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碩 士 論 文

Lit-only σ -games 的代數結構

The Algebra Behind Lit-only σ -games

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摘要

令 $S = \{s_1, s_2, \dots, s_n\}$ 是一個有限的集合。如果給定一個函數 $m: S \times S \rightarrow \mathbb{N} \cup \{\infty\}$ 定義為 $m(s, s) = 1$ ；而對不同的 s, s' 滿足 $m(s, s') = m(s', s) \in \{2, 3\}$ 。那麼此集合 S 可以被聯想成一個圖(也把此圖用 S 表示)，圖的點集合為集合 S ，邊集合為 $\{ss' \mid m(s, s') = 3\}$ 。一個 *simply-laced Coxeter group* W_S 是一個跟 (S, m) 有關的群。在此篇論文中證明了當圖 S 是一個有 n 個頂點的路徑(path)時， W_S 是同構(isomorphic)於一個對稱群 S_{n+1} 的群。我們考慮一個很自然的同態函數(homomorphism) $\sigma: W_S \rightarrow GL(R^n)$ 將 W_S 對映到線性群 $GL(R^n)$ 中，使得 $\sigma(W_S)$ 是一個可以作用在 R^n 空間上的線性群(矩陣所構成的群)。當我們把 $\sigma(W_S)$ 中的矩陣都轉置後，可得到這些轉置矩陣形成的群 $\sigma^*(W_S)$ 。若將群 $\sigma^*(W_S)$ 作用在 R^n 上，可證明群 $\sigma^*(W_S)$ 會同構(isomorphic)於一個對稱群 S_{n+1} 。因為群 $\sigma^*(W_S)$ 中的矩陣都是整係數矩陣，若將這些整係數矩陣的係數同餘(modulo) 2，則可得到一些新的矩陣形成一個新的群。在此篇論文中，我們規定這個新的群只有左乘運算，且將這個群作用在一個二元體(binary field) F_2 所形成的 n 維空間 F_2^n ，並佈於一個二元體 F_2 上。我們稱這個新的群作用在 F_2^n 上是一個作用在圖 S 的 *lit only σ -game*。我們討論當圖 S 是 3 個頂點的 cycle 時， W_S 中的子群 G 之生成集的樣子且 G 滿足 $\sigma^*(G) = \{I\} \pmod{2}$ 。

The Algebra Behind Lit-only σ -games

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Abstract

Let $S = \{s_1, s_2, \dots, s_n\}$ be a finite set and m be a function with $m: S \times S \rightarrow \mathbb{N} \cup \{\infty\}$ satisfying $m(s, s) = 1$ and $m(s, s') = m(s', s) \in \{2, 3\}$ for distinct $s, s' \in S$. The set S is associated with the graph, also denoted by S , with the vertex set S and the edge set $\{ss' \mid m(s, s') = 3\}$. A *simply-laced Coxeter group* W_S associated with (S, m) is the group generated by S subject to the relations

$$(ss')^{m(s, s')}$$

for $s, s' \in S$. We consider a homomorphism $\sigma: W_S \rightarrow GL(R^n)$, which is referred as *canonical representation* of W_S , where $GL(R^n)$ is the group of invertible linear transformations of R^n into itself. We consider the canonical representation σ of W_S into R^n and use its dual representation σ^* to show that W_S is isomorphic to the symmetric group S_{n+1} if the graph S is an n -vertex path. The matrices $\sigma^*(W_S)$ have integral coefficients. The left multiplication of these matrices modulo 2 on the n -dimensional space F_2^n over a binary field is usually called the *lit only σ -game* on the graph S in literatures. In the special case when S is a 3-vertex cycle, we determine the subgroup G of W_S with $\sigma^*(G) = \{I\} \pmod{2}$.

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1 Introduction

Assume that S is a finite set, and $m : S \times S \rightarrow \mathbb{N} \cup \{\infty\}$ is a function satisfying $m(s, s) = 1$ and $m(s, s') = m(s', s) \geq 2$ for distinct $s, s' \in S$. Let $F(S)$ be the free group on the set S and N be the normal subgroup of $F(S)$ generated by all elements

$$(ss')^{m(s,s')},$$

where $s, s' \in S$. The group $W := F(S)/N$ is called the *Coxeter group associated with (S, m)* , and the pair (W, S) is called a *Coxeter system*. A Coxeter group W can be represented by a *Coxeter graph* $\Gamma = (V, E)$ whose vertex set $V = S$ and edge set $E = \{ss' \mid m(s, s') \geq 3, s \neq s' \in S\}$. The edges with $m(s, s') > 3$ are labeled by the number but the label 3 be omitted. The Coxeter group is *simply-laced* if $m(s, s') \in \{1, 2, 3\}$ for $s, s' \in S$. The Coxeter graph of simply-laced Coxeter groups exactly coincide with simply-laced Dynkin diagrams[6]. For example a Coxeter group of type A_n has its Coxeter graph a path of order n . Figure 1 lists the simply-laced Dynkin diagrams.

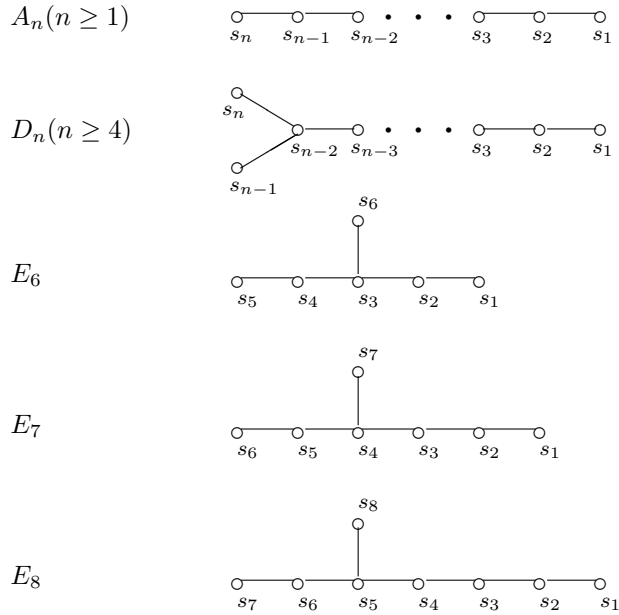


Figure 1: simply-laced Dynkin diagrams.

Throughout this paper, we assume W_S is a simply-laced Coxeter group associated with (S, m) and the set S is associated with the graph, also denoted by S , with the vertex set S and the edge set $\{ss' \mid m(s, s') = 3\}$. Note that, in section 3, the graph S of the Coxeter group W_S of type A_n is the Coxeter graph of W_S .

It is well known that W_S is finite if and only if S is simply-laced Dynkin diagrams [6]. The following is a well-known property, called the universal property of free groups. See [1, page 219] for details.

Theorem 1.1 (The universal property of free groups). *If G is a group with identity 1, and ϕ is a map from S into G , then there is a unique homomorphism $\phi' : F(S) \rightarrow G$ such that $\phi'(s) = \phi(s)$ for $s \in S$.*

The next theorem is a direct result of Theorem 1.1

Theorem 1.2. *If G is a group with identity 1, and ϕ is a map from S into G such that $(\phi(s)\phi(s'))^{m(s,s')} = 1$ for $s, s' \in S$, then there is a unique homomorphism $\phi' : W_S \rightarrow G$ such that $\phi'(s) = \phi(s)$ for $s \in S$.*

Proof. By Theorem 1.1, there is a unique homomorphism $\phi' : F(S) \rightarrow G$ such that

$$\phi'((ss')^{m(s,s')}) = (\phi'(s)\phi'(s'))^{m(s,s')} = (\phi(s)\phi(s'))^{m(s,s')} = 1,$$

i.e. $N \subseteq \text{Ker}(\phi')$. Hence ϕ' induces a unique homomorphism from $W_S = F(S)/N$ into G , which is still denoted by ϕ' , $\phi' : W_S \rightarrow G$. \square

We will use the same notation ϕ for ϕ' in the above theorem and say that *the domain S of the map ϕ lifts to the domain W_S .*

Let V_S denote the vector space over \mathbb{R} with a given basis $\{\alpha_s \mid s \in S\}$ and $V_S^* := \{f \mid f : V_S \rightarrow \mathbb{R} \text{ is linear}\}$ be the *dual space* of V_S with the dual basis $\{\alpha_s^* \mid s \in S\}$, where $\alpha_s^* : V_S \rightarrow \mathbb{R}$ is the map satisfying

$$\alpha_s^* \alpha_{s'} = \begin{cases} 1, & \text{if } s' = s; \\ 0, & \text{else,} \end{cases}$$

for any $s, s' \in S$. The *linear representation* of W_S is a homomorphism $\sigma : W_S \rightarrow GL(V_S)$, where $GL(V_S)$ is the group of invertible linear transformations from V_S into V_S , with the composition. Since V_S, V_S^* are $|S|$ -dimensional

vector spaces, we may regard V_S, V_S^* as \mathbb{R}^S . In section 2, we introduce the linear representation σ of the Coxeter group W_S as described in [6, page 110] and replace $GL(V_S)$ with $GL(\mathbb{R}^n)$. In order to find the transpositions act on \mathbb{R}^n , we consider its dual representation σ^* . In Section 3, we use the dual representation σ^* of the Coxeter group W_S of type A_n into \mathbb{R}^n to show that W_S is isomorphic to the symmetric group S_{n+1} . The matrices $\sigma^*(W_S)$ have integral coefficients as shown in Proposition 2.3. Let $\{e_s \mid s \in S\}$ denote the standard basis of \mathbb{R}^S . Then

$$\sigma^*(s)e_{s'} = \begin{cases} e_{s'}, & \text{if } s \neq s'; \\ -e_{s'} + \sum_{m(s,s'')=3} e_{s''}, & \text{if } s = s'. \end{cases}$$

The left multiplication of the matrices $\sigma^*(s)$ modulo 2 on the set F_2^n is called the *lit only σ -game* on S , which was first studied in [2], and independently in [3, 4, 5, 7, 8].

Let G be a group with a generating set S such that $e \notin S$. The *Cayley graph* $\text{Cay}(G, S)$ of G with respect to S has the vertex set G and the edge set $\{g(gs) \mid g \in G, s \in S\}$. The thesis focus on the special case when S is a 3-vertex cycle, and determines the subgroup G of W_S with $\sigma^*(G) = \{I\} \pmod{2}$ in Theorem 4.9. The Cayley graph $\text{Cay}(W_S/G, \{s_1, s_2, s_3\})$ is described in the end.

2 Representation of W_S

Let $S = \{s_1, s_2, \dots, s_n\}$ be a finite set and let V_S denote the vector space over \mathbb{R} with a given basis $\{\alpha_s \mid s \in S\}$. For $s, s' \in S$, we define a symmetric bilinear form B on V_S by

$$B(\alpha_s, \alpha_{s'}) := -\cos \frac{\pi}{m(s, s')},$$

and define a *reflection* $\sigma_s : V_S \rightarrow V_S$ by the rule:

$$\sigma_s \lambda := \lambda - 2B(\alpha_s, \lambda)\alpha_s,$$

where $\lambda \in V_S$. We have

$$\sigma_s(\alpha_{s'}) = \begin{cases} -\alpha_{s'}, & \text{if } m(s, s') = 1; \\ \alpha_{s'} + \alpha_s, & \text{if } m(s, s') = 3; \\ \alpha_{s'}, & \text{if } m(s, s') = 2, \end{cases} \quad (1)$$

where $\alpha_{s'}$ a basis vector of V_S . Note that $\sigma_s^2 = I$ and hence $\sigma_s \in GL(V_S)$ for $s \in S$.

Theorem 2.1. *The domain of map $\sigma : S \rightarrow GL(V_S)$, defined by $s \rightarrow \sigma_s$ for $s \in S$, lifts to W_S .*

Proof. By Theorem 1.2, we need to check that $(\sigma(s)\sigma(s'))^{m(s, s')} = 1$. Assume $m(s, s') = 1$ (i.e. $s' = s$). The result is hold, since $\sigma(s) = \sigma_s$ has order 2 in $GL(V_S)$. For any λ belongs to V_S , assume $m(s, s') = 2$. We have:

$$\begin{aligned} (\sigma_s \sigma_{s'})^{m(s, s')} \lambda &= (\sigma_s \sigma_{s'})^2 \lambda \\ &= (\sigma_s \sigma_{s'})[\sigma_s \lambda - 2B(\alpha_{s'}, \lambda)\sigma_s \alpha_{s'}] \\ &= (\sigma_s \sigma_{s'})[\sigma_s \lambda - 2B(\alpha_{s'}, \lambda)\alpha_{s'}] \\ &= (\sigma_s \sigma_{s'})[\lambda - 2(B(\alpha_s, \lambda)\alpha_s + B(\alpha_{s'}, \lambda)\alpha_{s'})] \\ &= \lambda. \end{aligned}$$

Finally, assume $m(s, s') = 3$. We may compute that :

$$\begin{aligned}
(\sigma_s \sigma_{s'})^{m(s, s')} \lambda &= (\sigma_s \sigma_{s'})^3 \lambda \\
&= (\sigma_s \sigma_{s'})^2 [\sigma_s \lambda - 2B(\alpha_{s'}, \lambda) \sigma_s \alpha_{s'}] \\
&= (\sigma_s \sigma_{s'}) [\lambda - 2B(\alpha_s, \lambda) (\alpha_s + \alpha_{s'}) - 2B(\alpha_{s'}, \lambda) \alpha_{s'}] \\
&= \lambda.
\end{aligned}$$

Thus the domain S of σ lifts to W_S . □

Indeed J.Humphreys proves the map σ is injective [6, Page 113].

Theorem 2.2. $\sigma : W_S \rightarrow GL(V_S)$ is injective.

We refer to the monomorphism σ as the *linear representation* of W_S .

As we fixed the Coxeter group W_S of type A_n (see in Section 3) and given an ordered basis $\{\alpha_{s_1}, \alpha_{s_2}, \dots, \alpha_{s_n}\}$ of V_S . By the equation (1), σ_{s_i} has the matrix form:

$$\sigma_{s_i} \sim \begin{pmatrix} 1 & & & & & & & & 0 \\ & \ddots & & & & & & & \\ & & 1 & & & & & & \\ & & 1 & -1 & 1 & & & & \\ & & & & 1 & & & & \\ & & & & & \ddots & & & \\ 0 & & & & & & & & 1 \end{pmatrix}. \quad (2)$$

Therefore, we consider the transpose of the above matrix. Back to the vector space V_S , if we let $L : V_S \rightarrow V_S$ be a linear transformation. Then the *dual map* $L^t : V_S^* \rightarrow V_S^*$ of L is defined by the following rule:

$$(L^t f)v = f(Lv)$$

for any $f \in V_S^*$ and $v \in V$. Note that if two maps $L_1, L_2 : V_S \rightarrow V_S$ are linear then it is easy to check $(L_1 L_2)^t = L_2^t L_1^t$.

Proposition 2.3. *For each $s \in S$, let $\sigma_s^t : V_S^* \rightarrow V_S^*$ be the dual map of the reflection $\sigma_s : V_S \rightarrow V_S$. Then for any $s, s' \in S$,*

$$\sigma_s^t(\alpha_{s'}^*) = \begin{cases} \alpha_{s'}^*, & \text{if } s \neq s'; \\ -\alpha_{s'}^* + \sum_{k:m(s',k)=3} \alpha_k^*, & \text{if } s = s', \end{cases} \quad (3)$$

for $k \in S$. In particular, $(\sigma_s^t)^2 = I$ and hence $\sigma_s^t \in GL(V_S^*)$ for $s \in S$.

Proof. Let $\lambda = \sum_{s'' \in S} c_{s''} \alpha_{s''} \in V_S$. First we may assume $s' = s$. Since σ_s^t is the dual map of σ_s and $\alpha_{s'}^*(\alpha_s) = 1$, then we have

$$\begin{aligned} [\sigma_s^t(\alpha_{s'}^*)](\lambda) &= \alpha_{s'}^*(\sigma_s \lambda) \\ &= \alpha_{s'}^*(\lambda) - 2B(\alpha_s, \lambda) \alpha_{s'}^*(\alpha_s) \\ &= \alpha_{s'}^*(\lambda) - 2B(\alpha_s, \lambda) \\ &= \alpha_{s'}^*(\lambda) - 2 \sum_{s'' \in S} c_{s''} B(\alpha_s, \alpha_{s''}) \\ &= \alpha_{s'}^*(\lambda) - 2 \sum_{\substack{s'' \in S \\ m(s, s'')=2}} c_s \cdot 0 - 2c_s \cdot 1 - 2 \sum_{\substack{s'' \in S \\ m(s, s'')=3}} c_{s''} B(\alpha_s, \alpha_{s''}) \\ &= \alpha_{s'}^*(\lambda) - 2c_s - 2 \sum_{\substack{s'' \in S \\ m(s, s'')=3}} c_{s''} B(\alpha_s, \alpha_{s''}). \end{aligned} \quad (4)$$

Since $\alpha_s^*(\lambda) = c_s$ and $B(\alpha_s, \alpha_{s''}) = -1/2$, for $m(s, s'') = 3$ and $s \in S$. Then

$$-2 \sum_{\substack{s'' \in S \\ m(s, s'')=3}} c_{s''} B(\alpha_s, \alpha_{s''}) = \sum_{\substack{s'' \in S \\ m(s, s'')=3}} c_{s''} \cdot 1 = \sum_{\substack{s'' \in S \\ m(s, s'')=3}} \alpha_{s''}^*(\lambda)$$

and the equation (4) equal to $(-\alpha_{s'}^* + \sum_{\substack{s'' \in S \\ m(s, s'')=3}} \alpha_{s''}^*)(\lambda)$

In the other case $s' \neq s$. Then $\alpha_{s'}^*(\alpha_s) = 0$. Thus,

$$[\sigma_s^t(\alpha_{s'}^*)](\lambda) = \alpha_{s'}^*(\lambda) - 2B(\alpha_s, \lambda)\alpha_{s'}^*(\alpha_s) = \alpha_{s'}^*(\lambda).$$

□

We shall call $\sigma_s^t : V_S^* \rightarrow V_S^*$ a *dual reflection* of σ_s and refer the basis $\{\alpha_s^* \mid s \in S\}$ the *standard basis* of V_S^* .

Definition 2.4. The *dual representation* $\sigma^* : W_S \rightarrow GL(V_S^*)$ of σ is defined by

$$\sigma^*(w) := \sigma(w^{-1})^t, \text{ for } w \in W_S.$$

Proposition 2.5. *Then σ^* is a monomorphism.*

Proof. For any $w_1, w_2 \in W_S$,

$$\begin{aligned} \sigma^*(w_1 w_2) &= \sigma((w_1 w_2)^{-1})^t \\ &= (\sigma(w_2^{-1})\sigma(w_1^{-1}))^t \\ &= \sigma(w_1^{-1})^t \sigma(w_2^{-1})^t \\ &= \sigma^*(w_1)\sigma^*(w_2). \end{aligned}$$

Hence the map $\sigma^* : W_S \rightarrow GL(V_S^*)$ is a homomorphism. Next we need to prove σ^* is injective.

Let $\sigma^*(w) \in GL(V_S^*)$ be the identity linear transformation for some $w \in W_S$. Let $f \in V_S^*$ and for any $v \in V_S$,

$$\begin{aligned} (\sigma^*(w)f)v &= (\sigma(w^{-1})^t f)v \\ &= f(\sigma(w^{-1})v). \end{aligned}$$

Since $\sigma^*(w) \in GL(V_S^*)$ is a identity map, $(\sigma^*(w)f)v = fv$. This implies $\sigma(w^{-1})v = v$ for any $v \in V_S$. Then $\sigma(w^{-1}) = e \in GL(V_S)$.

And by Theorem 2.2, $\sigma : W_S \rightarrow GL(V_S)$ is a monomorphism, we must have $w = e \in W_S$. This shows that the map $\sigma^* : W_S \rightarrow GL(V_S^*)$ is injective. \square

The following lemma describes the mapping of σ^* .

Lemma 2.6. *For each $s \in S$, $\sigma^*(s) = \sigma_s^t$, where $\sigma^* : W_S \rightarrow GL(V_S^*)$.*

Proof. For each $s \in S$, s has order 2 in W_S , then $\sigma^*(s) := \sigma(s^{-1})^t = \sigma(s)^t$. Since $\sigma(s) = \sigma_s$, we have $\sigma(s)^t = \sigma_s^t$. Thus, $\sigma^* : W_S \rightarrow GL(V_S^*)$ by sending s to σ_s^t . \square

From Proposition 2.5 and Lemma 2.6, we had known that $\sigma^*(s_i s_j) = \sigma_{s_i}^t \sigma_{s_j}^t$, where $s_i, s_j \in S$. In the next section, we shall introduce the simply-laced Coxeter group W_S of type A_n and give a proof of W_S is the symmetric group on a set S^* .

3 Coxeter group W_S of type A_n

In this section, we consider the Coxeter group W_S of type A_n . We shall prove W_S is isomorphic to the symmetric group S_{n+1} on $n+1$ elements. Throughout this section, set $S = \{s_1, s_2, \dots, s_n\}$, and $m : S \times S \rightarrow \{1, 2, 3\}$ is the function satisfying $m(s_i, s_j) = m(s_j, s_i)$ and

$$m(s_i, s_j) = \begin{cases} 1, & \text{if } j = i \text{ and } i \in \{1, 2, \dots, n\}; \\ 2, & \text{if } j \notin \{i-1, i+1\}, i, j \in \{1, 2, \dots, n\}; \\ 3, & \text{if } j \in \{i-1, i+1\}, i \in \{2, 3, \dots, n-1\}. \end{cases}$$

The (W_S, S) denotes the Coxeter system of type A_n and the set S is associated with the graph with the vertex set S and the edge set $\{ss' \mid m(s, s') = 3\}$, which exactly coincide with the Coxeter graph of type A_n as shown in Figure 1. Recall from Proposition 2.5, Lemma 2.6 and (3), there exists a monomorphism $\sigma^* : W_S \rightarrow GL(V_S^*)$ with $\sigma^*(s_i) = \sigma_{s_i}^t$ satisfying

$$\sigma_{s_i}^t(\alpha_{s_j}^*) = \begin{cases} \alpha_{s_j}^* & \text{if } j \neq i; \\ -\alpha_{s_j}^* + \sum_{k:m(s_i, s_k)=3} \alpha_{s_k}^* & \text{if } j = i, \end{cases} \quad (5)$$

In the ordered basis $\{\alpha_{s_1}^*, \alpha_{s_2}^*, \dots, \alpha_{s_n}^*\}$ of V_S^* , $\sigma_{s_i}^t$ has a matrix form:

$$\sigma_s^t \sim \begin{pmatrix} 1 & & & & & & 0 \\ & \ddots & & & & & \\ & & 1 & 1 & & & \\ & & & -1 & & & \\ & & & & 1 & 1 & \\ & & & & & \ddots & \\ 0 & & & & & & 1 \end{pmatrix}. \quad (6)$$

If we define $\epsilon_1 = \alpha_{s_1}^*$, and for $2 \leq i \leq n+1$,

$$\epsilon_i = \sigma_{s_{i-1}}^t \dots \sigma_{s_2}^t \sigma_{s_1}^t \alpha_{s_1}^*.$$

We call $S^* = \{\epsilon_1, \epsilon_2, \dots, \epsilon_{n+1}\}$ a *simple set* of V_S^* . We will show later that any n elements of S^* form a basis of V_S^* .

Theorem 3.1. *If V_S^* is a dual space of vector space V_S , $\{\alpha_s^* \mid s \in S\}$ is a standard basis of V_S^* , and $S^* = \{\epsilon_1, \epsilon_2, \dots, \epsilon_{n+1}\}$ is a subset of V_S^* satisfying*

the following relations:

$$\begin{aligned}\epsilon_1 &= \alpha_{s_1}^*, \\ \epsilon_i &= -\alpha_{s_{i-1}}^* + \alpha_{s_i}^*, \text{ for } 2 \leq i \leq n, \\ \epsilon_{n+1} &= -\alpha_{s_n}^*,\end{aligned}$$

then any n elements of S^* is also a basis of V_S^* .

Proof. Let the subset $S_1^* = \{\epsilon_1, \epsilon_2, \dots, \epsilon_n\}$ of S . First we prove that S_1^* is a linear independent set. Assume

$$\begin{aligned}0 &= \sum_{i=1}^n a_i \epsilon_i \\ &= a_1 \alpha_{s_1}^* + a_2 (-\alpha_{s_1}^* + \alpha_{s_2}^*) + \dots + a_n (-\alpha_{s_{n-1}}^* + \alpha_{s_n}^*) \\ &= (a_1 - a_2) \alpha_{s_1}^* + (a_2 - a_3) \alpha_{s_2}^* + \dots + a_n \alpha_{s_n}^*.\end{aligned}$$

Since $\{\alpha_s^* \mid s \in S\}$ is a basis of V_S^* , we have

$$\begin{cases} 0 = a_1 - a_2, \\ \vdots \\ 0 = a_{n-1} - a_n, \\ 0 = a_n. \end{cases}$$

Thus, $a_1 = a_2 = \dots = a_n = 0$.

By definition, ϵ_i can be written as a linear combinations of $\{\alpha_{s_i}^* \mid s_i \in S\}$ for $1 \leq i \leq n$, and S_1^* is a linear independent set. Thus, S_1^* is also a basis of V_S^* .

Next, by the definition of ϵ_i for $1 \leq i \leq n + 1$, we had known that $\epsilon_1 + \epsilon_2 + \dots + \epsilon_{n+1} = 0$. Hence any n elements of $S^* = \{\epsilon_1, \epsilon_2, \dots, \epsilon_{n+1}\}$ form a basis of V_S^* . \square

Theorem 3.2. *Let (W_S, S) denote the Coxeter system of type A_n . Then W_S is the symmetric group on S^* .*

Proof. By proposition 2.5, we had known that $\sigma^*(W_S)$ is isomorphic to W_S . Thus, we just prove that $\sigma^*(W_S)$ is the symmetric group on S^* . For each $\sigma^*(s_i) = \sigma_{s_i}^t \in GL(V_S^*), 1 \leq i \leq n$.

$$\sigma_{s_i}^t \epsilon_j = \begin{cases} \epsilon_{i+1}, & \text{if } j = i; \\ \epsilon_i, & \text{if } j = i + 1; \\ \epsilon_j, & \text{others.} \end{cases}$$

for $1 \leq j \leq n + 1$. Thus, $\sigma_{s_i}^t$ is a transposition $(\epsilon_i, \epsilon_{i+1})$ for $1 \leq i \leq n$. Since $\{\sigma_{s_i}^t \mid 1 \leq i \leq n\}$ is a generating set of $\sigma^*(W_S)$. Hence $\sigma^*(W_S)$ is the symmetric group on S^* . \square

Then W_S is isomorphic to the symmetric group S_{n+1} , since $|S^*| = n + 1$.

4 The Coxeter group associated with K_3

In this section, we consider the Coxeter group W with its associated graph K_3 of three vertices and three edges. That is $W = W_S$, where $S = \{s_1, s_2, s_3\}$ and $m(s_i, s_i) = 1, m(s_i, s_j) = m(s_j, s_i) = 3$ for distinct $i, j \in \{1, 2, 3\}$. Recall from Proposition 2.5 and (3), there exists a monomorphism $\sigma^* : W_S \rightarrow GL(V_S^*)$ with the matrices of $\sigma^*(s_1), \sigma^*(s_2), \sigma^*(s_3)$ as

$$\begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \quad (7)$$

with respect to the standard ordered basis $\{\alpha_{s_1}^*, \alpha_{s_2}^*, \alpha_{s_3}^*\}$ of V_S^* . Note that the above three matrices generate an infinite group over \mathbb{R} . Let F_2 be the field of two elements 0, 1. We define an action of W on F_2^3 by $w \cdot u = \sigma^*(w)u \pmod{2}$ for $w \in W$ and $u \in F_2^3$. Let $\{e_1, e_2, e_3\}$ be the standard ordered basis of F_2^3 . We shall determine the stabilizer W_{e_i} of e_i under the above action, and then determine $W_{e_1} \cap W_{e_2} \cap W_{e_3}$. Note that

$$s_i \cdot e_j := \begin{cases} e_1 + e_2 + e_3, & \text{if } i = j; \\ e_j, & \text{otherwise,} \end{cases} \quad (8)$$

for $1 \leq i, j \leq 3$. Hence with respect to the ordered basis $\{e_1, e_2, e_3\}$, the action of s_1, s_2, s_3 has the following matrix form

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad (9)$$

respectively. The action of a Coxeter group W on F_2^S is also called *lit-only σ -game on Γ* , where Γ is the Coxeter graph associated with (S, m) .

From (8), we have the following two Lemmas.

Lemma 4.1. *Let $S = \{s_1, s_2, s_3\}$. Then for distinct two elements $s_i, s_j \in S$,*

$$s_i s_j s_i(e_t) = \begin{cases} e_j, & \text{if } t = i; \\ e_i, & \text{if } t = j; \\ e_t, & \text{otherwise.} \end{cases}$$

In particular, $s_i s_j s_i = s_j s_i s_j$ in the Coxeter group W . Thus, $s_i s_j s_i$ and $s_j s_i s_j$ act on the subset $\{e_1, e_2, e_3\}$ of F_2^3 as the same transpositions (e_i, e_j) of the symmetric group on $\{e_1, e_2, e_3\}$ for distinct $i, j \in \{1, 2, 3\}$.

Lemma 4.2. Let $S = \{s_1, s_2, s_3\}$. Then for distinct three elements $s_i, s_j, s_t \in S$,

$$s_i s_t s_j s_i(e_k) = \begin{cases} e_j, & \text{if } k = i; \\ e_t, & \text{if } k = j; \\ e_i, & \text{if } k = t. \end{cases}$$

Hence, $s_i s_t s_j s_i$ and $s_i s_j s_t s_i$ are permutations (e_i, e_j, e_t) and (e_i, e_t, e_j) respectively for distinct three numbers $i, j, t \in \{1, 2, 3\}$. \square

Definition 4.3. We use the following notations.

$$\begin{aligned} H_{s_1} &= \{s_2, s_3, s_1 s_2 s_3 s_2 s_1\}, \\ H_{s_2} &= \{s_3, s_1, s_2 s_3 s_1 s_3 s_2\}, \\ H_{s_3} &= \{s_1, s_2, s_3 s_1 s_2 s_1 s_3\}. \end{aligned}$$

From the above, we may discover that H_{s_2} can be obtained from H_{s_1} by replacing 1 with 2, 2 with 3 and 3 with 1. In the same way, H_{s_3} can be obtained from H_{s_1} by replacing 1 with 3, 2 with 1 and 3 with 2.

Proposition 4.4. For $1 \leq i \leq 3$, $H_{s_i} \subseteq W_{e_i}$

Proof. Without loss of generality, we may assume $i = 1$, then by the action of W on F_2^3 . We may check that s_2, s_3 fix the vector e_1 . Next we check that $s_1 s_2 s_3 s_2 s_1 \in W_{e_1}$. By the equation (8), $s_2 s_1 \cdot e_1 = s_2(e_1 + e_2 + e_3) = e_2$. Then, we have that $s_1 s_2 s_3 \cdot e_2 = s_1 s_2 \cdot e_2 = e_1$. Thus $H_{s_1} \subseteq W_{e_1}$. \square

We shall prove that H_{s_i} is a generating set of W_{e_i} , for $1 \leq i \leq 3$. Before this, we introduce the *length function* and the *reduced form* of an element in W .

Definition 4.5. Let $S = \{s_1, s_2, s_3\}$ and W be the Coxeter group associated with (S, m) . For each $w \in W$, let r be the smallest integer such that

$$w = s'_1 s'_2 \cdots s'_r$$

for some $s'_i \in S$. r is called the *length* of w , denoted by $\ell(w)$, and call any expression of w as a product of r elements of S a *reduced form*. By convention, $\ell(1) = 0$, and $\ell(s) = 1$ for $s \neq 1$ and $s \in S$. Note that for any reduced form $s'_1 s'_2 \cdots s'_r$, $s'_i \neq s'_{i+1}$ for $i \in \{1, 2, \dots, r-1\}$.

Proposition 4.6. For $1 \leq i \leq 3$, H_{s_i} generates W_{e_i} .

Proof. We provide the case $i = 1$, and the remaining can be done by symmetry. By Proposition 4.4, $H_{s_1} \subseteq W_{e_1}$.

To prove $W_{e_1} \subseteq \langle H_{s_1} \rangle$, we pick $u \in W_{e_1}$. We show $u \in \langle H_{s_1} \rangle$. Proved by the length $\ell(u)$ of u . This is clear when $\ell(u) = 0$ since $u = 1$ in this case.

By induction, assume that $u \in \langle H_{s_1} \rangle$ if $\ell(u) \leq k-1$. Suppose $\ell(u) = k$ and $u = s_{i_1} s_{i_2} \cdots s_{i_k} \in W_{e_1}$ in a reduced form, for some $s_{i_j} \in S$. We divide the argument into two cases: $i_k = 2$ or 3 and the other case $i_k = 1$.

Case1 Suppose $s_{i_k} = s_3$ (or s_2). We choose $\alpha = s_3 \in H_{s_1}$ (resp. $s_2 \in H_{s_1}$).

Thus, $\ell(u\alpha) < \ell(u)$, and we have $u\alpha \in \langle H_{s_1} \rangle$. Hence $u\alpha\alpha^{-1} \in \langle H_{s_1} \rangle$ i.e. $u \in \langle H_{s_1} \rangle$.

Case2 Suppose $s_{i_k} = s_1$. Clearly $i_k \neq 1$ since $s_1 \notin W_{e_1}$. Now we discuss two cases $s_{i_{k-1}} = s_2$ and $s_{i_{k-1}} = s_3$.

1. Suppose $s_{i_{k-1}} = s_2$. Note that $s_2 s_1 \notin W_{e_1}$, since the first column of the matrix $\sigma^*(s_2 s_1)$ is not e_1 . Hence $k \geq 3$. Then $s_{i_{k-2}} = s_1$ or s_3 .

(a) Suppose $s_{i_{k-2}} = s_1$. We choose $\alpha = s_2 \in H_{s_1}$. Then

$$\begin{aligned} u\alpha &= (s_{i_1} \cdots s_{i_{k-3}} s_1 s_2 s_1) s_1 s_2 s_1 s_2 s_1 \\ &= s_{i_1} \cdots s_{i_{k-3}} s_2 s_1. \end{aligned}$$

Since $s_2 = s_1 s_2 s_1 s_2 s_1$ in the Coxeter group W . Thus $\ell(u\alpha) < \ell(u)$, $u\alpha \in \langle H_{s_1} \rangle$. Hence $u \in \langle H_{s_1} \rangle$, since $\alpha^{-1} \in \langle H_{s_1} \rangle$.

(b) Suppose $s_{i_{k-2}} = s_3$. We choose $\alpha = s_1 s_2 s_3 s_2 s_1 \in H_{s_1}$. Then

$$\begin{aligned} u\alpha &= (s_{i_1} \cdots s_{i_{k-3}} s_3 s_2 s_1) s_1 s_2 s_3 s_2 s_1 \\ &= s_{i_1} \cdots s_{i_{k-3}} s_2 s_1. \end{aligned}$$

Thus $\ell(u\alpha) < \ell(u)$, $u\alpha \in \langle H_{s_1} \rangle \Rightarrow u \in \langle H_{s_1} \rangle$.

2. Suppose $s_{i_{k-1}} = s_3$. This can be done similarly by replacing s_2 by s_3 in the above proof, and notice that $s_1 s_2 s_3 s_2 s_1 = s_1 s_3 s_2 s_3 s_1$ in the Coxeter group W_S .

□

Definition 4.7. For a subgroup $G \subseteq W$, let G^i be the set of elements of length i in G . In particular, we list the elements of $W_{e_j}^i$ for $0 \leq i \leq 4$, $1 \leq j \leq 3$ as following table.

Table:

i	$W_{e_1}^i$	$W_{e_2}^i$	$W_{e_3}^i$
0	$\{e\}$	$\{e\}$	$\{e\}$
1	$\{s_2, s_3\}$	$\{s_3, s_1\}$	$\{s_1, s_2\}$
2	$\{s_2 s_3, s_3 s_2\}$	$\{s_3 s_1, s_1 s_3\}$	$\{s_1 s_2, s_2 s_1\}$
3	$\{s_2 s_3 s_2\}$	$\{s_3 s_1 s_3\}$	$\{s_1 s_2 s_1\}$
4	\emptyset	\emptyset	\emptyset

From the above, $W_{e_2}^i$ can be obtained from $W_{e_1}^i$ by replacing 1 with 2, 2 with 3 and 3 with 1. In the same way, $W_{e_3}^i$ can be obtained from $W_{e_1}^i$ by replacing 1 with 3, 2 with 1 and 3 with 2. We may use $W_{e_1}^i \cap W_{e_2}^i \cap W_{e_3}^i$ to find the set of element of length i in W to fix $\{e_1, e_2, e_3\}$. In addition, we may prove that $W_{e_j}^4 = \emptyset$, for $j = 1, 2, 3$. Assume that $e_j = e_1$ and $W_{e_1}^4 \neq \emptyset$. Then there is a $s_i s_j s_k s_t$ satisfying $(s_i s_j s_k s_t)e_1 = e_1$ and any adjacent s_i, s_i are distinct. Then $(s_j s_k s_t)e_1 = e_1 + e_2 + e_3$ and $s_i = s_1$. That is, $(s_k s_t)e_1 = e_k, e_k \neq e_1$ for $s_t = s_1$; otherwise $(s_j s_k s_t)e_1 \neq e_1 + e_2 + e_3$. Thus $(s_j s_k s_t)e_1 = e_1 + e_2 + e_3$, for $s_j = s_k$, contradiction.

Definition 4.8.

$$H = \{(s_i s_j s_i s_t)^2 \mid \text{for distinct } i, j, t \in \{1, 2, 3\}\}.$$

In particular, $|H| = 3$ since $s_i s_j s_i = s_j s_i s_j$ in the Coxeter group W for distinct $i, j \in \{1, 2, 3\}$.

Theorem 4.9. $G = W_{e_1} \cap W_{e_2} \cap W_{e_3}$ is the normal subgroup of W generated by the set H .

Proof. First we prove that G is generated by the set H . By the group action of W on F_2^3 , then we can easy to check that H is contained in $W_{e_1} \cap W_{e_2} \cap W_{e_3}$. Thus, $H \subseteq G$.

To prove $G \subseteq \langle H \rangle$, we pick $w \in G$. We show $w \in \langle H \rangle$. Proved by the length $\ell(w)$ of w . This is clear when $\ell(w) = 0$ since $w = 1$ in this case. For any element w of length, $\ell(w) \in \{1, 2, 3, 4\}$ one can check that $w \notin G$ by above Table. So we have known the assertion holds for $\ell(w) \leq 4$.

By induction, assume that $w \in \langle H \rangle$ if $\ell(w) \leq k-1$. Suppose $\ell(w) = k \geq 5$ and $w = s_{i_1} s_{i_2} \cdots s_{i_k} \in G$ in a reduced form, for some $s_{i_j} \in S$. For any $i \in \{1, 2, 3\}$, we let $s_{i_k} = s_i$ then $s_{i_{k-1}} = s_j$ or s_t , where $j \neq t \in \{1, 2, 3\} - \{i\}$. By symmetry, we may assume $s_{i_{k-1}} = s_j$. Then we divide $s_{i_{k-4}} \cdots s_{i_{k-1}} s_{i_k}$ into the following 8 cases: (1) $s_i s_j s_i s_j s_i$, (2) $s_t s_j s_i s_j s_i$, (3) $s_i s_t s_i s_j s_i$, (4) $s_j s_t s_i s_j s_i$, (5) $s_j s_i s_t s_j s_i$, (6) $s_t s_i s_t s_j s_i$, (7) $s_i s_j s_t s_j s_i$, (8) $s_t s_j s_t s_j s_i$, for distinct $i, j, t \in \{1, 2, 3\}$.

(1) Suppose $s_{i_{k-4}} \cdots s_{i_{k-1}} s_{i_k} = s_i s_j s_i s_j s_i$. We choose $\alpha = e \in \langle H \rangle$. Then

$$\begin{aligned} w\alpha &= we \\ &= s_{i_1} \cdots s_{i_{k-5}} s_i s_j s_i s_j s_i (s_i s_j)^3 \\ &= s_{i_1} \cdots s_{i_{k-5}} s_j s_i s_j. \end{aligned}$$

Since $(s_i s_j)^3 = e$ in the Coxeter group W . Then, $\ell(w) = \ell(s_{i_1} \cdots s_{i_{k-5}} s_j s_i s_j) < \ell(w)$, contradiction. Thus, $s_{i_{k-4}} \cdots s_{i_k} \neq s_i s_j s_i s_j s_i$.

For the case (2), we may use the same way to prove that $s_{i_{k-4}} \cdots s_{i_k} \neq s_t s_j s_i s_j s_i$.

(3) Suppose $s_{i_{k-4}} \cdots s_{i_{k-1}} s_{i_k} = s_i s_t s_i s_j s_i$. We replace $\alpha = (s_i s_j s_i s_t)^2 \in \langle H \rangle$. Then

$$\begin{aligned} w\alpha &= s_{i_1} \cdots s_{i_{k-5}} s_i s_t s_i s_j s_i (s_i s_j s_i s_t)^2 \\ &= s_{i_1} \cdots s_{i_{k-5}} s_j s_i s_t. \end{aligned}$$

Thus, $\ell(w\alpha) = \ell(s_{i_1} \cdots s_{i_{k-5}} s_j s_i s_t) < \ell(w)$. Then $w\alpha \in \langle H \rangle$. Hence $w \in \langle H \rangle$, since $\alpha^{-1} \in \langle H \rangle$.

(4) Suppose $s_{i_{k-4}} \cdots s_{i_{k-1}} s_{i_k} = s_j s_t s_i s_j s_i$. We replace $\alpha = (s_j s_i s_j s_t)^2 \in \langle H \rangle$.

$\langle H \rangle$. Then

$$\begin{aligned}
w\alpha &= s_{i_1} \cdots s_{i_{k-5}} s_j s_t s_i s_j s_i (s_j s_i s_j s_t)^2 \\
&= s_{i_1} \cdots s_{i_{k-5}} s_j s_t s_i s_j s_i (s_i s_j)^3 (s_j s_i s_j s_t)^2 \\
&= s_{i_1} \cdots s_{i_{k-5}} s_i s_j s_t.
\end{aligned}$$

Since $(s_j s_i s_j s_t)^2 = (s_i s_j)^3 (s_j s_i s_j s_t)^2$ in the Coxeter group W . Thus, $\ell(w\alpha) = \ell(s_{i_1} \cdots s_{i_{k-5}} s_i s_j s_t) < \ell(w)$. Hence $w \in \langle H \rangle$.

(5) Suppose $s_{i_{k-4}} \cdots s_{i_{k-1}} s_{i_k} = s_j s_i s_t s_j s_i$. We replace $\alpha = (s_i s_j s_t s_j)^2 \in \langle H \rangle$. Then

$$\begin{aligned}
w\alpha &= s_{i_1} \cdots s_{i_{k-5}} s_j s_i s_t s_j s_i (s_i s_j s_t s_j)^2 \\
&= s_{i_1} \cdots s_{i_{k-5}} s_j s_i s_t s_j s_i (s_i s_j s_t s_j s_i s_j s_t s_j) \\
&= s_{i_1} \cdots s_{i_{k-5}} s_j s_i s_t s_j s_i (s_i s_j s_t s_i s_j s_i s_t s_j) \\
&= s_{i_1} \cdots s_{i_{k-5}} s_i s_t s_j.
\end{aligned}$$

Since $(s_i s_j s_t s_j)^2 = ((s_j s_t s_j s_i)^2)^{-1} \in \langle H \rangle$ and $s_i s_j s_i = s_j s_i s_j$ in the Coxeter group W . Thus, $\ell(w\alpha) = \ell(s_{i_1} \cdots s_{i_{k-5}} s_i s_t s_j) < \ell(w)$. Hence $w \in \langle H \rangle$.

(6) Suppose $s_{i_{k-4}} \cdots s_{i_{k-1}} s_{i_k} = s_t s_i s_t s_j s_i$. We replace $\alpha = (s_i s_j s_i s_t)^2 \in \langle H \rangle$. Then

$$\begin{aligned}
w\alpha &= s_{i_1} \cdots s_{i_{k-5}} s_t s_i s_t s_j s_i (s_i s_j s_i s_t)^2 \\
&= s_{i_1} \cdots s_{i_{k-5}} s_t s_i s_t s_j s_i (s_i s_j s_i s_t s_i s_j s_i s_t) \\
&= s_{i_1} \cdots s_{i_{k-5}} s_t s_i s_t s_j s_i (s_i s_j s_t s_i s_t s_j s_i s_t) \\
&= s_{i_1} \cdots s_{i_{k-5}} s_j s_i s_t.
\end{aligned}$$

Since $s_i s_j s_i = s_j s_i s_j$ in the Coxeter group W . Thus, $\ell(w\alpha) = \ell(s_{i_1} \cdots s_{i_{k-5}} s_j s_i s_t) < \ell(w)$. Hence $w \in \langle H \rangle$.

(7) Suppose $s_{i_{k-4}} \cdots s_{i_{k-1}} s_{i_k} = s_i s_j s_t s_j s_i$. We replace $\alpha = (s_i s_j s_t s_j)^2 \in \langle H \rangle$. Then

$$\begin{aligned} w\alpha &= s_{i_1} \cdots s_{i_{k-5}} s_i s_j s_t s_j s_i (s_i s_j s_t s_j)^2 \\ &= s_{i_1} \cdots s_{i_{k-5}} s_j s_t s_j. \end{aligned}$$

Since $(s_i s_j s_t s_j)^2 = ((s_j s_t s_j s_i)^2)^{-1} \in \langle H \rangle$ in the Coxeter group W . Thus, $\ell(w\alpha) = \ell(s_{i_1} \cdots s_{i_{k-5}} s_j s_t s_j) < \ell(w)$. Hence $w \in \langle H \rangle$.

(8) Suppose $s_{i_{k-4}} \cdots s_{i_{k-1}} s_{i_k} = s_t s_j s_t s_j s_i$. We replace $\alpha = e \in \langle H \rangle$. Then

$$\begin{aligned} w\alpha &= we \\ &= s_{i_1} \cdots s_{i_{k-5}} s_t s_j s_t s_j s_i (s_i (s_j s_t)^3 s_i) \\ &= s_{i_1} \cdots s_{i_{k-5}} s_j s_t s_i. \end{aligned}$$

Since $e = s_i (s_j s_t)^3 s_i$ in the Coxeter group W . Then, $\ell(w) = \ell(s_{i_1} \cdots s_{i_{k-5}} s_j s_t s_i) < \ell(w)$, contradiction. Thus, $s_{i_{k-4}} \cdots s_{i_k} \neq s_t s_j s_t s_j s_i$.

Next we need to prove that G is normal in W . Recall $\sigma^* : W \rightarrow GL(V_S^*)$ is a monomorphism and W acts on F_2^3 by $w \cdot u = \sigma^*(w)u \pmod{2}$ for $w \in W$ and $u \in F_2^3$. Let $x \in G, w \in W$. Then for any $i \in \{1, 2, 3\}$

$$\begin{aligned} w^{-1} x w \cdot e_i &= \sigma^*(w^{-1} x w) e_i \pmod{2} \\ &= \sigma^*(w^{-1}) \sigma^*(x) \sigma^*(w) e_i \pmod{2} \\ &= \sigma^*(w^{-1}) \sigma^*(w) e_i \pmod{2} \\ &= e_i. \end{aligned}$$

Since $x \in G = W_{e_1} \cap W_{e_2} \cap W_{e_3} \Rightarrow \sigma^*(x)$ fixes any vectors in F_2^3 . Thus, $w^{-1} x w \in G$. \square

Remark 4.10. By Theorem 4.9, we may discover some relations in the group W/G as follows. For distinct $i, j, t \in \{1, 2, 3\}$.

(i) $s_i s_j s_i s_t = s_j s_i s_j s_t = s_t s_i s_j s_i$.

(ii) $s_i s_j s_t s_i = s_j s_t s_i s_j = s_t s_i s_j s_t$.

Then we have the following Cayley graph $\text{Cay}(W/G, \{s_1, s_2, s_3\})$ of group W/G with respect to S has vertex set W/G and edge set $\{g(gs) \mid g \in G, s \in S\}$.

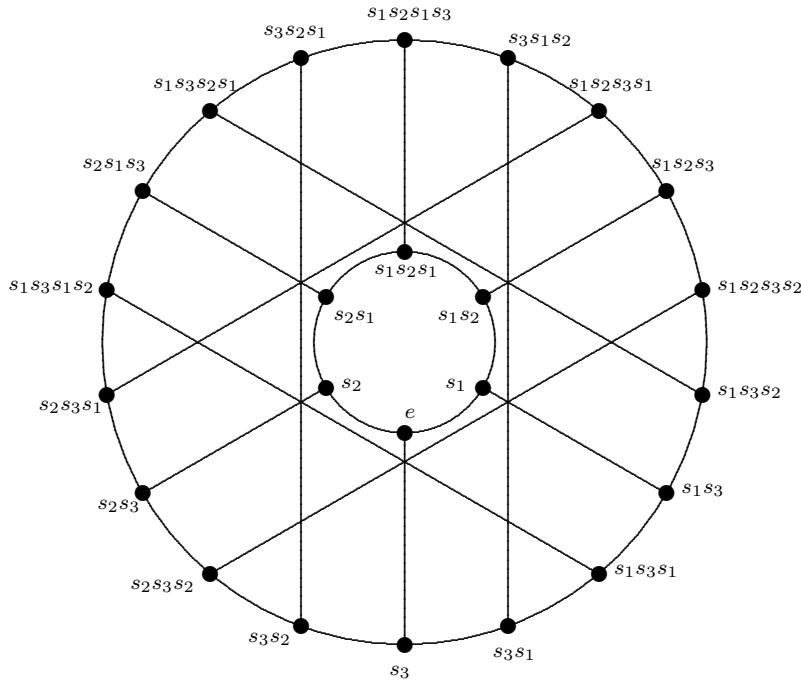


Figure 3. The Cayley graph $\text{Cay}(W/G, \{s_1, s_2, s_3\})$.

References

- [1] David S. Dummit and Richard M. Foote, *Abstract algebra*, Prentice-Hall Inc, (1991)
- [2] Henrik Eriksson, Kimmo Eriksson, Jonas Sjöstrand, Note on the Lamp Lighting Problem, *Advances in Applied Mathematics* 27(2001), (Pages 357-366)
- [3] J. Goldwasser, X. Wang, Y. Wu, Does the lit-only restriction make any difference for the sigma-game and sigma-plus game?, *European Journal of Combinatorics*,(2008), doi:10.1016/j.ejc.2008.09.020.
- [4] Hau-wen Huang, Chih-wen Weng, Combinatorial representations of Coxeter groups over a field of two elements, preprint
- [5] Hau-wen Huang and Chih-wen Weng, The edge-flipping group of a graph, *European Journal of Combinatorics*, Volume 31, Issue 3, (April 2010), (Pages 932-942)
- [6] James E. Humphreys, *Reflection Groups and Coxeter Groups*, Cambridge University Press, (1990)
- [7] Xinmao Wang and Yaokun Wu, Minimum light number of lit-only σ -game on a tree, *Theoretical Computer Science* 381(2007), (Pages 292–300).
- [8] Hsin-Jung Wu and Gerard J. Chang, *A study on equivalence classes of painted graphs*, Master Thesis, NTU, Taiwan, (2006).