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Note

Notes on Chvátal's conjecture

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Abstract

Using Kleitman's lemma and results of Schönheim and Miklós it is shown that if $w(\mathscr{D})=|\mathscr{D}|/2$, then every maximum-sized intersecting family in $\mathscr{D}$ contains all base elements of $\mathscr{D}$. Then, the converse of this statement is conjectured and shown that this is equivalent to that of Chvátal.

Keywords: Finite set; Downset; Intersecting family

Let X be a finite set, $\mathscr{P}(X)$ the power set of X and $\mathscr{A}\subseteq\mathscr{P}(X)$. $\mathscr{A}$ is called a downset on X if $A\in\mathscr{A}$ and $B\subseteq A$ imply that $B\in\mathscr{A}$. $\mathscr{F}\subseteq\mathscr{A}$ is called an intersecting family in $\mathscr{A}$ if $A\cap B\neq\emptyset$ for all $A,B\in\mathscr{A}$. $S\subseteq\mathscr{A}$ is called a star in $\mathscr{A}$ if there exists an element $x\in X$ such that $x\in A$ for all $A\in S$. Let $w(\mathscr{A})$ denote the maximum size of an intersecting family in $\mathscr{A}$ and let $s(\mathscr{A})$ denote the maximum size of a star in $\mathscr{A}$. Clearly $w(\mathscr{A})\geqslant s(\mathscr{A})$. Chvátal conjectured that there is equality if $\mathscr{A}$ is a downset.

Conjecture 1 (*Chvátal* [3]). If $\mathscr{A}$ is a downset, then $w(\mathscr{A})=s(\mathscr{A})$.

There are two particular interesting cases where the conjecture is true. If $\mathscr{A}=\mathscr{P}(X)$, then $w(\mathscr{A})=s(\mathscr{A})=2^{n-1}$, which is the simplest result in intersection theory [1]. If $\mathscr{A}=\{A\subseteq X: |A|\leqslant k\}$ for some positive integer $k\leqslant n/2$, then $w(\mathscr{A})=s(\mathscr{A})=\sum_{i=1}^{k}\binom{n-1}{i-1}$, which is the corollary of the famous Erdös–Ko–Rado Theorem [5]. So far Chvátal's conjecture has been verified for some further classes of downsets [2,4,7–10]. For example, Schönheim [8] showed that if the intersection of all maximal members of a

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downset $\mathscr{A}$ is non-empty, then $w(\mathscr{A})=|\mathscr{A}|/2$ and Chvátal's conjecture is true for $\mathscr{A}$. On the other hand, Miklós [7] showed that if $w(\mathscr{A})=|\mathscr{A}|/2$ then the intersection of all maximal members of $\mathscr{A}$ is non-empty. In this note, we explore Chvátal's conjecture by means of a classical lemma of Kleitman. As an application, we give a simple proof of Miklós's result. Moreover, we formulate a conjecture and show that it is equivalent to Chvátal's conjecture.

The following lemma is due to Kleitman. We include the proof of it for our purpose. We first need several notations. For $\mathscr{A} \subseteq \mathscr{P}(X)$ and $x \in X$, let $\mathscr{A}(x)=\{A\backslash\{x\}: x \in A \in \mathscr{A}\}$, $\mathscr{A}(\bar{x})=\{A \in \mathscr{A}: x \notin A\}$. $\mathscr{A}$ is called an upset on X if $A \in \mathscr{A}$ and $B \supseteq A$ imply that $B \in \mathscr{A}$.

Lemma 1 (Kleitman [6]). *If $\mathscr{U}$ is an upset on X and $\mathscr{D}$ is a downset on X, where $|X|=n$, then*

$$|\mathscr{U} \cap \mathscr{D}| \leqslant |\mathscr{U}||\mathscr{D}|/2^n. \tag{1}$$

Proof. We proceed by induction on n, assuming that the result is true for $n-1$ and considering the case n. Let $x \in X$. Then both $\mathscr{U}(x)$ and $\mathscr{U}(\bar{x})$ are upsets, and both $\mathscr{D}(x)$ and $\mathscr{D}(\bar{x})$ are downsets on the $(n-1)$-set $X\backslash\{x\}$. Note that $\mathscr{U}(x) \supseteq \mathscr{U}(\bar{x})$ and $\mathscr{D}(x) \subseteq \mathscr{D}(\bar{x})$, we have

$$\begin{aligned}
|\mathscr{U} \cap \mathscr{D}| &= |\mathscr{U}(x) \cap \mathscr{D}(x)| + |\mathscr{U}(\bar{x}) \cap \mathscr{D}(\bar{x})| \\
&\leqslant |\mathscr{U}(x)||\mathscr{D}(x)|/2^{n-1} + |\mathscr{U}(\bar{x})||\mathscr{D}(\bar{x})|/2^{n-1} \\
&= \frac{(|\mathscr{U}(x)| + |\mathscr{U}(\bar{x})|)(|\mathscr{D}(x)| + |\mathscr{D}(\bar{x})|)}{2^n} \\
&\quad + \frac{(|\mathscr{U}(x)| - |\mathscr{U}(\bar{x})|)(|\mathscr{D}(x)| - |\mathscr{D}(\bar{x})|)}{2^n} \\
&\leqslant |\mathscr{U}||\mathscr{D}|/2^n.
\end{aligned}$$

Thus the proof is completed by induction. $\square$

Remark 1. It is not difficult to see that equality holds if and only if for all $x \in X$,

$$|\mathscr{U}(\bar{x}) \cap \mathscr{D}(\bar{x})| = |\mathscr{U}(\bar{x})||\mathscr{D}(\bar{x})|/2^{n-1},$$

$$|\mathscr{U}(x) \cap \mathscr{D}(x)| = |\mathscr{U}(x)||\mathscr{D}(x)|/2^{n-1}$$

and either $\mathscr{U}(\bar{x})=\mathscr{U}(x)$ or $\mathscr{D}(\bar{x})=\mathscr{D}(x)$.

In what follows, we always let X be a set of n elements and let $\mathscr{D}$ denote a downset on X. Let $\mathscr{F}$ be a maximum-sized intersecting family in $\mathscr{D}$. The set $\mathscr{T}$ of minimal members of $\mathscr{F}$ is called a bottom of $\mathscr{D}$. Denote $\nabla\mathscr{T}=\{A \subseteq X: A \supseteq T$ for some $T \in \mathscr{T}\}$. Clearly $\nabla\mathscr{T}$ is an upset on X and $\mathscr{F}=\nabla\mathscr{T} \cap \mathscr{D}$. Let $\mathrm{Bot}(\mathscr{D})$ denote the set of bottoms of $\mathscr{D}$ for all the maximum-sized intersecting families.

An immediate consequence of Lemma 1 is the following corollary, which has been formulated and proved in [8].

Corollary 1. *If $\mathscr{D}$ is a downset, then* $w(\mathscr{D}) \leqslant |\mathscr{D}|/2$.

Proof. Let $\mathscr{T} \in \mathrm{Bot}(\mathscr{D})$ and $\mathscr{U} = \nabla\mathscr{T}$. Then $|\mathscr{U}| \leqslant 2^{n-1}$ since $\mathscr{U}$ is intersecting. Hence by Lemma 1, we have

$$w(\mathscr{D}) = |\mathscr{U} \cap \mathscr{D}| \leqslant |\mathscr{U}||\mathscr{D}|/2^n \leqslant |\mathscr{D}|/2. \qquad \square$$

A maximal member of a downset $\mathscr{D}$ is called a base of $\mathscr{D}$. Let $\mathscr{B} = \mathscr{B}(\mathscr{D})$ denote the set of bases of $\mathscr{D}$ and let $M = M(\mathscr{D})$ denote the intersection of bases of $\mathscr{D}$. We say that $\mathscr{D}$ is a perfect downset if $w(\mathscr{D}) = |\mathscr{D}|/2$.

Lemma 2 (Schönheim [8]). *If $M \neq \emptyset$, then $\mathscr{D}$ is perfect and Chvátal's conjecture is true for $\mathscr{D}$.*

Now we present a characterization for perfect downsets by means of Lemma 1.

Theorem 1. *A downset $\mathscr{D}$ is perfect if and only if for any $\mathscr{T} \in \mathrm{Bot}(\mathscr{D})$ and $\mathscr{U} = \nabla\mathscr{T}$,*

$$|\mathscr{U} \cap \mathscr{D}| = |\mathscr{U}||\mathscr{D}|/2^n.$$

Proof. Suppose that $\mathscr{D}$ is perfect. Let $\mathscr{T} \in \mathrm{Bot}(\mathscr{D})$ and $\mathscr{U} = \nabla\mathscr{T}$. Then $|\mathscr{U} \cap \mathscr{D}| = |\mathscr{D}|/2$. Thus by Lemma 1, we have

$$|\mathscr{U}| \geqslant \frac{2^n|\mathscr{U} \cap \mathscr{D}|}{|\mathscr{D}|} = 2^{n-1}. \qquad (2)$$

But $|\mathscr{U}| \leqslant 2^{n-1}$, hence equality in (2) holds, i.e., $|\mathscr{U} \cap \mathscr{D}| = |\mathscr{U}||\mathscr{D}|/2^n$.

Conversely, suppose that $|\mathscr{U} \cap \mathscr{D}| = |\mathscr{U}||\mathscr{D}|/2^n$ where $\mathscr{T} \in \mathrm{Bot}(\mathscr{D})$ and $\mathscr{U} = \nabla\mathscr{T}$. Then by Remark 1, either $\mathscr{U}(\bar{x}) = \mathscr{U}(x)$ or $\mathscr{D}(\bar{x}) = \mathscr{D}(x)$ holds for any $x \in X$. But $\mathscr{U}(\bar{x}) = \mathscr{U}(x)$ cannot hold for all $x \in X$ since $\mathscr{U} \neq \mathscr{P}(X)$. Hence there exists an $x \in X$ such that $\mathscr{D}(\bar{x}) = \mathscr{D}(x)$, which implies that $\mathscr{D}$ is perfect. $\square$

Corollary 2. *If $\mathscr{D}$ is perfect, then* $\mathrm{Bot}(\mathscr{D}) \subseteq \mathrm{Bot}(\mathscr{P}(X))$. *In particular, if $Y \subseteq X$ then* $\mathrm{Bot}(\mathscr{P}(Y)) \subseteq \mathrm{Bot}(\mathscr{P}(X))$.

Lemma 3. *If $\mathscr{T} \in \mathrm{Bot}(\mathscr{P}(X))$, $N = \bigcup\{T : T \in \mathscr{T}\}$ and $N \subseteq Y \subseteq X$, then $\mathscr{T} \in \mathrm{Bot}(\mathscr{P}(Y))$.*

Proof. Denote $\mathscr{U} = \nabla\mathscr{T}$. Let $x \notin Y$. Then $x \notin N$, which implies that $\mathscr{U}(x) = \mathscr{U}(\bar{x})$. So $|\mathscr{U}(\bar{x})| = |\mathscr{U}|/2 = 2^{n-2}$. Note that $\mathscr{U}(\bar{x})$ is still intersecting, hence $\mathscr{U}(\bar{x})$ is a maximum-sized intersecting family in $\mathscr{P}(X(\bar{x}))$, and therefore $\mathscr{T} \in \mathrm{Bot}(\mathscr{P}(X(\bar{x})))$. Continuing this process, we can finally conclude that $\mathscr{T} \in \mathrm{Bot}(\mathscr{P}(Y))$ as required. $\square$

As an application of the above discussion, we may give a simple proof of the following result of Miklós.

Theorem 2 (Miklós [7]). *If* $\mathscr{D}$ *is perfect, then* $M \neq \emptyset$ *and* $\mathrm{Bot}(\mathscr{D}) = \mathrm{Bot}(\mathscr{P}(M))$.

Proof. Let $\mathscr{T} \in \mathrm{Bot}(\mathscr{D})$ and $\mathscr{U} = \nabla\mathscr{T}$. Then $|\mathscr{U} \cap \mathscr{D}| = |\mathscr{U}||\mathscr{D}|/2^n$ from Theorem 1. Denote $N = \bigcup\{T: T \in \mathscr{T}\}$ and let $x \in N$. Then $\mathscr{U}(\bar{x}) \neq \mathscr{U}(x)$, which implies that $\mathscr{D}(\bar{x}) = \mathscr{D}(x)$ from Remark 1. So $x \in M$. Thus $N \subseteq M$. It follows that $\mathscr{T} \in \mathrm{Bot}(\mathscr{P}(M))$ from Corollary 2 and Lemma 3. Consequently $M \neq \emptyset$ and $\mathrm{Bot}(\mathscr{D}) \subseteq \mathrm{Bot}(\mathscr{P}(M))$.

We next show that $\mathrm{Bot}(\mathscr{P}(M)) \subseteq \mathrm{Bot}(\mathscr{D})$ by induction on n.

Let $\mathscr{T} \in \mathrm{Bot}(\mathscr{P}(M))$ and $x \in X\backslash M$. Denote $\mathscr{D}_1 = \mathscr{D}(x)$ and $\mathscr{D}_2 = \mathscr{D}(\bar{x})$. Then $\mathscr{D}_j$ is a downset on the $(n-1)$-set $X\backslash\{x\}$ and $M(\mathscr{D}_j) \supseteq M$ $(j = 1, 2)$. Hence $\mathscr{T} \in \mathrm{Bot}(\mathscr{P}(M(\mathscr{D}_j)))$ by Corollary 2. Applying the induction hypothesis to $\mathscr{D}_j$, we obtain $\mathscr{T} \in \mathrm{Bot}(\mathscr{D}_j)$. So

$$|\nabla\mathscr{T} \cap \mathscr{D}| = |\nabla\mathscr{T} \cap \mathscr{D}_1| + |\nabla\mathscr{T} \cap \mathscr{D}_2| = |\mathscr{D}_1|/2 + |\mathscr{D}_2|/2 = |\mathscr{D}|/2,$$

which implies that $\mathscr{T} \in \mathrm{Bot}(\mathscr{D})$. Consequently $\mathrm{Bot}(\mathscr{P}(M)) \subseteq \mathrm{Bot}(\mathscr{D})$.

Thus $\mathrm{Bot}(\mathscr{P}(M)) = \mathrm{Bot}(\mathscr{D})$ and the proof is complete. □

We say that $\mathscr{D}$ is a full downset if every maximum-sized intersecting family in $\mathscr{D}$ contains all base elements of $\mathscr{D}$.

Corollary 3. *Every perfect downset is full.*

Proof. Suppose that $\mathscr{D}$ is a perfect downset and that $\mathscr{F}$ is a maximum-sized intersecting family in $\mathscr{D}$. Let $\mathscr{T}$ be the set of minimal members of $\mathscr{F}$. Then $\mathscr{T} \in \mathrm{Bot}(\mathscr{D})$, which follows that $\mathscr{T} \in \mathrm{Bot}(\mathscr{P}(M))$ from Theorem 2. Hence $T \subseteq M$ for any $T \in \mathscr{T}$, and therefore $T \subseteq B$ for any $B \in \mathscr{B}$. Thus $B \in \mathscr{F}$, and $\mathscr{D}$ is therefore full. □

Finally, we formulate a conjecture, which has been observed by Miklós in [7], and show that this is equivalent to that of Chvátal.

Conjecture 2. *Every full downset is perfect.*

Theorem 3. *Conjecture* 1 *holds if and only if Conjecture* 2 *holds.*

Proof. Suppose that Conjecture 1 is true and that $\mathscr{D}$ is a full downset. Then $M \neq \emptyset$ since there exists a star as a maximum-sized intersecting family in $\mathscr{D}$. Thus $\mathscr{D}$ is perfect by Lemma 2 and Conjecture 2 is therefore true.

Now suppose that Conjecture 1 is false. Then we will find a downset which is full but not perfect. In fact, let $\mathscr{D}$ be a minimum-sized downset violating Conjecture 1. Then $\mathscr{D}$ is not perfect. On the other hand, assume that $\mathscr{D}$ is not full, then there must be a maximum-sized intersecting family in $\mathscr{D}$ which does not contain some base B

of $\mathscr{D}$. Note that $\mathscr{D}'=\mathscr{D}\backslash\{B\}$ is still a downset, hence Conjecture 1 is true for $\mathscr{D}'$ by the minimality of $\mathscr{D}$, i.e., $w(\mathscr{D}')=s(\mathscr{D}')$. However, $w(\mathscr{D})\geqslant s(\mathscr{D})$, $s(\mathscr{D})\geqslant s(\mathscr{D}')$ and $w(\mathscr{D})=w(\mathscr{D}')$. So $w(\mathscr{D})=s(\mathscr{D})$, which contradicts the hypothesis to $\mathscr{D}$. Thus $\mathscr{D}$ is full and Conjecture 2 is therefore false. □

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