國立交通大學

應用數學系

碩 士 論 文

嚴格正矩陣的研究
Strictly Totally Positive Matrices

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摘　要

嚴格正矩陣在很多數學領域扮演著重要的角色。然而，即使對專門研究線性代數的數學家而言，他們也不是很熟悉嚴格正矩陣。這篇論文最主要的目的是介紹嚴格正矩陣的基本性質，在論文的最後則包含對此種矩陣的觀察以及一個開放性的問題。

Strictly Totally Positive Matrices

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Abstract

Strictly totally positive matrices play an important role in various mathematical branches, but there are not very familiar even to linear algebraists. It is the purpose of this thesis to introduce the known properties of these matrices and their self-contained proofs. Some observations and open problems are given in the end.

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Contents

1 Introduction

A *positive* matrix is a matrix with all its entries positive. The important applications of positive matrices are spread in many areas. For instance, positive matrices play a crucial role in theoretical economics. A *strictly totally positive* matrix is a matrix with all its minors positive. The traces of strictly totally positive matrices can be found in many mathematical branches such as probability theory, approximation theory, facilitating numerical procedures in solving certain types of differential equations, in the analysis of certain integral and differential operators and so on. However, strictly totally positive matrices are not very familiar even to linear algebraists though these matrices have important role in various mathematical branches.

In this paper, we organize the basic properties of strictly totally positive matrices. The proofs are all selfcontained. Most of the results can be found in [6], [7]. At first, we shall introduce *Tensor product spaces, skew-symmetric spaces*, linear transformations over above two spaces, *sign variations of vectors* which all play important roles with the strictly totally positive matrices. Also, we will present relations between skew-symmetric spaces and sign variation of vectors, between sign variation of vectors and strictly sign regular matrices, and between strictly sign regular matrices and strictly totally positive matrices.

Now we introduce some notations, we view $\mathbb{R}^n$ as the set of column vectors. Let V be a real vector space with a subspace H. We denote $\langle\ ,\ \rangle$ the standard inner product function of V, and $H^\perp$ an orthogonal complement of H with respect to the inner product function $\langle\ ,\ \rangle$. We denote $M_{n\times m}(\mathbb{R})$

the set of all $n \times m$ matrices with real entries. In particular, we also denote $M_n(\mathbb{R})$ the set of all $n \times n$ matrices with real entries. The transpose of a matrix A is denoted by A^t. If $A = [a_{ij}] \in M_{n\times n}(\mathbb{R})$, then we define $\det A = \sum_{\sigma \in S_n} sgn(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$, where S_n is the permutation group on the set $\{1, \cdots, n\}$.

Let $A \in M_{n\times m}(\mathbb{R})$. For the index set $\alpha \subseteq \{1, \cdots, n\}$ and $\beta \subseteq \{1, \cdots, m\}$, we denote the submatrix of A lies in the rows indexed by α and the columns indexed by β as $A[\alpha|\beta]$. Often it is convenient to indicate a submatrix of A via deletion of rows or columns. For example, we denote $A(\alpha|\,\beta)$ is the result of deleting the rows of A indexed by α and the columns indexed by β. In this paper, if $A \in M_{n\times m}(\mathbb{R})$, then we view $A[-|\beta]$ as $A[\,\{1, \cdots, n\}|\,\beta]$. Similarly $A[\alpha|-]$ as $A[\alpha|\,\{1, \cdots, m\}]$.

2 Tensor Products

Definition 2.1. Let $e_1, e_2, \cdots, e_n$ be the standard basis of $\mathbb{R}^n$. The tensor product of $\mathbb{R}^n$ and $\mathbb{R}^m$ is the vector space of dimension $n \times m$ defined by

$$\mathbb{R}^n \otimes \mathbb{R}^m \equiv \left\{ \sum_{i,j=1}^{n,m} c_{ij} e_i \otimes e_j \Big|\; c_{ij} \in \mathbb{R} \right\}, \tag{1}$$

where $\{\, e_i \otimes e_j|\; i = 1, \cdots, n, j = 1, \cdots m\}$ is an orthonormal basis.

We do not always write elements in $\mathbb{R}^n \otimes \mathbb{R}^m$ as in (1). In fact, we use the following notation:

$$\sum_{i=1}^{n} c_i e_i \otimes \sum_{j=1}^{m} d_j e_j \equiv \sum_{i,j=1}^{n,m} c_i d_j e_i \otimes e_j. \tag{$*$}$$

Remark 2.2.

1. Since $\{e_i \otimes e_j |\ i = 1, \cdots, n, j = 1, \cdots m\}$ is an orthonormal basis,

$$\langle e_i \otimes e_j, e_k \otimes e_l \rangle = \delta_{ik}\delta_{jl} = \langle e_i, e_k \rangle \langle e_j, e_l \rangle.$$

2. Using the convention in $(*)$, $\langle x_1 \otimes x_2, y_1 \otimes y_2 \rangle = \langle x_1, y_1 \rangle \langle x_2, y_2 \rangle$ for all $x_1, y_1 \in \mathbb{R}^n$, $x_2, y_2 \in \mathbb{R}^m$.

Similarly, we can define k-tensor product

$$\bigotimes^k \mathbb{R}^n \equiv \underbrace{((\mathbb{R}^n \otimes \mathbb{R}^n) \otimes \mathbb{R}^n) \cdots \otimes \mathbb{R}^n}_{\text{k copies}}.$$

The basis of $\bigotimes^k \mathbb{R}^n$ is $\{e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k} |\ i_1, \cdots, i_k \in \{1, \cdots, n\}\}$. In particular, $\bigotimes^k \mathbb{R}^n$ has dimension n^k. Let $L : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $M : \mathbb{R}^s \longrightarrow \mathbb{R}^t$ be linear transformations. Define $L \otimes M : \mathbb{R}^n \otimes \mathbb{R}^s \longrightarrow \mathbb{R}^m \otimes \mathbb{R}^t$ linearly by

$$L \otimes M(x \otimes y) = Lx \otimes My \tag{2}$$

for $x \in \mathbb{R}^n$, $y \in \mathbb{R}^s$.

Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ be the standard bases of $\mathbb{R}^n$, $\mathbb{R}^m$, $\mathbb{R}^s$, $\mathbb{R}^t$ respectively. Let $A = [L]^{\mathcal{A}}_{\mathcal{B}}$ be the matrix representation of L w.r.t. $\mathcal{A}$, $\mathcal{B}$ and $B = [M]^{\mathcal{C}}_{\mathcal{D}}$ is the matrix representation of M w.r.t $\mathcal{C}$, $\mathcal{D}$. Then the matrix representation of

$L \otimes M$ w.r.t. the standard bases of $\mathbb{R}^n \otimes \mathbb{R}^s$, $\mathbb{R}^m \otimes \mathbb{R}^t$ in the lexicographical order is denoted by

$$A \otimes B \equiv \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \cdots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}$$

where $A = [a_{ij}]_{m \times n}$.

Throughout this paper we identify a linear transformation and its matrix representation. The following remark is concerning about the basic properties of matrices. We leave the proofs of the first two statements to the reader.

Remark 2.3.

1. Assume A, B are square matrices of the same size. Then $P(A \otimes B)P^t = B \otimes A$ for some permutation matrix P.

2. Assume A, B are square matrices of the same size. Then $(A \otimes B)^t = A^t \otimes B^t$.

3. Let $A \in M_{m \times n}(\mathbb{R})$, $B \in M_{t \times s}(\mathbb{R})$, $C \in M_{n \times u}(\mathbb{R})$, $D \in M_{s \times u}(\mathbb{R})$. Then

$$(A \otimes B)(C \otimes D) = AC \otimes BD.$$

4. Suppose A, B are invertible square matrices. Then $A \otimes B$ is invertible and $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$.

5. If A is similar to A' and B is similar to B', then $A \otimes B$ is similar to $\mathrm{A}' \otimes \mathrm{B}'$.

6. If $A \in M_{n\times n}(\mathbb{R})$ with eigenvalues $\lambda_1, \cdots, \lambda_n$, $B \in M_{m\times m}(\mathbb{R})$ with eigenvalues $\mu_1, \cdots, \mu_n$, then

   (a) $A \otimes B$ has eigenvalues $\lambda_i\mu_j$, $i = 1, \cdots, n$, $j = 1, \cdots, m$;

   (b) If u is an eigenvector of A for λ_i, v is an eigenvector of B for μ_j, then $u \otimes v$ is an eigenvector of $A \otimes B$ for $\lambda_i\mu_j$.

Proof of 3. By (2),

$$\begin{aligned}(A \otimes B)(C \otimes D)e_i \otimes e_j &= (A \otimes B)(Ce_i \otimes De_j) = ACe_i \otimes BDe_j \\ &= (AC \otimes BD)(e_i \otimes e_j)\end{aligned}$$

for all $1 \leq i \leq u, 1 \leq j \leq v$.

Q.E.D.

Proof of 4. By Remark 2.3.3,

$$\begin{aligned}(A \otimes B)(A^{-1} \otimes B^{-1}) &= AA^{-1} \otimes BB^{-1} \\ &= I \otimes I = I.\end{aligned}$$

Q.E.D.

Proof of 5. Suppose $A = PA'P^{-1}$, $B = QB'Q^{-1}$. Then

$$\begin{aligned}(A \otimes B) &= PA'P^{-1} \otimes QB'Q^{-1} \\ &= (P \otimes Q)(A' \otimes B')(P \otimes Q)^{-1}\end{aligned}$$

by remark 2.3.3.

Q.E.D.

Proof. of 6(a) $A \sim \begin{bmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$, $B \sim \begin{bmatrix} \mu_1 & & * \\ & \ddots & \\ 0 & & \mu_n \end{bmatrix}$,

hence

$$\begin{aligned} A \otimes B &\sim \begin{bmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \otimes \begin{bmatrix} \mu_1 & & * \\ & \ddots & \\ 0 & & \mu_n \end{bmatrix} \\ &= \begin{bmatrix} \begin{bmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \mu_1 & & * \\ & \ddots & \\ 0 & & \begin{bmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \mu_m \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1\mu_1 & & * & & & * & & & * \\ 0 & \ddots & & & & & & & \\ 0 & 0 & \lambda_n\mu_1 & & & & & & \\ 0 & 0 & 0 & \lambda_1\mu_2 & & * & & & * \\ 0 & 0 & 0 & 0 & \ddots & & & & \\ 0 & 0 & 0 & 0 & 0 & \lambda_n\mu_2 & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1\mu_m & & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_n\mu_m \end{bmatrix}. \end{aligned}$$

Hence $\lambda_i\mu_j$ are eigenvalues of $A \otimes B$ for $i = 1, \cdots, n$, $j = 1, \cdots, m$.

Q.E.D.

Proof. of 6(b)

$$\begin{aligned}(A \otimes B)(u \otimes v) &= Au \otimes Bv \\ &= \lambda_i u \otimes \mu_j v \\ &= \lambda_i \mu_j (u \otimes v).\end{aligned}$$

Q.E.D.

3 Skew-symmetric power

In this section, we will introduce an important subspace of the tensor product space which is called the skew-symmetric space.

Definition 3.1. Let S_k be the permutation group on the set $\{1, \cdots, k\}$. For $\pi \in S_k$, define $\pi : \bigotimes^k \mathbb{R}^n \longrightarrow \bigotimes^k \mathbb{R}^n$ linearly by

$$x_1 \otimes \cdots \otimes x_k \longrightarrow \pi(x_1 \otimes \cdots \otimes x_k) = x_{\pi(1)} \otimes \cdots \otimes x_{\pi(k)}.$$

Then the k-th skew-symmetric space over $\mathbb{R}^n$ is denoted by

$$\bigwedge^k \mathbb{R}^n \equiv \left\{ x \in \bigotimes^k \mathbb{R}^n \middle| \pi(x) = \text{sgn}(\pi)x \quad \text{for all } \pi \in S_k \right\}.$$

Example 3.2. Show that

$$e_1 \otimes e_2 - e_2 \otimes e_1 \in \bigwedge^2 \mathbb{R}^n.$$

Sol. Let $\pi = (1, 2)$, then

$$\begin{aligned}\pi(e_1 \otimes e_2 - e_2 \otimes e_1) &= e_2 \otimes e_1 - e_1 \otimes e_2 \\ &= -(e_1 \otimes e_2 - e_2 \otimes e_1) \\ &= \text{sgn}(\pi)(e_1 \otimes e_2 - e_2 \otimes e_1). \qquad (3)\end{aligned}$$

If $\pi = e$, the identity permutation, then equation (3) hold clearly.

Q.E.D.

Definition 3.3. For $x_i \in \mathbb{R}^n$, $i = 1, \cdots, k$,

$$x_1 \wedge \cdots \wedge x_k \equiv \frac{1}{k!} \sum_{\pi \in S_k} \mathrm{sgn}(\pi)\pi(x_1 \otimes \cdots \otimes x_k) \in \bigotimes^k \mathbb{R}^n$$

is called the skew-symmetric product of $x_1, \cdots, x_k$.

It is easy to check the following remark.

Remark 3.4.

$$\text{For } x_i \in \mathbb{R}^n,\ 1 \leq i \leq k,\ x_1 \wedge \cdots \wedge x_k \in \bigwedge^k \mathbb{R}^n.$$

Theorem 3.5. $k!\langle x_1 \wedge \cdots \wedge x_k, y_1 \wedge \cdots \wedge y_k \rangle = \det[\langle x_i, y_j \rangle]_{k \times k}$.

Proof.

$$k!\langle x_1 \wedge \cdots \wedge x_k, y_1 \wedge \cdots \wedge y_k \rangle$$

$$\begin{aligned}
&= \frac{1}{k!} \sum_{\pi \in S_k} \sum_{\sigma \in S_k} \mathrm{sgn}(\pi)\mathrm{sgn}(\sigma) \langle x_{\pi(1)}, y_{\sigma(1)} \rangle \cdots \langle x_{\pi(k)}, y_{\sigma(k)} \rangle \\
&= \frac{1}{k!} \sum_{\pi \in S_k} \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma\pi^{-1}) \langle x_{\pi(1)}, y_{(\sigma\pi^{-1})\pi(1)} \rangle \cdots \langle x_{\pi(k)}, y_{(\sigma\pi^{-1})\pi(k)} \rangle \\
&= \frac{1}{k!} \sum_{\pi \in S_k} \det\left[\langle x_i, y_j \rangle\right]_{k \times k} \\
&= \det\left[\langle x_i, y_j \rangle\right]_{k \times k}.
\end{aligned}$$

Q.E.D.

Corollary 3.6. $x_1 \wedge \cdots \wedge x_k = 0 \Longleftrightarrow x_1, \cdots, x_k$ are linear dependent.

Proof. Let $A = [x_1, x_2, \cdots, x_k]$. Observe $\text{rank}A^tA = \text{rank}A$ for all $A \in M_{n\times k}(\mathbb{R})$. Hence

$$\begin{aligned} x_1 \wedge \cdots \wedge x_k = 0 &\Longleftrightarrow \det\left[\langle x_i, x_j\rangle\right]_{k\times k} = 0. \\ &\Longleftrightarrow \det(A^tA) = 0. \\ &\Longleftrightarrow \text{rank}A < k. \\ &\Longleftrightarrow x_1, \cdots, x_k \text{ are linear dependent.} \end{aligned}$$

Q.E.D.

Now we are ready to give a basis of $\bigwedge^k \mathbb{R}^n$.

Remark 3.7.

1. Let $\alpha = \{\alpha_1 < \alpha_2 < \cdots < \alpha_k\} \subseteq \{1, 2, \cdots, n\}$

$$e_\alpha \equiv e_{\alpha_1} \wedge e_{\alpha_2} \wedge \cdots \wedge e_{\alpha_k}.$$

Then the set $\{\ e_\alpha \mid |\alpha| = k,\ \alpha \subseteq \{1, 2, \cdots, n\}\}$ spans $\bigwedge^k \mathbb{R}^n$.

2. If $\alpha, \beta \subseteq \{1, 2, \cdots, n\}$, $|\alpha| = |\beta| = k$, then $\langle e_\alpha, e_\beta\rangle = \frac{1}{k!}\ \delta_{\alpha\beta}$.

3. The set $\left\{\ \sqrt{k!}\ e_\alpha \mid |\alpha| = k, \alpha \subseteq \{1, 2, \cdots, n\}\right\}$ is an orthonormal basis of $\bigwedge^k \mathbb{R}^n$. In particular, $\bigwedge^k \mathbb{R}^n$ has dimension $\binom{n}{k}$.

Next, we study the linear transformations between skew-symmetric spaces.

Definition 3.8. Let $A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, the k-th exterior power $\bigwedge^k A$ of A is defined linearly by

$$\bigwedge^k A : \bigwedge^k \mathbb{R}^n \longrightarrow \bigwedge^k \mathbb{R}^m,$$

$$\text{where} \quad (x_1 \wedge \cdots \wedge x_k) \mapsto Ax_1 \wedge \cdots \wedge Ax_k.$$

Lemma 3.9. Suppose $A \in M_{m\times n}(\mathbb{R})$, $B \in M_{n\times p}(\mathbb{R})$. Then

$$\bigwedge^k AB = (\bigwedge^k A)(\bigwedge^k B).$$

Proof.

$$\begin{aligned} \bigwedge^k AB(x_1 \wedge \cdots \wedge x_k) &= ABx_1 \wedge \cdots \wedge ABx_k \\ &= \bigwedge^k A(Bx_1 \wedge \cdots \wedge Bx_k) \\ &= \bigwedge^k A(\bigwedge^k B(x_1 \wedge \cdots \wedge x_k)) \\ &= (\bigwedge^k A \circ \bigwedge^k B)(x_1 \wedge \cdots \wedge x_k). \end{aligned}$$

Q.E.D.

Corollary 3.10. Suppose $A \in M_n(\mathbb{R})$ is invertible. Then $\bigwedge^k A$ is invertible and $(\bigwedge^k A)^{-1} = \bigwedge^k A^{-1}$.

Proof. By lemma 3.9

$$\bigwedge^k A \circ \bigwedge^k A^{-1} = \bigwedge^k AA^{-1} = \bigwedge^k I = I.$$

Q.E.D.

Remark 3.11. Suppose $A \in M_{m\times n}(\mathbb{R})$. Then by theorem 3.5,

$$\begin{aligned}
(\bigwedge^k A)_{\alpha\beta} &= \langle \bigwedge^k A\sqrt{k!}\, e_\beta, \sqrt{k!}\, e_\alpha\rangle \\
&= k!\langle \bigwedge^k Ae_\beta, e_\alpha\rangle \\
&= k!\langle Ae_{\beta_1}\wedge\cdots\wedge Ae_{\beta_k}, e_{\alpha_1}\wedge\cdots\wedge e_{\alpha_k}\rangle \\
&= \det\left[\langle Ae_{\beta_i}, e_{\alpha_j}\rangle\right]_{k\times k} \\
&= \det A\left[\alpha|\beta\right],
\end{aligned}$$

$$\begin{aligned}
\text{where}\quad \alpha &= \{\alpha_1 < \cdots < \alpha_k\} \subseteq \{1,\cdots,m\},\ |\alpha| = |\beta| = k, \\
\beta &= \{\beta_1 < \cdots < \beta_k\} \subseteq \{1,\cdots,n\}.
\end{aligned}$$

Next, we will introduce a class of special matrices which are called strictly totally positive matrices. The traces of these matrices can be found in many mathematical branches such as probability theory, approximation theory, facilitating numerical procedures in solving certain types of differential equations, in the analysis of certain integral and differential operators and so on.

Definition 3.12.

1. $A = [\,a_{ij}\,] \in M_{n\times m}(\mathbb{R})$ is called strictly positive (resp. positive)

   if $a_{ij} > 0$ (resp. $a_{ij} \geq 0$) for all $i \in \{1,\cdots,n\}, j \in \{1,\cdots,m\}$.

   In this situation, A is denoted by $A > 0$ (resp. $A \geq 0$).

2. $A \in M_{n\times m}(\mathbb{R})$ is called strictly totally positive (resp. totally positive) if $\det A[\alpha|\beta] > 0$ (resp. $\det A[\alpha|\beta] \geq 0$) for all $\alpha \subseteq \{1, \cdots, n\}$, $\beta \subseteq \{1, \cdots, m\}$ $|\alpha| = |\beta|$.

Remark 3.13.

1. STP (resp. TP) is stand for strictly totally positive (resp. totally positive) .
2. A STP (resp. TP) matrix is strictly positive (resp. positive) .
3. If $A \in M_{n\times m}(\mathbb{R})$ is a STP matrix, then A^t is also a STP matrix .
4. $A \in M_{n\times m}(\mathbb{R})$ is a STP (resp. TP) matrix $\Longleftrightarrow \bigwedge^k A > 0$ (resp. ≥ 0) for all $k = 1, \cdots, \min\{m, n\}$.

Theorem 3.14. *(Binet-Cauchy Theorem)*
Suppose $A, B \in M_n(\mathbb{R})$ and for all $\alpha, \beta \subseteq \{1, \cdots, n\}, |\alpha| = |\beta| = k$. Then

$$\det AB[\alpha|\beta] = \sum_{\substack{\omega \subseteq \{1,\cdots,n\} \\ |\omega|=k}} \det A[\alpha|\omega] \det B[\omega|\beta].$$

Proof. By remark 3.11, .

$$\begin{aligned} \det AB[\alpha|\beta] &= (\bigwedge^k AB)_{\alpha\beta} = \left[(\bigwedge^k A)(\bigwedge^k B)\right]_{\alpha\beta} \\ &= \sum_{\substack{\omega \subseteq \{1,\cdots,n\} \\ |\omega|=k}} (\bigwedge^k A)_{\alpha\omega}(\bigwedge^k B)_{\omega\beta}) \\ &= \sum_{\substack{\omega \subseteq \{1,\cdots,n\} \\ |\omega|=k}} \det A[\alpha|\omega] \det B[\omega|\beta]. \end{aligned}$$

Q.E.D.

Corollary 3.15.

1. If $A, B \in M_n(\mathbb{R})$, then $\det AB = \det A \cdot \det B$.

2. If $A \in M_{n\times m}(\mathbb{R})$, $B \in M_{m\times p}(\mathbb{R})$ are TP (STP) matrices, then AB is a TP (STP) matrix.

Proof. of 1. This is immediate from Binet-Cauchy theorem with $k = n$, $\alpha = \beta = \{1, \cdots, n\}$.

Q.E.D.

2. This is immediate from definition 3.12.2 Binet-Cauchy theorem.

Q.E.D.

Remark 3.16.

$$\begin{aligned} \left(\bigwedge^k \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \right)_{\alpha\beta} &= \det \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} [\alpha|\beta] \\ &= \delta_{\alpha\beta} \lambda_{\alpha_1} \cdots \lambda_{\alpha_k}, \end{aligned}$$

where $\alpha = \{\alpha_1, \cdots, \alpha_k\}$.

Theorem 3.17 (Kronecker Theorem).

Suppose $A \in M_n(\mathbb{R})$ with eigenvalues $\lambda_1(A), \cdots, \lambda_n(A) \in \mathbb{C}$, such that $|\lambda_1(A)| \geq \cdots \geq |\lambda_n(A)|$. Then $\bigwedge^k A$ is a $\binom{n}{k} \times \binom{n}{k}$ matrix and the following 1. 2. hold:

1. $\bigwedge^k A$ has eigenvalues $\prod_{1 \le i \le k} \lambda_{\alpha_i}(A)$, where $\alpha = \{\alpha_1 < \cdots < \alpha_k\} \subseteq \{1, \cdots, n\}$;

2. If u_i is an eigenvector of A for $\lambda_{\alpha_i}(A)$, then $u_1 \wedge \cdots \wedge u_k$ is an eigenvector of $\bigwedge^k A$ for $\prod_{1 \le i \le k} \lambda_{\alpha_i}(A)$, where the set $\alpha = \{\alpha_1 < \cdots < \alpha_k\} \subseteq \{1, \cdots, n\}$.

Proof. 1. Suppose A is diagonalizable. Then $A = P^{-1} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} P$ for some $P \in M_n(\mathbb{R})$. Hence by lemma 3.9,

$$\bigwedge^k A = \bigwedge^k P^{-1} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} P = (\bigwedge^k P)^{-1} \bigwedge^k \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \bigwedge^k P.$$

Hence $\bigwedge^k A$ has eigenvalues $\prod_{1 \le i \le k} \lambda_{\alpha_i}(A)$, where $\alpha = \{\alpha_1 < \cdots < \alpha_k\} \subseteq \{1, \cdots, n\}$ by remark 3.16.

Since diagonalizable matrices are dense in the space of square matrices, and the spectrum depends continuously on matrix entries. Hence 1. holds in general.

Q.E.D.

2. By definition 3.8,

$$\begin{aligned}\bigwedge^k A(u_1 \wedge \cdots \wedge u_k) &= Au_1 \wedge \cdots \wedge Au_k \\ &= \lambda_{\alpha_1} u_1 \wedge \cdots \wedge \lambda_{\alpha_k} u_k \\ &= \lambda_{\alpha_1} \cdots \lambda_{\alpha_k} (u_1 \wedge \cdots \wedge u_k).\end{aligned}$$

Q.E.D.

Using theorem 3.17, we can have the following remark. The proof is left to the reader.

Remark 3.18.

1. $\bigwedge^k A$ is also called the k-th *compound* of A.
2. Rank $\bigwedge^k A = \binom{\text{rank}A}{k}$.
3. $\det \bigwedge^k A = (\det A)^{\binom{n-1}{k-1}}$.

4 Schur Complement

Definition 4.1. Let $A \in M_n(\mathbb{R})$, fix k $(1 \leq k \leq n)$ and $\alpha, \beta \subseteq \{1, \cdots, n\}$ such that $|\alpha| = |\beta| = k$. Assume $A[\alpha|\beta]$ is invertible. The Schur complement of $A[\alpha|\beta]$ in A is the matrix $A \setminus [\alpha|\beta]$ which is defined by

$$A \setminus [\alpha|\beta] \equiv A(\alpha|\beta) - A(\alpha|\beta]A[\alpha|\beta]^{-1}A[\alpha|\beta). \tag{4}$$

Example 4.2. Let $A = \begin{bmatrix} 2 & 2 & 3 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$. Then

$$\begin{aligned} A \setminus [1|\,1] &= \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} 2^{-1}[2,3] \\ &= \begin{bmatrix} -1 & \frac{-1}{2} \\ -1 & \frac{-1}{2} \end{bmatrix}. \end{aligned}$$

Observe that $\det A = 0$, $\det A[1|\,1] = 2$, $\det A \setminus [1|\,1] = 0$, and $\det A = \det A[1|1] \cdot \det A \setminus [1|1]$.

Definition 4.3. Let $\alpha,\, \alpha' \subseteq \{1, \cdots, n\}$, $\alpha \cup \alpha' = \{1, \cdots, n\}, \alpha \cap \alpha' = \emptyset$, $\alpha = \{\alpha_1 < \cdots < \alpha_k\} \subseteq \{1, \cdots, n\}$, $\alpha' = \{\alpha'_1 < \cdots < \alpha'_{n-k}\} \subseteq \{1, \cdots, n\}$. Then define

$$\Pi_\alpha \equiv \begin{pmatrix} 1 & \cdots & k & k+1 & \cdots & n \\ \alpha_1 & \cdots & \alpha_k & \alpha'_1 & \cdots & \alpha'_{n-k} \end{pmatrix} \in S_k.$$

Remark 4.4.

1. $\mathrm{sgn}(\alpha) \equiv \mathrm{sgn}(\Pi_\alpha)$ for $\alpha \subseteq \{1, \cdots, n\}$.

2. $A[\alpha] \equiv A[\alpha|\alpha]$, $A(\alpha) \equiv A(\alpha|\alpha)$, $A \setminus \alpha \equiv A \setminus [\alpha|\,\alpha]$ for $A \in M_n(\mathbb{R})$ and $\alpha \subseteq \{1, \cdots, n\}$.

3. Let T_α be the permutation matrix representing Π_α^{-1}. Then

$$\begin{aligned}
A[\alpha|\beta] &= T_\alpha A T_\beta^{-1}[1,\cdots,k], \\
A(\alpha|\beta] &= T_\alpha A T_\beta^{-1}\left(\{1,\cdots,k\}|\{1,\cdots,k\}\right], \\
A(\alpha|\beta) &= T_\alpha A T_\beta^{-1}\left(\{1,\cdots,k\}\right), \\
A[\alpha|\beta) &= T_\alpha A T_\beta^{-1}\left[\{1,\cdots,k\}|\{1,\cdots,k\}\right),
\end{aligned}$$

hence

$$A\setminus[\alpha|\beta] = A(\alpha|\beta) - A(\alpha|\beta]A[\alpha|\beta]^{-1}A[\alpha|\,\beta) = T_\alpha A T_\beta^{-1}\setminus\{1,\cdots,k\},$$

for $A\in M_n(\mathbb{R})$ and $\alpha,\beta\subseteq\{1,\cdots,n\}$ with $|\,\alpha| = |\,\beta| = k$.

Theorem 4.5. Suppose $A\in M_n(\mathbb{R})$, $\alpha,\beta\subseteq\{1,\cdots,n\}$ such that $|\,\alpha| = |\beta| = k$. Assume $A[\alpha|\beta]$ is invertible. Then

$$\det A = \mathrm{sgn}(\alpha)\mathrm{sgn}(\beta)\det A[\alpha|\beta]\det A\setminus[\alpha|\,\beta].$$

Proof. First, assume $\alpha=\beta=\{1,\cdots,k\}$. Then

$$\begin{aligned}
&\begin{bmatrix} I_n[\alpha] & 0 \\ A(\alpha|\alpha]A[\alpha|\,\alpha]^{-1} & I_n(\alpha)\end{bmatrix}\begin{bmatrix} A[\alpha] & 0 \\ 0 & A\setminus\alpha\end{bmatrix}\begin{bmatrix} I_n[\alpha] & A[\alpha]^{-1}A[\alpha|\,\alpha) \\ 0 & I_n(\alpha)\end{bmatrix} \\
&= \begin{bmatrix} I_n[\alpha]A[\alpha] & 0 \\ A(\alpha|\alpha]A[\alpha]^{-1}A[\alpha] & I_n(\alpha)A\setminus\alpha\end{bmatrix}\begin{bmatrix} I_n[\alpha] & A[\alpha]^{-1}A[\alpha|\alpha) \\ 0 & I_n(\alpha)\end{bmatrix} \\
&= \begin{bmatrix} A[\alpha] & A[\alpha|\alpha) \\ A(\alpha|\alpha] & A(\alpha|\alpha]A[\alpha]^{-1}A[\alpha|\alpha) + A\setminus\alpha\end{bmatrix} \qquad (5) \\
&= A.
\end{aligned}$$

by (4). Hence

$$\det A = 1 \times \det A[\alpha] \times (\det A \setminus \alpha) \times 1 = \det A[\alpha] \cdot \det A \setminus \alpha.$$

In general; for all $\alpha, \beta \subseteq \{1, \cdots, n\}$ with $|\alpha| = |\beta| = k$, by above argument,

$$\det(T_\alpha A T_\beta^{-1}) = \det(T_\alpha A T_\beta^{-1}[\{1, \cdots, k\}]) \cdot \det(T_\alpha A T_\beta^{-1} \setminus \{1, \cdots, k\}).$$

Hence

$$\begin{aligned}
\det A &= \operatorname{sgn}(\alpha)\operatorname{sgn}(\beta)\det T_\alpha A T_\beta^{-1} \\
&= \operatorname{sgn}(\alpha)\operatorname{sgn}(\beta)\det T_\alpha A T_\beta^{-1}[\{1, \cdots, k\}] \cdot \det(T_\alpha A T_\beta^{-1} \setminus \{1, \cdots, k\}) \\
&= \operatorname{sgn}(\alpha)\operatorname{sgn}(\beta)\det A[\alpha|\beta]\det A \setminus [\alpha|\beta].
\end{aligned}$$

Q.E.D.

Corollary 4.6. Assume $A \in M_n(\mathbb{R})$ is invertible and $A[\alpha|\, \beta]$ is invertible, where $\alpha, \beta \subseteq \{1, \cdots, n\}$, $|\alpha| = |\beta|$. Then $A \setminus [\alpha|\beta]$ is invertible and

$$(A \setminus [\alpha|\beta])^{-1} = A^{-1}(\beta|\, \alpha)$$

Proof. Observe

$$\begin{aligned}
A^{-1}(\beta|\alpha) &= T_\beta A^{-1} T_\alpha^{-1}(\{1, \cdots, k\}) \\
&= (T_\alpha A T_\beta^{-1})^{-1}(\{1, \cdots, k\}) \\
&= (T_\alpha A T_\beta^{-1} \setminus \{1, \cdots, k\})^{-1} \\
&= (A \setminus [\alpha|\beta])^{-1}.
\end{aligned}$$

The third equality holds by taking inverse on both sides of (5).

Q.E.D.

Corollary 4.7. If $A \in M_n(\mathbb{R})$ is invertible, $\alpha, \beta \subseteq \{1, \cdots, n\}$, $|\alpha| = |\beta|$, then

$$\det A^{-1}[\alpha|\beta] = \operatorname{sgn}(\alpha)\operatorname{sgn}(\beta)\frac{\det A(\beta|\alpha)}{\det A}.$$

Proof. By theorem 4.5 and corollary 4.6,

$$\begin{aligned} \det A^{-1} &= \operatorname{sgn}(\alpha)\operatorname{sgn}(\beta)\det A^{-1}[\alpha|\beta]\det\left[(A^{-1} \setminus [\alpha|\beta])^{-1}\right]^{-1} \\ &= \operatorname{sgn}(\alpha)\operatorname{sgn}(\beta)\det A^{-1}[\alpha|\beta]\det\left[A(\beta|\alpha)\right]^{-1}. \end{aligned}$$

Hence

$$\frac{1}{\det \mathrm{A}} = \operatorname{sgn}(\alpha)\operatorname{sgn}(\beta)\det A^{-1}[\alpha|\beta]\frac{1}{\det\left[A(\beta|\alpha)\right]}.$$

Then

$$\det A^{-1}[\alpha|\beta] = \operatorname{sgn}(\alpha)\operatorname{sgn}(\beta)\frac{\det A(\beta|\alpha)}{\det A}.$$

Q.E.D.

Corollary 4.8. Suppose $A \in M_n(\mathbb{R})$ is invertible and $\alpha = \{i\}$, $\beta = \{j\}$. Then

$$(A^{-1})_{ij} = \det A^{-1}[\,\alpha|\beta\,] = (-1)^{i+j}\frac{\det A(j|i)}{\det A}.$$

5 Variation of signs

In this section, we will introduce a new concept which is the variation of signs of a vector. There is an important connection between the skew-symmetric space and the sign variations of vectors.

Definition 5.1. For $x = (\alpha_1, \cdots, \alpha_n)^t \in \mathbb{R}^n$, let $S^-(x)$ be the sign changes in $\alpha_1, \cdots, \alpha_n$, with zero terms discarded, and let $S^+(x)$ be the maximum sign changes in $\alpha_1, \cdots, \alpha_n$, where zeros are assigned $+1$, -1 arbitrarily.

Example 5.2. $S^+\begin{pmatrix}1\\0\\-1\\0\end{pmatrix} = 2,\ S^-\begin{pmatrix}1\\0\\-1\\0\end{pmatrix} = 1.$

It is immediate from definition of S^+ and S^-, we have the following:

Remark 5.3.

1. $0 \le S^-(x) \le S^+(x) \le n-1$ for all $x \in \mathbb{R}^n$.

2. Let x_p be a sequence in $\mathbb{R}^n$ and x_p convergent to x. Then

$$\overline{\lim}\ S^+(x_p) \le S^+(x) \ \text{ and } \ \underline{\lim}\ S^-(x_p) \ge S^-(x).$$

Definition 5.4. Suppose $x_i \in \mathbb{R}^n$, $i = 1, \cdots, k$. Then

$$x_1 \wedge \cdots \wedge x_k = \sum_{\substack{\alpha \subseteq \{1,\cdots,n\} \\ |\alpha|=k}} \varepsilon_\alpha e_\alpha \ge 0 \text{ if } \varepsilon_\alpha \ge 0 \text{ for all } \alpha.$$

Remark 5.5.

$$x_1 \wedge \cdots \wedge x_k \ge 0 \text{ (resp.} > 0).$$

$\Leftrightarrow$ $\varepsilon_\alpha \ge 0$ (resp. > 0) for all $\alpha \subseteq \{1, \cdots, n\}, |\alpha| = k$.

$\Leftrightarrow$ $\langle x_1 \wedge \cdots \wedge x_k, e_\alpha \rangle \ge 0$ (resp. > 0) for all $\alpha \subseteq \{1, \cdots, n\}, |\alpha| = k$.

$\Leftrightarrow$ $\det[\langle x_i, e_{\alpha_j} \rangle]_{k\times k} \ge 0$ (resp. > 0) for all $\alpha = \{\alpha_1 < \cdots < \alpha_k\} \subseteq \{1, \cdots, n\}$.

$\Leftrightarrow$ $\det[x_1, \cdots, x_k][\alpha|-] \ge 0$ (resp. > 0) for all $\alpha = \{\alpha_1 < \cdots < \alpha_k\} \subseteq \{1, \cdots, n\}$.

$\Leftrightarrow$ $\bigwedge^k [x_1, \cdots, x_k] \ge 0$ (resp. > 0) for all $\alpha = \{\alpha_1 < \cdots < \alpha_k\} \subseteq \{1, \cdots, n\}$, by definition5.4, theorem 3.5 and remark 3.11.

Theorem 5.6. Let $n > m$ be positive integers and $A \in M_{n\times m}(\mathbb{R})$. Then

$$S^+(Ax) \le m-1 \text{ for all nonzero } x \in \mathbb{R}^m \Longleftrightarrow \bigwedge^m A > 0 \text{ or } \bigwedge^m A < 0.$$

Proof.($\Longleftarrow$) By the truncation of rows, without loss of generality, let $n = m+1$ and there exists nonzero $x \in \mathbb{R}^n$ with

$$(Ax)_1 \ge 0\,,\ (Ax)_2 \le\ 0,\ (Ax)_3 \ge 0\ \cdots. \tag{6}$$

Suppose $A = [a_1, \cdots, a_m]$ and $\bigwedge^m A > 0$ (similarly for $\bigwedge^m A < 0$). Then

$$\det A[\alpha|-] > 0\ ,\ \text{ for all}\, \alpha \subseteq \{1, \cdots, n\}\ |\alpha| = m.$$

In particular, rank$A = m$.

Suppose $a_1 \wedge \cdots \wedge a_m = \sum_{i=1}^n \varepsilon_i e_{\widehat{i}}$, where $e_{\widehat{i}} = e_1 \wedge \cdots \wedge \widehat{e_i} \wedge \cdots \wedge e_n$. Then $\varepsilon_i > 0$, for all $i \in \{1, \cdots, n\}$. Hence $Ax \in \text{span}\{a_1, \cdots, a_m\}$ and by corollary 3.6

$$\begin{aligned} 0 &= Ax \wedge a_1 \wedge \cdots \wedge a_m \\ &= Ax \wedge \sum_{i=1}^n \varepsilon_i e_{\widehat{i}} \\ &= [(Ax)_1 e_1 + (Ax)_2 e_2 + \cdots + (Ax)_n e_n] \wedge \sum_{i=1}^n \varepsilon_i e_{\widehat{i}} \\ &= \sum_{i=1}^n (-1)^{i-1} (Ax)_i \varepsilon_i (e_1 \wedge \cdots \wedge e_n). \end{aligned}$$

Then

$$\sum_{i=1}^n (-1)^{i-1} (Ax)_i \varepsilon_i = 0. \tag{7}$$

By (6) and (7), $(-1)^{i-1}(Ax)_i\varepsilon_i = 0$ for all $i \in \{1, \cdots, n\}$. Since $\varepsilon_i > 0$ for all $i \in \{1, \cdots, n\}$, $(Ax)_i = 0$ for all $i = \{1, \cdots, n\}$, this contradicts $\text{rank}A = m$.

$(\Longrightarrow)$ First, we show $\det A[\alpha|-]\det A[\beta|-] > 0$ for all $\alpha, \beta \subseteq \{1, \cdots, n\}$, $|\alpha| = |\beta| = m$, and $|\alpha \cup \beta| = m + 1$.

Without loss of generality, let $n = m + 1$ (i.e. $\alpha \cup \beta = \{1, \cdots, n\}$). Suppose $A = [a_1, \cdots, a_m]$ and $S^+(Ax) \leq m-1$ for all nonzero $x \in \mathbb{R}^n$. Then $S^+(Ax) \neq n - 1$ for all $x \in \mathbb{R}^n$.

Suppose $a_1 \wedge \cdots \wedge a_m = \sum_{i=1}^{n} \varepsilon_i e_{\hat{i}}$ and for some $l = 1, \cdots, n-1$. ε_l, ε_{l+1} are not both 0 with $\varepsilon_l\varepsilon_{l+1} \leq 0$. Then

$$
\begin{aligned}
&(\varepsilon_{l+1}e_l + \varepsilon_l e_{l+1}) \wedge \sum_{i=1}^{n} \varepsilon_i e_{\hat{i}} \\
= \; & [(-1)^{l-1}\varepsilon_{l+1}\varepsilon_l + (-1)^l \varepsilon_l \varepsilon_{l+1}](e_1 \wedge \cdots \wedge e_n) \\
= \; & 0.
\end{aligned}
$$

Hence, $\varepsilon_{l+1}e_l + \varepsilon_l e_{l+1} \in \text{span}\{a_1, \cdots, a_m\}$. So

$$n - 1 = S^+(\varepsilon_{l+1}e_l + \varepsilon_l e_{l+1}) \leq m$$

by assumption. This contradicts $n > m$.

In general, for all $\alpha, \beta \subseteq \{1, \cdots, n\}$, $|\alpha| = |\beta| = m$. We can find subsets $\omega_1 = \alpha$, $\omega_2, \cdots$, $\omega_t = \beta$ such that $|\omega_i| = m$, $\omega_i \subseteq \{1, \cdots, n\}$, $|\omega_i \cup \omega_{i+1}| = m + 1$, $i \in \{1, \cdots, t-1\}$.

Q.E.D.

Definition 5.7.

$$J_n \equiv \begin{bmatrix} 1 & & & & \\ & -1 & & \Large 0 & \\ & & 1 & & \\ & & & -1 & \\ & \Large 0 & & & \ddots \end{bmatrix}_{n\times n}.$$

Example 5.8.

$$J_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \ J_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Remark 5.9. Suppose $\alpha,\ \omega \subseteq \{1, \cdots, n\},\ \alpha = \{\alpha_1 < \cdots < \alpha_k\}, |\alpha| = |\omega|$. Then

$$\begin{aligned} \det J_n[\alpha|\omega] &= \delta_{\alpha\omega}(-1)^{e(\alpha)} \\ &= \delta_{\alpha\omega} \prod_{i=1}^{k} (-1)^{\alpha_i - 1} \\ &= \delta_{\alpha\omega}(-1)^{\sum_{i=1}^{k} \alpha_i - i + i - 1} \\ &= \delta_{\alpha\omega}\mathrm{sgn}(\alpha)(-1)^{\frac{k(k-1)}{2}}, \end{aligned}$$

where $e(\alpha)$ is the number of even numbers in α.

Lemma 5.10. Suppose $A \in M_{n\times n}(\mathbb{R}),\ \alpha\ , \beta \subseteq \{1, \cdots, n\},\ |\alpha| = |\beta|$. Then

$$\det J_n A^{-1} J_n[\alpha|\beta] = \frac{\det A(\beta|\alpha)}{\det A}.$$

Proof. By theorem 3.14 and corollary 4.7,

$$\begin{aligned}
\det J_n A^{-1} J_n[\alpha|\beta] &= \sum_{\omega,\omega' \subseteq \{1,\cdots,n\}} \det J_n[\alpha|\omega] \det A^{-1}[\omega|\omega'] \det J_n[\omega'|\beta] \\
&= \det J_n[\alpha|\alpha] \det A^{-1}[\alpha|\beta] \det J_n[\beta|\beta] \\
&= \mathrm{sgn}(\alpha)\mathrm{sgn}(\beta)\mathrm{sgn}(\alpha)\mathrm{sgn}(\beta) \frac{\det A(\beta|\alpha)}{\det A}. \\
&= \frac{\det A(\beta|\alpha)}{\det A}.
\end{aligned}$$

Q.E.D.

Replacing A by $J_n A J_n$ and recalling $J_n^2 = I$, we have the following remark.

Remark 5.11. Suppose $A \in M_{n\times n}(\mathbb{R})$, $\alpha, \beta \subseteq \{1, \cdots, n\}$, $|\alpha| = |\beta|$. Then

$$\det A^{-1}[\alpha|\beta] = \frac{\det J_n A J_n(\beta|\alpha)}{\det A}.$$

The following lemma can be easily proved by induction. We leave it to the reader.

Lemma 5.12.

$$S^+(x) + S^-(J_n x) = n - 1 \quad \text{for all } x \in \mathbb{R}^n.$$

The following theorem describes the relation of sign variations in vectors of orthogonal spaces.

Theorem 5.13. Suppose $M \subseteq \mathbb{R}^n$ with $1 \le \dim M = m < n$. Then

$$S^+(x) \le m - 1 \quad \text{for all nonzero } x \in M \iff S^-(y) \ge m \quad \text{for all nonzero } y \in M^\perp.$$

Proof. ($\Longrightarrow$) Choose an orthonormal basis $a_1, \cdots, a_n$ of $\mathbb{R}^n$ such that $a_1, \cdots, a_m \in M$ and $a_{m+1}, \cdots, a_n \in M^\perp$, $A \equiv [a_1, \cdots, a_n]$. Note $A^{-1} = A^t$ and $\det A = 1$. Suppose $S^+(x) \leq m-1$ for all nonzero $x \in M$. By Theorem 5.6,

$$\bigwedge^m A[-|1, \cdots, m] > 0 \text{ or } < 0.$$

Without loss of generality, suppose $\bigwedge^m A[-|1, \cdots, m] > 0$. Then by remark 5.11,

$$\begin{aligned} 0 &< \det A[\alpha|1, \cdots, m] \\ &= \det(A^{-1})^t[\alpha|1, \cdots, m] \\ &= \det A^{-1}[1, \cdots, m|\alpha] \\ &= \det J_n A J_n(\alpha|1, \cdots, m) \\ &= \det J_n A J_n[\alpha'|m+1, \cdots, n], \end{aligned}$$

for all $\alpha \subseteq \{1, \cdots, n\}, |\alpha| = m$, where α' is a complemental set in $\{1, \cdots, n\}$. Hence $\bigwedge^{n-m} J_n A J_n[-|m+1, \cdots, n] > 0$. By Theorem 5.6,

$$S^+(z) \leq (n-m) - 1 \tag{8}$$

for all $z \in \text{span}\{\ J_n A J_n e_i \mid i = m+1, \cdots, n\}$. Let $y \in M^\perp$. Then $y = \beta_{m+1} a_{m+1} + \cdots + \beta_n a_n$ for some $\beta_i \in \mathbb{R}$, where $i \in \{m+1, \cdots, n\}$. Note $J_n A J_n e_i = (-1)^{i+1} J_n a_i$, $i \in \{m+1, \cdots, n\}$. Hence $J_n y = \beta_{m+1} J_n a_{m+1} + \cdots + \beta_n J_n a_n \in \text{span}\{\ J_n A J_n e_i \mid i = m+1, \cdots, n\}$. Then,

$$n - m - 1 \geq S^+(J_n y) = n - 1 - S^-(y)$$

by (8) and lemma 5.12. Hence $S^-(y) \geq m$.

($\Longleftarrow$) If $S^-(y) \geq m$ for all nonzero $y \in M^\perp$, then by lemma 5.12,

$$S^+(J_n y) \leq (n-1) - m.$$

Hence $S^-(x) \geq n - m$ for all nonzero $x \in (J_n M^\perp)^\perp$ by previous direction. We have

$$S^+(J_n x) \leq m - 1 \tag{9}$$

for all nonzero $x \in (J_n M^\perp)^\perp$. Equation (9) is equivalent to $S^+(x) \leq m-1$ for all nonzero $x \in M$.

Q.E.D.

Definition 5.14.

By a signature, we mean an infinite real sequence $\epsilon = (\varepsilon_i)$, such that $\varepsilon_i = \pm 1$.

Example 5.15.

$$\epsilon = \{1, -1, 1, 1, 1, -1, -1, \cdots\} \text{ is a signature.}$$

Definition 5.16. $A \in M_{n\times m}(\mathbb{R})$ is sign regular (resp. strictly sign regular) with signature ϵ if

$$\varepsilon_k \bigwedge^k A \geq 0 \text{ (resp. } > 0) \text{ for all } k,\ \ 1 \leq k \leq \min\{m, n\}.$$

From remark 3.11, $A \in M_{n\times m}(\mathbb{R})$ is sign regular (resp. strictly sign regular) iff $\varepsilon_k \det A[\alpha|\beta] \geq 0$ (resp. $\varepsilon_k \det A[\alpha|\beta] > 0$), for all $\alpha \subseteq \{1, \cdots, n\}$, $\beta \subseteq \{1, \cdots, m\}, 1 \leq k \leq \min\{m, n\}, |\alpha| = |\beta| = k$.

It is immediate from definition 3.12, we have the following:

Remark 5.17.

A is totally positive $\Longleftrightarrow$ A is sign regular with signature $\epsilon = \{1, 1, 1, \cdots\}$.

we give a characterization of strictly sign regular matrices in terms of the variation of signs of vectors on their image.

Theorem 5.18. Suppose $n \geq m$ and $A = [a_1, \cdots, a_m] \in M_{n\times m}(\mathbb{R})$. Then A is strictly sign regular with some signature ϵ if and only if $S^+(Ax) \leq S^-(x)$ for all nonzero $x \in \mathbb{R}^m$.

Proof.($\Longrightarrow$) Pick nonzero $x \in \mathbb{R}^m$. Suppose $k = S^-(x)$. Then

$$Ax = x_1a_1 + \cdots + x_ma_m = y_1 + \cdots + y_{k+1},$$

where $y_i = \sum_{\beta_i \leq j \leq \omega_i} x_ja_j$ and $\beta_i, \omega_i \in \{1, \cdots, n\}$, $1 \leq i \leq k+1$, such that

$$\begin{cases} x_jx_{j'} < 0, & \text{if } j \in [\beta_i, \omega_i] \text{ } and \text{ } j' \in [\beta_{i+1}, \omega_{i+1}]. \\ x_jx_{j'} > 0, & \text{if } j, j' \in [\beta_i, \omega_i]. \end{cases}$$

Since

$$y_1 \wedge \cdots \wedge y_{k+1} = \sum_{j_i \in [\beta_i, \omega_i]} (x_{j_1} \cdots x_{j_{k+1}}) a_{j_1} \wedge \cdots \wedge a_{j_{k+1}}$$

$$\begin{cases} > 0, & \text{if } (-1)^{k+1}\varepsilon_{k+1} > 0, \\ < 0, & \text{if } (-1)^{k+1}\varepsilon_{k+1} < 0, \end{cases}$$

where $x_{j_s}x_{j_{s+1}} < 0$ for all $s \in \{1, \cdots, k\}$. By theorem 5.6 and remark 5.5, $S^+(\ [y_1, \cdots, y_{k+1}] \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}) \leq k + 1 - 1 = k$. That is

$$S^+(Ax) \leq k = S^-(x).$$

($\Longleftarrow$) To prove A is strictly sign regular with some signature ϵ, we need to show a_α and a_β both > 0 or < 0 for all $\alpha, \beta \subseteq \{1, \cdots, m\}$ with $|\alpha| = |\beta|$.

Without loss of generality, suppose

$$\alpha = \{\omega_1 < \cdots < \widehat{\omega_i} < \cdots < \omega_{k+1}\}$$

and

$$\beta = \{\omega_1 < \cdots < \omega_i < \widehat{\omega_{i+1}} < \cdots < \omega_{k+1}\}.$$

Then for $t \in [0,1]$ and $c_i \in \mathbb{R}$,

$$S^+(c_1 a_{\omega_1} + \cdots + c_{i-1} a_{\omega_{i-1}} + c_i((1-t)a_{\omega_i} + t a_{\omega_{i+1}}) + c_{i+2} a_{\omega_{i+2}} + \cdots + c_{k+1} a_{\omega_{k+1}})$$

$$\begin{aligned}
&= S^+(A(c_1 e_{\omega_1} + \cdots + c_{i-1} e_{\omega_{i-1}} + c_i((1-t)e_{\omega_i} + t e_{\omega_{i+1}}) + \\
&\quad c_{i+2} e_{\omega_{i+2}} + \cdots + c_{k+1} e_{\omega_{k+1}})) \\
&\leq S^-(c_1 e_{\omega_1} + \cdots + c_{i-1} e_{\omega_{i-1}} + c_i((1-t)e_{\omega_i} + t e_{\omega_{i+1}}) + c_{i+2} e_{\omega_{i+2}} + \\
&\quad \cdots + c_{k+1} e_{\omega_{k+1}}) \\
&\leq i - 2 + 1 + 1 + k + 1 - (i+2) = k - 1.
\end{aligned}$$

Hence by theorem 5.6 and remark 5.5,

$$a_{\omega_1} \wedge \cdots \wedge a_{\omega_{i-1}} \wedge ((1-t)a_{\omega_i} + t a_{\omega_{i+1}}) \wedge a_{i+2} \wedge \cdots \wedge a_{\omega_{i+k}} > 0$$

for all $t \in [0,1]$, or

$$a_{\omega_1} \wedge \cdots \wedge a_{\omega_{i-1}} \wedge ((1-t)a_{\omega_i} + t a_{\omega_{i+1}}) \wedge a_{i+2} \wedge \cdots \wedge a_{\omega_{i+k}} < 0$$

for all $t \in [0,1]$. That is $(1-t)a_\alpha + t a_\beta > 0$ for all $t \in [0,1]$ or < 0. For $t = 0$, we obtain $a_\alpha > 0$ or $a_\alpha < 0$.

For $t = 1$, we obtain $a_\beta > 0$ or $a_\beta < 0$. By the function $(1-t)a_\alpha + ta_\beta$ is continuous on t, a_α, a_β have the same sign.

Q.E.D.

Corollary 5.19. Let A be an $n \times m$ STP matrix. Then

$$S^+(Ax) \leq S^-(x)$$

for all nonzero $x \in \mathbb{R}^m$.

Proof. An $n \times m$ STP matrix is strictly sign-regular with signature $\epsilon = \{1, 1, \cdots\}$. Hence by previous theorem, this corollary is obvious.

Q.E.D.

6 Eigenvalues

We will study the eigenvalues of a strictly sign regular matrix in this section. First, we need a well known theorem.

Theorem 6.1 (Perron's Theorem). Let $A \in M_n(\mathbb{R})$ and $A > 0$ with eigenvalues $\rho_i(A)$, $i \in \{1, \cdots, n\}$. Suppose $|\rho_1(A)| \geq |\rho_2(A)| \geq \cdots \geq |\rho_n(A)|$. Then

(a) $\rho_1(A) > 0$ is a real.

(b) There is an $x \in \mathbb{R}^n$ with $x > 0$ and $Ax = \rho(A)x$.

(c) $\rho_1(A) > |\rho_i(A)|$, where $i \in \{2, \cdots, n\}$.

The following theorem may be the most important theorem in this section. It can be used to prove that any STP matrix has all positive, simple eigenvalues.

Theorem 6.2. Let $A \in M_n(\mathbb{R})$ be strictly sign regular with some signature ϵ. Suppose $|\rho_1(A)| \geq |\rho_2(A)| \geq \cdots \geq |\rho_n(A)|$, where $\rho_i(A) \in \mathbb{C}$ are eigenvalues of A. Then $\rho_i(A)$ are reals, $\frac{\varepsilon_i}{\varepsilon_{i-1}}\rho_i(A) > |\rho_{i+1}(A)|$ and the eigenvectors $u_1, \cdots, u_n$ of $\rho_1(A), \cdots, \rho_n(A)$ respectively can be chosen such that $u_1 \wedge \cdots \wedge u_i > 0$ for all $i \in \{1, \cdots, n\}$. (Here $\varepsilon_0 \equiv 1, \rho_{n+1}(A) \equiv 0$).

Proof. We will prove this theorem by induction on i. When $i = 1$, we get $\varepsilon_1 A > 0$. By Perron's theorem, $|\varepsilon_1\rho_1(A)| > |\varepsilon_1\rho_k(A)| = |\rho_k(A)|$ for all $k > 1$ and we can choose eigenvector $u_1 > 0$. In general, suppose $\rho_1(A), \cdots, \rho_{i-1}(A)$ are reals with $\frac{\varepsilon_j}{\varepsilon_{j-1}}\rho_j(A) > |\rho_{j+1}(A)|$ for all $j \leq i-1$ and corresponding eigenvectors $u_1, \cdots, u_{i-1}$ such that $u_1 \wedge \cdots \wedge u_{i-1} > 0$. Since $\varepsilon_i \bigwedge^i A > 0$, by Perron's theorem,

$$\begin{aligned}
\varepsilon_i\rho_1(A)\cdots\rho_i(A) &> |\rho_1(A)\cdots\rho_{i-1}(A)\rho_{i+1}(A)| \\
&= |\rho_1(A)|\cdots|\rho_{i-1}(A)||\rho_{i+1}(A)|. \\
&= \frac{\varepsilon_1}{\varepsilon_0}\rho_1(A)\cdots\frac{\varepsilon_{i-1}}{\varepsilon_{i-2}}\rho_{i-1}(A)|\,\rho_{i+1}(A)|
\end{aligned}$$

Hence

$$\frac{\varepsilon_i}{\varepsilon_{i-1}}\rho_i(A) > |\rho_{i+1}(A)|.$$

Note that by Perron's theorem, $\varepsilon_i u_1 \wedge \cdots \wedge u_i > 0$ or < 0. Hence we can choose u_i such that $u_1 \wedge \cdots \wedge u_i > 0$.

Q.E.D.

Theorem 6.3. Let $A \in M_n(\mathbb{R})$ be strictly sign-regular with some signature ϵ. Let $\rho_1(A), \cdots, \rho_n(A)$ be eigenvalues of A with corresponding eigenvectors $u_1, \cdots, u_n$. Then $S^-(u_k) = S^+(u_k) = k-1$.

Proof. By previous theorem, $u_1 \wedge \cdots \wedge u_k > 0$ or < 0, then $S^+(u_k) \leq k-1$ by Remark 5.5 and Theorem 5.6. $J_n A^{-1} J_n$ is strictly sign regular by Lemma 5.10, and

$$\begin{aligned} J_n A^{-1} J_n (J_n u_k) &= J_n \frac{1}{\rho_k(A)} u_k \\ &= \frac{1}{\rho_k(A)} (J_n u_k) \\ &= \rho_{n+1-k}(J_n A^{-1} J_n) J_n u_k. \end{aligned}$$

Hence $S^+(J_n u_k) \leq n+1-k-1 = n-k$ by Remark 5.5 and Theorem 5.6. Then by lemma 5.12, $S^-(u_k) \geq k-1$. Hence $S^-(u_k) = S^+(u_k) = k-1$.

Q.E.D.

We can generalize theorem 6.3 to the following theorem.

Theorem 6.4. Let $A \in M_n(\mathbb{R})$ be strictly sign-regular with eigenvalues $\rho_1 > \rho_2 > \cdots > \rho_n$ and corresponding eigenvectors $u_1, \cdots, u_n$. Fix s, k such that $1 \leq s \leq k \leq n$. Then

$$s - 1 \leq S^-(\sum_{i=s}^{k} c_i u_i) \leq S^+(\sum_{i=s}^{k} c_i u_i) \leq k-1$$

for each $1 \leq s \leq k \leq n$ and c_i not all zeros.

Proof. By theorem 6.2, $u_1 \wedge \cdots \wedge u_k > 0$ or < 0. By remark 5.5 and theorem 5.6 , $S^+(\sum_{i=s}^{k} c_i u_i) \leq k-1$. By lemma 5.10, $J_n A^{-1} J_n$ is strictly sign-regular.

It has eigenvalues $\frac{1}{\rho_1}, \cdots, \frac{1}{\rho_n}$ and eigenvectors $J_n u_n, J_n u_{n-1}, \cdots, J_n u_1$. Note $|\frac{1}{\rho_n}| > |\frac{1}{\rho_{n-1}}| > \cdots > |\frac{1}{\rho_1}| > 0$. Again $J_n u_n \wedge \cdots \wedge J_n u_s > 0$ or < 0, then

$$S^+(\sum_{i=s}^{k} J_n u_i) \leq n - (s-1) - 1 = n - s.$$

Hence $S^-(\sum_{i=s}^{k} u_i) \leq s - 1$ by lemma 5.12.

Q.E.D.

Theorem 6.5. Let A be an $n \times n$ STP matrix and $\rho_1 > \cdots > \rho_n$ are eigenvalues of A. Fix an integer k $(1 \leq k \leq n)$. Let $A^{(k)}$ denote the principal submatrix of A obtained by deleting its kth row and column and $\mu_1^{(k)} > \cdots > \mu_{n-1}^{(k)} > 0$ be the eigenvalues of $A^{(k)}$. Then for all $j \in \{1, \cdots, n-1\}$,

$$\rho_{j-1} > \mu_j^{(k)} > \rho_{j+1} \text{ (where } \rho_0 = \rho_1).$$

Proof. First, we will prove $\mu_j^{(k)} > \rho_{j+1}$ for all $j \in \{1, \cdots, n-1\}$. Let $\mathbf{p} = (p_1, \cdots, p_n)^t$ be an eigenvector of A with eigenvalues ρ_{j+1}. Then $A\mathbf{p}^t = \rho_{j+1}\mathbf{p}^t$. By theorem 6.3,

$$S^-(\mathbf{p}) = S^+(\mathbf{p}) = j.$$

Let $\mathbf{q}$ denote a real eigenvector of $A^{(k)}$ according to eigenvalue $\mu_j^{(k)}$. Then $A^{(k)}\mathbf{q} = \mu_j^{(k)}\mathbf{q}$. Again, by theorem 6.3,

$$S^-(\mathbf{q}) = S^+(\mathbf{q}) = j - 1.$$

Suppose $\mathbf{q} = (q_1, \cdots, q_{k-1}, q_{k+1}, \cdots, q_n)^t$. Let $\mathbf{q}' \equiv (q_1, \cdots, q_{k-1}, 0, q_{k+1}, \cdots, q_n)^t$ and $\mathbf{q}'' \equiv \dfrac{A\mathbf{q}'}{\mu_j^{(k)}}$. Thus $\mathbf{q}'' = (q_1, \cdots, q_{k-1}, r_k, q_{k+1}, \cdots, q_n)^t$ for some $r_k \in \mathbb{R}$. From corollary 5.19 and theorem 6.3, we have

$$j - 1 = S^+(\mathbf{q}) \leq S^+(\mathbf{q}'') \leq S^-(\mathbf{q}') = S^-(\mathbf{q}) = j - 1.$$

Then

$$S^+(\mathbf{q}'') = S^-(\mathbf{q}') = j - 1.$$

If $p_k = 0$, then $\mathbf{p}' = (p_1, \cdots, p_{k-1}, p_{k+1}, \cdots, p_n)^t$ is an eigenvector of $A^{(k)}$ according to the eigenvalue ρ_{j+1}. Since the vector $\mathbf{p}'$ also has exactly j sign changes, by theorem 6.3,

$$\rho_{j+1} = \mu_{j+1}^{(k)} < \mu_j^{(k)}.$$

If $r_k = 0$, then $\mathbf{q}' = \mathbf{q}''$ is an eigenvector of A with $j - 1$ sign changes. Hence by theorem 6.3. , we have

$$\mu_j^{(k)} = \rho_j > \rho_{j+1}.$$

Thus we may assume p_k, r_k all are nonzero.

Without loss of generality, we may assume p_k, r_k are both positive. Define

$$\mathbf{F} = \{f : f > 0, S^-(f\mathbf{q}' + \mathbf{p}) \leq j - 1\}$$

and $f^* = \inf \mathbf{F}$.

We claim f^* is a positive number. This is due to the following two reasons. First, let f be large enough such that $f|q_i| > |p_i|$ if $q_i \neq 0$. We have

$$S^-(f\mathbf{q}' + \mathbf{p}) \leq S^+(f\mathbf{q}' + \mathbf{p}) \leq S^+(\mathbf{q}'') = j - 1.$$

Hence $\mathbf{F} \neq \emptyset$. Second, let f be small enough and positive such that $f|\,q_i| < |p_i|$ if $p_i \neq 0$. We have

$$S^-(f\mathbf{q}' + \mathbf{p}) \geq S^-(\mathbf{p}) = j.$$

Hence $\mathbf{F}$ has the greatest lower bound $f^* > 0$.

From the definition of f^* and Remark 5.3.2, we get

$$S^-(f^*\mathbf{q}' + \mathbf{p}) \leq j - 1$$

and

$$S^+(f\mathbf{q}' + \mathbf{p}) \geq j \tag{10}$$

for all positive $f \leq f^*$. Now, $A(f^*\mathbf{q}' + \mathbf{p}) = f^*\mu_j^{(k)}\mathbf{q}'' + \rho_{j+1}\mathbf{q}$. Let $\widehat{f} = \dfrac{f^*\mu_j^{(k)}}{\rho_{j+1}} > 0$. From Corollary 5.19,

$$S^+(\widehat{f}\mathbf{q}'' + \mathbf{p}) \leq S^-(f^*\mathbf{q}' + \mathbf{p}) \leq j - 1.$$

Since $\widehat{f}\mathbf{q}'' + \mathbf{p}$ and $\widehat{f}\mathbf{q}' + \mathbf{p}$ differ only in the kth coordinate, where both are positive. It follows that

$$S^+(\widehat{f}\mathbf{q}' + \mathbf{p}) = S^+(\widehat{f}\mathbf{q}'' + \mathbf{p}) \leq j - 1.$$

From (10), this implies $\widehat{f} > f^*$. Thus $\mu_j^{(k)} > \rho_{j+1}$.

Next, we will prove for all $j = 1, \cdots, n-1$

$$\mu_j^{(k)} < \rho_{j-1}.$$

Let $\mathbf{y}$ be a real eigenvector of A with eigenvalues ρ_{j-1}. Then

$$A\mathbf{y} = \rho_{j-1}\mathbf{y} \text{ and } S^+(\mathbf{y}) = S^-(\mathbf{y}) = j - 2.$$

The vectors $\mathbf{q}, \mathbf{q}', \mathbf{q}''$ are as above. If $r_k = 0$, then $\mu_j^{(k)} = \rho_j < \rho_{j-1}$. If $y_k = 0$, then $\rho_{j-1} = \mu_{j-1}^{(k)} > \mu_j^{(k)}$. We may assume r_k and y_k are both positive. Let

$$d^* = \inf\{d : d > 0, S^-(d\mathbf{y} + \mathbf{q}') \leq j - 2\}.$$

Then we follow the previous argument, $\mu_j^{(k)} < \rho_{j-1}$ for all $1 \leq j \leq n-1$.

Q.E.D.

Lemma 6.6.

1. The set of all STP matrices are dense in the set of all totally positive matrices.

2. Let $f : M_n(\mathbb{R}) \longrightarrow \mathbb{C}^n$ be a function such that

$$f(A) = (\rho_1(A), \rho_2(A), \cdots, \rho_n(A)).$$

Then f is continuous, where $\rho_i(A)$ $(1 \leq i \leq n)$ are eigenvalues of A and $|\rho_1(A)| \geq \cdots \geq |\rho_n(A)|$.

Proof. of 1. See A.M. Whitney, A reduction theorem for totally positive matrices, *J. Analyse Math.* 2:88–92(1952).

Proof. of 2. It is clear from the fact that eigenvalues of a matrix are the zeros of its characteristic polynomial, which are continuous.

Theorem 6.7. Let A be an $n \times n$ TP matrix and $\rho_1 \geq \cdots \geq \rho_n$ are eigenvalues of A. Fix an integer k $(1 \leq k \leq n)$. Let $A^{(k)}$ denote the principal submatrix of A obtained by deleting its kth row and column and let $\mu_1^{(k)} \geq \cdots \geq \mu_{n-1}^{(k)}$ be the eigenvalues of $A^{(k)}$. Then for all j, $1 \leq j \leq n-1$,

$$\rho_{j-1} \geq \mu_j^{(k)} \geq \rho_{j+1} \text{ (where } \rho_0 = \rho_1).$$

Proof. Apply above Lemma and theorem 6.5.

7 Some results about STP matrices

Definition 7.1. Let $A = [a_{ij}] \in M_{n\times m}(\mathbb{R})$ and $B = [b_{ij}] \in M_{n\times m}(\mathbb{R})$. Then the *Hardmard Product* of A and B is the matrix

$$A \odot B \equiv [a_{ij}b_{ij}] \in M_{n\times m}(\mathbb{R}).$$

Lemma 7.2. (Minkowski's inequality)
If $A, B \in M_{n\times n}(\mathbb{R})$ are symmetric and positive definite, then

$$[\det(A+B)^{\frac{1}{n}}] \geq \det A^{\frac{1}{n}} + \det B^{\frac{1}{n}}.$$

Proof. See Matrix Analysis (Horn and Johnson) p482.

The following examples are concerning about STP matrices.

Remark 7.3.

1. Let $A, B \in M_2(\mathbb{R})$ be STP matrices. Then $A \odot B$ is a STP matrix.
2. Let $A, B \in M_2(\mathbb{R})$ be symmetric STP matrices. Then $A + B$ is a STP matrix.
3. For all $n \in \mathbb{N}$, there exists an $n \times n$ STP matrix.

Proof. of 1. It is easy to check. We leave it to the reader.

Q.E.D.

Proof. of 2. Just apply Lemma 7.2 for $n = 1$. We have $\det(A+B) \geq \det A + \det B > 0$.

Q.E.D.

Proof. of 3. Consider the *Vandermonde* matrix $V = [a_{ij}]_{n\times n}$, $a_{ij} = (j+1)^{i-1}$ $(1 \leq i, j \leq n)$. By using of the *Vandermonde* matrix always have positive determinant $\prod_{n\geq j>i\geq 1} (j-i)$, V is a STP matrix.

Remark 7.4.

1. If $A, B \in M_{n\times m}(\mathbb{R})$ are STP matrices, then $A \odot B$ is not necessary to be a STP matrix.

2. If A, B are STP matrices, then $A \otimes B$ is not necessary to be a STP matrix.

3. If $A, B \in M_n(\mathbb{R})$ are STP matrices, then $A + B$ is not necessary to be a STP matrix.

4. $A \in M_n(\mathbb{R})$ is a STP matrix, then A^{-1} is not necessary to be a STP matrix.

Counterexample of 1.

Let $A = \begin{bmatrix} 1 & 1.01 & 1.02 \\ 1 & 1.02 & 1.403 \\ 1 & 1.03 & 1.061 \end{bmatrix}$, $B = \begin{bmatrix} 1.1 & 1 & 0.1 \\ 2.2 & 2.1 & 1 \\ 1.99 & 1.9 & 1 \end{bmatrix}$. Then

$$A \odot B = \begin{bmatrix} 1.1 & 1.01 & 0.102 \\ 2.2 & 2.142 & 1.403 \\ 1.99 & 1.957 & 1.061 \end{bmatrix}.$$

Note $\det A \odot B = -0.0536$. Hence $A \odot B$ is not STP.

Counterexample of 2.

Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$. Then

$$A \otimes B = \begin{bmatrix} 2 & 1 & 4 & 2 \\ 1 & 1 & 2 & 2 \\ 4 & 2 & 10 & 5 \\ 2 & 2 & 5 & 5 \end{bmatrix}.$$

Note

$$(A \otimes B)\,[\{1,2\}|\{2,3\}] = \begin{bmatrix} 1 & 4 \\ 1 & 2 \end{bmatrix}$$

has determinant -2. Hence it is not a strictly totally positive matrix.

Counterexample of 3.

Let $A = \begin{bmatrix} 11 & 3 & 5 \\ 3 & 1 & 2 \\ 5 & 2 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 2 \end{bmatrix}$. Then

$$A + B = \begin{bmatrix} 15 & 5 & 6 \\ 5 & 4 & 4 \\ 6 & 4 & 7 \end{bmatrix}.$$

Note

$$(A \otimes B)\,[\{1,2\}|\{2,3\}] = \begin{bmatrix} 5 & 6 \\ 4 & 4 \end{bmatrix}$$

has determinant -2. Hence it is not a strictly totally positive matrix.

Counterexample of 4.

Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$. Then $A^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$. Hence it is not a strictly totally positive matrix.

Remark 7.5. As notations in theorem 6.5, in fact, let us fix $k = 1, n$. Then $\rho_j > \mu_j^{(k)} > \rho_{j+1}$ for all $j \in \{1, \cdots, n-1\}$.

We now given a counterexample for that $k = 2$.

Example 7.6. Let

$$A = \begin{bmatrix} 1.1 & 1 & 0.1 \\ 2.2 & 2.1 & 1 \\ 1.99 & 1.9 & 1 \end{bmatrix}.$$

Then the approximate eigenvalues of A are 3.8893 0.3021 0.0086. But the approximate eigenvalues of $A^{(2)}$ is 1.4989 0.6011.

Definition 7.7. Let $A = [a_{ij}]_{n\times m}$. We denote $|A| \equiv [|a_{ij}|\,]_{n\times m}$.

Question 7.8. Let $A \in M_{n\times n}(\mathbb{R})$ be a STP matrix. Is $|A^{-1}_{n\times n}|$ a STP matrix for $n > 2$?

Now we check $|A^{-1}_{n\times n}|$ is STP for $n = 1, 2$. For $n = 1$, suppose $A = [\,a\,]$ and $a > 0$. Then $A^{-1} = [\,\frac{1}{a}\,] > 0$ is a STP matrix. For $n = 2$, suppose $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a STP matrix. Then $|\,A^{-1}| = \frac{1}{\det A}\begin{bmatrix} d & b \\ c & a \end{bmatrix}$ and $\det A = 1$.

Hence A is a STP matrix. But what happened when $n > 2$?

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