}\left((bs_k + a)\prod_{i=1}^{n}(s_k - r_i)\right) = (-1)^{k+1}$$

since $bs_k + a < br_k + a = \frac{ad-bc}{d} < 0$. Make a list of signs of $F$ as follows:

| $x:$ | $r_n$ | $\cdots$ | $r_{k+1}$ | $s_k$ | $r_k$ | $r_{k-1}$ | $\cdots$ | $r_1$ | $+\infty$ |
|---|---|---|---|---|---|---|---|---|---|
| $r_k = -c/d$ | $(-1)^{n+1}$ | $\cdots$ | $(-1)^{k+2}$ | $(-1)^{k+1}$ | $0$ | $(-1)^{k-1}$ | $\cdots$ | $-$ | $+$ |

From this we see that $F(x)$ has $n-k$ zeros in the interval $(r_n, s_k)$ and $k-1$ ones in the interval $(r_{k-1}, +\infty)$. It remains to show that the interval $(s_k, r_{k-1})$ contains two zeros of $F(x)$. Actually, since $F(s_k)$ and $F(r_{k-1})$ have the same sign, the interval $(s_k, r_{k-1})$ contains an even number of zeros of $F(x)$ (see [21, Part V, Prob.8] for instance). In this interval, $F(x)$ has one zero $-c/d$ and thus at least two zeros, as desired.

This completes the proof. $\square$

We are now in a position to prove Theorem 1. The proof consists of the following two propositions.

**Proposition 1** *Let $f(x)$ and $g(x)$ be two real polynomials whose leading coefficients have the same sign and* $\deg g = \deg f - 1$. *Suppose that $f, g \in \mathbf{RZ}$ and $g$ interlaces $f$. Then, if $ad \ge bc$, $F(x) = (bx+a)f(x) + (dx+c)g(x) \in \mathbf{RZ}$.*

**Proof** Without loss of generality, we may assume that $f$ and $g$ are monic and have no zeros in common. Clearly, it suffices to consider the case $\deg f \geq 1$.

Let $r$ be a real number larger than all zeros of $f(x)$. Define $f_1(x) = f(x+r), g_1(x) = g(x+r)$ and $F_1(x) = F(x+r)$. Then $f_1(x), g_1(x)$ have only negative zeros and $g_1 \rightsquigarrow f_1$. Also,

$$F_1(x) = [bx + (br+a)]f_1(x) + [dx + (dr+c)]g_1(x).$$

From Lemma 4 it follows that $F_1(x) \in \mathbf{RZ}$ since

$$(br+a)d = bdr + ad \geq bdr + bc = b(dr+c).$$

Thus $F(x) \in \mathbf{RZ}$. $\Box$

**Proposition 2** *Let $f(x)$ and $g(x)$ be two real polynomials whose leading coefficients have the same sign and $\deg g = \deg f$. Suppose that $f, g \in \mathbf{RZ}$ and $g$ alternates left of $f$. Then, if $ad \geq bc$, $F(x) = (bx+a)f(x)+(dx+c)g(x) \in \mathbf{RZ}$.*

**Proof** Without loss of generality, we may assume that $f(x) = \prod_{i=1}^{n}(x - r_i)$ and $g(x) = \prod_{i=1}^{n}(x - s_i)$ where $n \geq 1$ and $s_n < r_n < \cdots < s_1 < r_1$. Let

$$\begin{aligned} f_1(x) &= \prod_{i=1}^{n}(x + r_1 - s_i) = g(x + r_1), \\ g_1(x) &= \prod_{i=1}^{n-1}(x + r_1 - r_i) = f(x + r_1)/x \end{aligned}$$

(set $g_1(x) = 1$ if $n = 1$), and $F_1(x) = F(x+r_1)$. Then $f_1, g_1$ have only negative zeros and $g_1$ interlaces $f_1$. Moreover,

$$F_1 = [dx + (dr+c)]f_1 + x[bx + (br+a)]g_1.$$

So

$$F_1^* = [(dr+c)x + d]f_1^* + [(br+a)x + b]g_1^*.$$

Clearly, $\deg f_1^* = n, \deg g_1^* = n-1$ and $g_1^*$ interlaces $f_1^*$. Also,

$$d(br+a) = bdr + ad \geq bdr + bc = (dr+c)b.$$

By Proposition 1, we have $F_1^* \in \mathbf{RZ}$. Hence $F_1 \in \mathbf{RZ}$, and so $F \in \mathbf{RZ}$. $\square$

# 3 Applications of Theorem 1 and Proof of Theorem 2

In this section we give several applications of Theorem 1. In particular, we will prove Theorem 2. We also present certain examples of triangular arrays which may be verified to be PF by Theorem 2.

**Proposition 3** *Suppose that $f(x), g(x) \in \mathbf{PF}$ and that $g$ interlaces $f$. Let*

$$F(x) = (ax+b)f(x) + x(cx+d)g(x).$$

*Then if $ad \geq bc$, $F(x) \in \mathbf{RZ}$.*

**Proof** Note that $g$ interlaces $f$ implies $f$ alternates left of $xg$ since all zeros of $f$ are nonpositive. Thus the statement follows from Proposition 2. $\square$

It is not difficult to see that Proposition 3 is actually equivalent to Proposition 2. Now we consider a special case of Proposition 3. Recall that if $f(x) \in \mathbf{RZ}$ then $f'(x) \in \mathbf{RZ}$ and $f'$ interlaces $f$. By Proposition 3 we have

**Proposition 4** *Let $f(x) \in \mathbf{PF}$ and $F(x) = (ax+b)f(x) + x(cx+d)f'(x)$. If $ad \geq bc$, then $F \in \mathbf{RZ}$.*

The preceding result can be restated in terms of sequences instead of polynomials and their derivatives.

**Proposition 5** *Let $x_0, x_1, \ldots, x_{n-1}$ be a PF sequence and $ad \geq bc$. Define*

$$y_k = [a + c(k-1)]x_{k-1} + (b + dk)x_k, \quad k = 0, 1, \ldots, n,$$

*where $x_{-1} = x_n = 0$. If all $y_k$ are nonnegative, then the sequence $y_0, y_1, \ldots, y_n$ is PF.*

In [6], Brenti investigated linear transformations that preserve the PF property. It is worth noticing that our results are closely related to those of Brenti. For example, Proposition 3 strengthens the following result of Brenti[6, Theorem 2.4.4].

**Brenti Theorem** *Let $f, g, F$ be the same as in* Proposition 3. *If $ac < 0, ad > \max\{bc, 0\}$ and all coefficients of $F(x)$ have the same sign, then $F(x) \in \mathbf{RZ}$.*

Proposition 5 actually gives linear transformations that preserve the PF property of sequences. It may be used to reproduce certain results of Brenti, e.g., [6, Theorem 2.4.2]. We now use it to prove Theorem 2.

**Proof of Theorem 2** We proceed by induction on $n$. The result is clearly true for $n = 0, 1$. So suppose that $n > 1$ and $A(n-1, 0), \ldots, A(n-1, n-1)$ is PF. By the assumption of the theorem, we have

$$\begin{aligned} (rn + s + t)b - (an + c)s &= (rb - as)n + (s + t)b - cs \\ &\geq (rb - as) + (s + t)b - cs \\ &= (r + s + t)b - (a + c)s \geq 0. \end{aligned}$$

It follows from Proposition 5 that $A(n, 0), \ldots, A(n, n)$ is PF, as desired. Thus the proof is complete. $\square$

**Remark 2** Recall that $A(0, 0) = 1$ and $A(n, k) = 0$ for $k < 0$ and $n > k$. To guarantee that all $A(n, k)$ are nonnegative, it is natural to require that

$rn+sk+t\geq 0$ for all $n\geq k\geq 1$ and $an+bk+c\geq 0$ for all $n-1\geq k\geq 0$, or equivalently,

$$\begin{cases} r\geq 0 \\ r+s\geq 0 \\ r+s+t\geq 0 \end{cases} \quad \text{and} \quad \begin{cases} a\geq 0 \\ a+b\geq 0 \\ a+c\geq 0. \end{cases}$$

We next give some examples of triangular arrays whose PF property have been proved by various techniques in the literature. All of these triangular arrays are easily seen to satisfy the assumption of Theorem 2. So the PF property is an immediate consequence of Theorem 2.

**Example 1** The binomial coefficients $\binom{n}{k}$, the (signless) Stirling numbers of the first kind $c(n,k)$, the Stirling numbers of the second kind $S(n,k)$ and the classical Eulerian numbers $A(n,k)$ (the numbers of permutations of $1,2,\ldots,n$ having $k-1$ descents) satisfy the recurrence

$$\begin{aligned} \binom{n}{k} &= \binom{n-1}{k-1}+\binom{n-1}{k}, \\ c(n,k) &= c(n-1,k-1)+(n-1)c(n-1,k), \\ S(n,k) &= S(n-1,k-1)+kS(n-1,k), \\ A(n,k) &= (n-k+1)A(n-1,k-1)+kA(n-1,k) \end{aligned}$$

respectively.(See [26] for instance). It is well known that generating functions of $\binom{n}{k}$ and $c(n,k)$ are $(x+1)^n$ and $x(x+1)\cdots(x+n-1)$, respectively. So the binomial coefficients and the (signless) Stirling numbers of the first kind are clearly PF. Harper[15] showed the PF property of the Stirling numbers of the second kind (see [11, 18, 19, 20] for log-concavity). For the proof of the PF property of the Eulerian numbers, see [10, p.292].

**Example 2** The associated Lah numbers defined by

$$L_m(n,k) = (n!/k!)\sum_{i=1}^{k}(-1)^{k-i}\binom{k}{i}\binom{n+mi-1}{n}$$

satisfy the recurrence

$$L_m(n,k) = mL_m(n-1,k-1) + (mk+n-1)L_m(n-1,k).$$

Ahuja and Enneking[2] showed that $\{L_m(n,k)\}_{0\le k\le n}$ is PF for all $m \ge 1$.

**Example 3** The associated Stirling numbers of two kinds, introduced by Jordan and Ward(see [9] for instance), satisfy the recurrence

$$c^*(n,k) = (2n-k-1)\left(c^*(n-1,k-1) + c^*(n-1,k)\right)$$

and

$$S^*(n,k) = (n-k)S^*(n-1,k-1) + (2n-k-1)S^*(n-1,k)$$

respectively. Kurtz[19] pointed out both of them are LC, and Ahuja[1] showed that they are PF.

**Example 4** The holiday numbers $\psi(n,k)$ and $\phi(n,k)$ of the first kind and the second kind, introduced by L. A. Székly in [31], satisfy the recurrence

$$\psi(n,k) = \psi(n-1,k-1) + (2n+k-1)\psi(n-1,k)$$

and

$$\phi(n,k) = \phi(n-1,k-1) + (2n+k)\phi(n-1,k)$$

respectively. Székly[32] showed that both of them are PF.

**Example 5** It is a long-standing conjecture that the Whitney numbers of the second kind of any finite geometric lattice are unimodal or even log-concave sequence(see [35]). Dowling[12] constructed a class of geometric lattices over a finite group of order $m$ and showed that the Whitney numbers $W_m(n,k)$ satisfy the recurrence

$$W_m(n,k) = W_m(n-1,k-1) + (1+mk)W_m(n-1,k).$$

Stonesifer[29] showed the log-concavity of $W_m(n,k)$. Benoumhani proved that $W_m(n,k)$ and $k!W_m(n,k)$ are PF in [5] and [4] respectively. It is easy to see that $k!W_m(n,k)$ actually satisfies the recurrence

$$A(n,k) = kA(n-1,k-1) + (1+mk)A(n-1,k).$$

It is worth mentioning that certain examples to which our propositions are not applicable can be made to be so by means of appreciates transformation. Let us examine a result of R. Simion which concerns the unimodality of the numbers $\Theta(\mathbf{n},k)$ of compositions of a multiset $\mathbf{n} = (n_1, n_2, \ldots)$ into exactly $k$ parts. It is known that the following recurrence holds (see [22, p.96]):

$$(n_j+1)\Theta(\mathbf{n}+e_j,k) = k\Theta(\mathbf{n},k-1) + (n_j+k)\Theta(\mathbf{n},k), \tag{2}$$

where $\mathbf{n}+e_j$ denotes the multiset obtained from $\mathbf{n}$ by adjoining one (additional) copy of the $j$th type element. Let $f_{\mathbf{n}}(x) = \sum_{k\geq 0} \Theta(\mathbf{n},k)x^k$ be the corresponding generating function. Then by (2),

$$(n_j+1)f_{\mathbf{n}+e_j}(x) = (x+n_j)f_{\mathbf{n}}(x) + x(x+1)f'_{\mathbf{n}}(x). \tag{3}$$

Simion showed that the multiplicity $m$ of $-1$ as a zero of $f_{\mathbf{n}}(x)$ is $\max_i\{n_i-1\}$ (see [24] for details). Based on this and the recurrence (3), he showed that

$f_{\mathbf{n}}(x) \in \mathbf{RZ}$ implies $f_{\mathbf{n}+e_j}(x) \in \mathbf{RZ}$. (Thus $f_{\mathbf{n}}(x) \in \mathbf{RZ}$ for any multiset $\mathbf{n}$ and $\Theta(\mathbf{n}, k)$ is therefore unimodal in $k$). Note that this result can not be followed directly from Proposition 4 since $n_j \geq 1$. However, if we write $f_{\mathbf{n}}(x) = (x+1)^m g_{\mathbf{n}}(x)$, then the problem can be reduced to show that $g_{\mathbf{n}}(x) \in \mathbf{RZ}$ implies $[(m+1)x + n_j]g_{\mathbf{n}}(x) + x(x+1)g'_{\mathbf{n}}(x) \in \mathbf{RZ}$. This follows immediately from Proposition 4 since $m + 1 \geq n_j$.

# 4 Concluding Remarks

There are various methods for showing that polynomials have only real zeros. One basic method for showing that a sequence $P_0(x), P_1(x), P_2(x), \ldots$ of polynomials has real zeros is to show by induction that the polynomials have interlaced simple (real) zeros, that is, they form a *Sturm sequence*[27]. The key part of this method is in the inductive step. The sequences of polynomials occurred in combinatorics often satisfy certain recurrence relations. Theorem 1 provides a paradigm for solving this kind of problems. For example, let $\{p_n(x)\}$ be a sequence of orthogonal polynomials associated with certain distribution. Then the following three-term recursion holds:

$$p_n(x) = (a_n x + b_n)p_{n-1}(x) - c_n p_{n-2}(x), \quad n = 1, 2, \ldots, \tag{4}$$

with $p_{-1}(x) = 0$, where $a_n, b_n$ and $c_n$ are real constants and $a_n, c_n > 0$ (see [30, Theorem 3.2.1] for instance). Well-known examples of orthogonal polynomials include the Jacobi polynomials, the Hermite polynomials and the Laguerre polynomials. By induction and Lemma 3(iii), it follows immediately that $p_n(x)$ has only real and simple zeros and $p_n$ interlaces $p_{n+1}$ for $n \geq 1$, which is a well-known result.

Many special cases of Theorem 1 have occurred in the literature. We refer the reader to [13, 16, 25, 33, 34].

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