國 立 交 通 大 學

應用數學系碩士班

碩士論文

$t \times (t+1)$ $\bar{d}$-可分離矩陣的最小t值: d=2 或 3 的情況

The minimum value of t for $t \times (t+1)$ $\bar{d}$-separable matrix: d=2 or 3

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中文摘要

群試設計 (group testing) 為應用數學的一個分支, 其應用層面包含了錯誤更正碼、基因 (DNA) 測試等。本論文著重在探討$t \times (t+1)$ $\bar{d}$-可分離群式設計的可能性。首先我們考慮投影平面的點線關係矩陣, 並證明刪除任一列可以產生$t \times (t+1)$ $\bar{d}$-可分離群矩陣, 當t等於d^2+d且d為質數的次方。接著我們證明當t小於d^2+d且d為2或3時並不存在$\bar{d}$-可分離群矩陣。

The minimum value of t for $t \times (t+1)$ $\bar{d}$-separable matrix: $d = 2$ or 3

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Abstract

Group testing is a branch of applied mathematics and has several applications, such as error correcting codes, DNA testing, etc. This thesis investigates the existence of a $t \times (t+1)$ $\bar{d}$-separable matrix for some t and d.

First, we consider the point-block incidence matrix of the projective plane of order d and show that removing any row from the matrix yields a $t \times (t+1)$ $\bar{d}$-separable matrix as $t = d^2 + d$ and d is a prime power. Then, we show that if $t < d^2 + d$ and $d = 2$ or 3, there is no $t \times (t+1)$ $\bar{d}$-separable matrix.

誌 謝

首先感謝我的指導教授 翁志文教授，謝謝教授始終耐心地指導我，沒有您的幫忙，這篇論文是無法完成的；也感謝系上諸位師長對我的包容，尤其是陳秋媛教授，不僅傳授專業知識，在生活方面更不吝給予關懷。當然，還有謝謝我家人的督促，使我順利完成學業。

在準備論文的這段期間，進展斷斷續續的，有些事不總是盡如人意，還讓身邊的親友們著急。我的確讓人等待太久，感謝大家在最後階段，適時以各種不同的方式，或忍讓或提醒或沈靜陪伴，讓我能夠專注把整件事圓滿完成。

確切來說，要道謝或道歉的對象太多，冷暖自知。無論是實質的援助，或是抽象的支持，我都深刻的感受到你們付出的溫暖，那麼，就讓我把其餘重要角色的名字都放在心底，盡在不言中。

謝謝這世界上美好的一切，還有我喜歡的人們。

Contents

Chapter 1

Introduction

Group testing is a branch of mathematics investing a problem about the determining a defective set among several items by testing several subsets of these items with some constraints. For instance, given n distinct items and the knowledge that there are at most d defective items among n ones, we want to find the all detective items via several testings, which is a prototype problem (called $(\overline{d}, n)$ problem) in combinatorial group testing. At the beginning, a fundamental problem is that the existence of a solution for the $(\overline{d}, n)$ problem and an trivial solution for the existence is to test a item each time and, after n testings, we can identify all defective items. However, when some constraints are imposed, the existence of a solution may no long be trivial. We may, for example, want to find the minimum number of testings in $(\overline{d}, n)$ problem such that all defective items can be identified.

As can be seen, there are two ways to do the testing and the group testing is usually categorized as adaptive or nonadaptive group testing. In adaptive group testing, a testing is done in order and a testing can be designed based on the previous results while all testing are done at once in nonadaptive group testing. Both of them are investigated in a lot of literatures and we focus on the later in this thesis. One application of nonadaptive group testing is the testing of DNA sequence. In this field, a nonadaptive group testing is preferred because it takes a lot of time to do a testing and we can do several testing at the same time in nonadaptive group testing. Hence, it takes less time to find defective items compared with that using adaptive group testing algorithms. For other applications, the readers can refer to [1] and reference therein.

In this thesis, we want to investigate the following conjecture:

Conjecture 1.1. There is no $\overline{d}$-separable matrix of size $t \times n$ with $t < n < d^2 + d + 1$.

The definition of $\overline{d}$-separable matrix in the following section. This conjecture is proposed in [2] and the case $d = 2$ is proved by considering all possibilities of a matrix. The main results of this thesis is that this conjecture is true if $d = 2$ or $d = 3$. Do notice that the case $d = 2$ has been proved before; however, it is quite long. In this thesis, we reprove this case in a shorter and elegant way.

Chapter 2

Preliminary results of group testing

In this chapter, we introduce the notations, terminology, and some properties about group testing we will used in this thesis. Most of the notations come from [1] and the reader who is familiar with them can skip this chapter.

2.1 Definitions and notations

Definition 2.1.1. *Boolean sum*

The boolean sum of two 0-1 column vectors C_1 and C_2 is the element-wise or operation of C_1 and C_2.

Example 2.1.2. Suppose $x = [0, 0, 1, 1]^T$, and $y = [0, 1, 0, 1]^T$, where T denotes the transpose operation. Then, the boolean sum of x and y is $[0, 1, 1, 1]^T$

In this thesis, a matrices we considered is 0-1 matrix unless stated otherwise. A $t \times n$ matrix is usually denoted by $\mathbf{M}$, where R_i and C_j denote the ith row and jth column of M, respectively. For convenience, $U(S)$ denotes the boolean sum of S, where S is the set of columns in a matrix M.

Example 2.1.3. Suppose

$$\mathbf{M} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Then, $C_1 = [0, 0, 1, 1]^T$, $C_2 = [1, 0, 1, 1]^T$, $C_3 = [0, 1, 0, 1]^T$ and $C_4 = [1, 0, 0, 1]^T$.

Suppose $S = \{C_1, C_3, C_4\}$. We have $U(S) = [1, 1, 1, 1]^T$.

Definition 2.1.4. (*d-separable and $\bar{d}$-separable*)

1. M is d-separable if the boolean sum of d columns are all distinct.

2. M is $\bar{d}$-separable if the boolean sum of i columns are all distinct, $1 \leq i \leq d$.

Example 2.1.5. Suppose

$$M = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

It is clear that M is 1-separable. However, it is not 2-separable because $U(\{C_1, C_2\}) = U(\{C_2, C_3\})$.

From the definition of d-separable and $\bar{d}$-separable, it is clear that if a matrix is $\bar{d}$-separable, it is also d-separable.

To investigate a property of a $\bar{d}$-separable matrix, it is found that the notion of d-disjunct is useful and the formal definition of d-disjunct is given below.

Definition 2.1.6. (*d-disjunct*)
A $t \times n$ matrix $\mathbf{M}$ is d-disjunct if the union of any d columns does not contain any other column.

Similar to the study of extremal set in combinatorics, it is interesting to study the existence of a $t \times n$ d-disjunct matrix $\mathbf{M}$ with some conditions. For instance, given d and n, what is the minimum value of t such that a $t \times n$ d-disjunct matrix $\mathbf{M}$ exists. For such problem, the extremal value is denoted by $t(d, n)$, which is summary in the following definition.

Definition 2.1.7. ($t(d, n)$)
$t(d, n)$ is the minimum number of rows required for a d-disjunct matrix with n columns.

2.2 Properties of d-separable, $\bar{d}$-separable, and d-disjunct

In this section, we show some properties of d-separable, $\bar{d}$-separable, and d-disjunct. Moreover, the relationship among d-separable, $\bar{d}$-separable, and d-disjunct will be summary in the end of this section.

Property 2.2.1. *If* $\mathbf{M}$ *is* $\bar{d}$*-separable, then* $\mathbf{M}$ *is* d*-separable. Moreover, if* $\mathbf{M}$ *is* d*-separable, then* M *is* k*-separable for* $1 \leq k \leq d$*.*

Proof. By the definition, it is clear. □

Property 2.2.2. *If all* $t \times n$ *matrix* $\mathbf{M}$ *are not* $\bar{d}-$*separable, then any* $k \times n$ $\mathbf{M}'$ *matrix is not* d*-separable for* $1 \leq k \leq t$*.*

Proof. Suppose these exists a $k \times n$ d-separable matrix $\mathbf{M}'$. Since $k = t$ is a trivial case, we assume $k < t$. Without loss of generality, we assume k is maximal such that all $(k+1) \times t$ matrix are not $\bar{d}-$separable. Then, add a all zero row at the bottom of $\mathbf{M}'$ and we get a $(k+1) \times n$ matrix which is $\bar{d}-$separable, a contradiction to that k is maximal. □

Property 2.2.3. *If a* $t \times (n+1)$ *matrix is* $\overline{d}$*-separable, then there exists a* $t \times n$ $\overline{d}$*-separable matrix.*

Proof. Suppose there exists a $t \times (n+1)$ matrix is $\overline{d}-$separable. Then, removing any one column yields a $\overline{d}-$separable $t \times n$ matrix. □

Lemma 2.2.4. $\overline{(d+1)}$*-separable implies* d*-disjunct.*

Proof. Suppose there is a $\overline{(d+1)}$-separable matrix $\mathbf{M}$ but not d-disjunct. Then there exists a column set S such that S contains C_i for some column $C_i \notin S$ and $|S| = d$. Notice that $U(S) = U(S \bigcup \{C_i\})$ and it implies that $\mathbf{M}$ is not $\overline{(d+1)}$-separable, a contradiction. □

Lemma 2.2.4 implies that if there exists no $t \times n$ d-disjunct matrix, there does not exist any $t \times n$ $\overline{(d+1)}$-separable matrix.

Lemma 2.2.5. *If a matrix* $\mathbf{M}$ *is* d*-disjunct, then* $\mathbf{M}$ *is* $\overline{d}$*-separable.*

Proof. Suppose $\mathbf{M}$ is not $\overline{d}$-separable. Then, there exists two distinct columns sets S_1 and S_2 such that $U(S_1) = U(S_2)$ and $1 \leq |S_1| \leq |S_2| \leq d$. Since $S_1 \neq S_2$, there is a column $C \in S_2 \setminus S_1$. Then, C is contained by $U(S_1)$, which implies that $\mathbf{M}$ is not $|S_1|$-disjunct. Because $|S_1| \leq d$, this also implies that $\mathbf{M}$ is not d disjunct, a contradiction. $\square$

Based on the definition 2.1.4 and lemma 2.2.5, the relationship among d-separable, $\bar{d}$-separable, and d-disjunct can be represented as below.

$$d - \text{disjunct} \quad \Rightarrow \quad \bar{d} - \text{separable} \quad \Rightarrow \quad d - \text{separable}$$

Chapter 3

Deterministic designs from symmetric BIBD

3.1 Construction method

Before we discuss the deterministic designs, we first give some basic definitions.

Definition 3.1.1. A design is a pair(X, A) such that the following properties are satisfied:

1. X is a set of elements called points.

2. A is a collection of nonempty subsets of X called blocks.

Definition 3.1.2. Let v, k, and λ be positive integers such that $v > k \geq 2$. A (v, k,λ)-balanced incomplete block design ((v, k,λ)-BIBD) is a design (X,A) such that the following properties are satisfied:

1. $|X| = v$ and $|A| = b$.

2. Each block contains exactly k points.

3. Each pair of distinct points is contained in exactly λ blocks.

A BIBD is called a symmetric BIBD if $b = v$.

Property 3.1.3. *For a symmetric (v,k,λ)-BIBD, each pair of distinct block has λ points in common.*

Proof. See the proof of Theorem 2.2 in [3]. □

The representation of a design through a 0-1 matrix is given in the following definition.

Definition 3.1.4. Let (X, A) be a design where $X = \{x_1, \cdots, x_v\}$ and $A = \{A_1, \cdots, A_b\}$. The incidence matrix of (X, A) is the $v \times b$ 0-1 matrix $\mathbf{M}$ defined by the rule

$$m_{i,j} = \begin{cases} 1 & if\ x_i \in A_j \\ 0 & if\ x_i \notin A_j. \end{cases}$$

where $m_{i,j}$ denotes the element in the ith row and the jth column of $\mathbf{M}$.

Now, we are ready to show the deterministic designs from symmetric BIBD. The construction method is first to find a (v, k, λ) symmetric BIBD and its corresponding incidence matrix $\mathbf{M}$. Then, remove one of row in $\mathbf{M}$ and we get a $(v-1) \times v$ $\overline{\left(\lceil \frac{k}{\lambda} \rceil - 1\right)}$-separable matrix. The rationality of this construction is based on the theorem 3.1.8.

Lemma 3.1.5. *A symmetric (v, k, λ)-BIBD is $\left(\lceil \frac{k}{\lambda} \rceil - 1\right)$-disjunct.*

Proof. According to the property of symmetric BIBD, any two columns intersect in exactly λ points and every column contains exactly k 1s. Suppose the statement is not true. Then, there exists a set S of $\lceil \frac{k}{\lambda} \rceil - 1$ columns and another column C_j with weight k such that $C_j \subseteq U(S)$. By Pigeonhold principle, there exists a column C_k in S such that the interesection of C_j and C_k has $\lambda + 1$ 1s, which is contradiction to the definition of λ. □

Lemma 3.1.6. *Removing any row from a d-disjunct matrix* $\mathbf{M}$ *yields a d-separable matrix.*

Proof. Without loss of generality, we assume that the first row is removed, yielding a submatrix $\mathbf{M}'$. Suppose that $\mathbf{M}'$ is not d-separable matrix, which means that there are two distinct columns sets S_1 and S_2 such that $|S_1| = |S_2| = d$ and $U(S_1) = U(S_2)$. Now, consider $U(S_1)$ and $U(S_2)$ in $\mathbf{M}$. Since $\mathbf{M}$ is d-disjunct, $U(S_1) \neq U(S_2)$ in $\mathbf{M}$.

In other words, they are difference in the first row in $\mathbf{M}$. Without loss of generality, we assume that the first row of $U(S_1)$ is 1 and that of $U(S_2)$ is 0 in $\mathbf{M}$. Then, we can find that $U(S_1)$ contains $U(S_2)$. Because $d \geq 1$, the above statement implies that $U(S_1)$ contains $U(S_2)$ and there is a column in $S_2 \setminus S_1$ (they are distinct and have the same size) contained in $U(S_1)$, a contradiction to that $\mathbf{M}$ is d-disjunct. □

Theorem 3.1.7. *Let* $\mathbf{M}$ *be a* d*-separable matrix and* $1 \leq k \leq d-1$. *Then* $\mathbf{M}$ *is* $\overline{(k+1)}$*-separable if and only if* $\mathbf{M}$ *is* k*-disjunct.*

Proof.

($\Rightarrow$) Suppose that $\mathbf{M}$ is not k-disjunct. Then, there exist a column C and a set S_1 of k other columns such that C is contained in $U(S_1)$. Let $S_2 = S_1 \bigcup \{C\}$. Then, $|S_2| = k+1$ and $U(S_1) = U(S_2)$. Hence, $\mathbf{M}$ is not $\overline{(k+1)}$-separable.

($\Leftarrow$) Suppose that $\mathbf{M}$ is not $\overline{(k+1)}$-separable. Then, there exist two distinct set S_1 and S_2 such that $U(S_1) = U(S_2)$ and $1 \leq |S_1| \leq |S_2| \leq k+1$. Because $\mathbf{M}$ be a d-separable matrix, the latter condition can be relaxed to $1 \leq |S_1| < |S_2| \leq k+1$. This implies that there exists a $C \in S_2 \setminus S_1$ and C is contained in $U(S_2) = U(S_1)$, which means that $\mathbf{M}$ is not k-disjunct as $|S_1| \leq k$. □

Theorem 3.1.8. *The matrix* $\mathbf{M}'$ *obtained by removing a row of the incidence matrix of a symmetric* (v,k,λ)*-BIBD is* $\overline{\left(\lceil \frac{k}{\lambda} \rceil - 1\right)}$*-separable.*

Proof. By lemma 3.1.6, $\mathbf{M}'$ is $\left(\lceil \frac{k}{\lambda} \rceil - 1\right)$-separable. If $\mathbf{M}'$ is $\overline{\left(\lceil \frac{k}{\lambda} \rceil - 2\right)}$-disjunct, then we are done by theorem 3.1.7. Suppose the statement is not true. Then, there exists a set S of $\left(\lceil \frac{k}{\lambda} \rceil - 2\right)$ columns in $\mathbf{M}$ and a column C_j of $\mathbf{M}'$ contained in S. Do notice that C_j contains k or $k-1$ 1s and any two columns of $\mathbf{M}'$ intersect in λ or $\lambda - 1$ positions (that is, the intersection of any two columns of $\mathbf{M}'$ has λ or $\lambda - 1$ 1s).

Case A: $k = a\lambda$ for some positive integer a,

If $k = a\lambda$ for some positive integer a, we have $\lceil \frac{k}{\lambda} \rceil \lambda = k$. Then, $\left(\lceil \frac{k}{\lambda} \rceil - 2\right)\lambda = k - 2\lambda$. Because $\lambda \geq 1$, by Pigenhold principle, there exists one of columns in S intersecting with

C_j in $\lambda + 1$ positions, which is a contradiction.

Case B: $k \neq a\lambda$ for some positive integer a

If $k \neq a\lambda$ for some positive integer a, we have $\left(\left\lceil \frac{k}{\lambda} \right\rceil - 2\right)\lambda = k + 1 - 2\lambda$. Do notice that $\lambda \geq 2$ because $\lambda = 1$ is considered in case A. Hence, $\left(\left\lceil \frac{k}{\lambda} \right\rceil - 2\right)\lambda = k+1-2\lambda \leq k-3$. Again, by Pigenhold principle, we have a contradiction. $\square$

3.2 Examples of the construction method

From theorem 3.1.8, we should find a (v, k, λ) symmetric BIBD and we get a $(v-1) \times v$ $\overline{\left(\left\lceil \frac{k}{\lambda} \right\rceil - 1\right)}$-separable matrix. Unfortunately, given arbitrary v, k, and λ, there is not necessary that a symmetric BIBD exits. However, some families of symmetric BIBD have been proposed and it is proved that there exists a symmetric BIBD under some conditions. In this thesis, we consider a well-known family of symmetric BIBD: finite projective plane.

Definition 3.2.1. A $(n^2 + n + 1, n + 1, 1)$-BIBD with $n \geq 2$ is called a finite projective plane of order n.

It can be proved that a projective plane of order n exists if n is a prime power [3]. First, we give a 6×7 $\overline{2}$-separable matrix through a finite projective plane of order 2.

Example 3.2.2. A finite projective plane of order 2.

A finite projective plane of order 2 is a $(7, 3, 1)$-BIBD. The point set is $X = \{1, 2, 3, 4, 5, 6, 7\}$ and the block set is $A = \{\{1, 2, 3\},\ \{1, 4, 5\},\ \{1, 6, 7\},\ \{2, 4, 6\},\ \{2, 5, 7\},\ \{3, 4, 7\},\ \{3, 5, 6\}\}$. The incidence matrix is

$$\mathbf{M} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}$$

Removing the last row of $\mathbf{M}$, we have

$$\mathbf{M}' = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}$$

Based on the theorem 3.1.8, $\mathbf{M}'$ is $\overline{2}$-separable.

The following example give a 12×13 $\overline{3}$-separable matrix.

Example 3.2.3. A finite projective plane of order 3.

A finite projective plane of order 3 is a $(13, 4, 1)$-BIBD. The point set is $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\}$ and the block set is
$A = \{\{1, 2, 4, 10\}, \{2, 3, 5, 11\}, \{3, 4, 6, 12\}, \{4, 5, 7, 13\}, \{5, 6, 8, 1\}, \{6, 7, 9, 2\}, \{7, 8, 10, 3\}, \{8, 9, 11, 4\}, \{9, 10, 12, 5\}, \{10, 11, 13, 6\}, \{11, 12, 1, 7\}, \{12, 13, 2, 8\}, \{13, 1, 3, 9\}\}$. The incidence matrix is

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

Removing the last row of $\mathbf{M}$, we have

$$\mathbf{M}' = \begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0
\end{pmatrix}$$

Based on the theorem 3.1.8, $\mathbf{M}'$ is $\overline{3}$-separable.

To sum up, theorem 3.1.8 and the existence of finite projective plane of order n imply that we can find a $(n^2+n)\times(n^2+n+1)$ $\overline{n}$-separable matrix if n is prime power. Hence, the minimium number of t for the existence of a $t\times(t+1)$ $\overline{d}$-separable matrix is smaller than d^2+d+1 when d is a prime power.

Chapter 4

Main result

In previous chapter, we show that the minimum number of t for the existence of a $t \times (t+1)$ $\overline{d}$-separable matrix is smaller than or equal to $d^2 + d$ when d is a prime power and it has been shown in [2] that the minimum number is $d^2 + d$ when $d = 2$. However, the proof in [2] is quite long and is done case by case. In this section, we reprove it by the properties of $\overline{2}$-separable matrix, which yields a more shorter and simple proof. Moreover, we show that $d = 3$ is also hold.

4.1 d=2

We start the proof with a property of 1-disjunct matrix. The following lemma is known as Sperner's theorem [4].

Lemma 4.1.1. *Given t, if there exists a $t \times n$ 1-disjunct matrix $\mathbf{M}$, then $n \leq \binom{t}{\lfloor \frac{t}{2} \rfloor}$.*

Proof. Let T be the set of row indices and A be a family of subsets of T such that no set in A is a subset of another set in A. Hence, A forms the 1-disjunct matrix. Denote the number of sets of size k in A by a_k. We want to show that

$$n = |A| = \sum_{k=0}^{t} a_k \leq \binom{t}{\lfloor \frac{t}{2} \rfloor}$$

by double counting the number of permutations of T. Clearly, there are $t!$ ways to permute T. On the other hand, given an element of A, say A_1, we can permute the elements in A_1 and then $A \setminus A_1$. There are $k!(t-k)!$ way to do so if the size of A_1 is

k. Do notice that we keep the permutation order of A_1 and then $A \setminus A_1$. By doing so, each such permutation will be correspond to an element in A. If an permutation could be correspond to at least two elements in A, then one will be a subset of the other, contradicting to the definition of A. Therefore, the number of permutations through this procedure is

$$\sum_{A_1 \in A} = |A_1|!(n - |A_!|)! = \sum_{k=0}^{t} a_k k!(t-k)!$$

This number is upper bounded by the number of all permutations, $t!$ and we have the following inequality:

$$\sum_{k=0}^{t} a_k k!(t-k)! \leq t!,$$

$$\text{or equivalently} \sum_{k=0}^{t} \frac{a_k}{\binom{t}{k}} \leq 1.$$

Because $\binom{t}{k}$ is maximum when $k = \lfloor t/2 \rfloor$, we have

$$\sum_{k=0}^{t} \frac{a_k}{\binom{t}{\lfloor t/2 \rfloor}} \leq \sum_{k=0}^{t} \frac{a_k}{\binom{t}{k}} \leq 1,$$

$$\text{or equivalently} \sum_{k=0}^{t} a_k \leq \binom{t}{\lfloor t/2 \rfloor}$$

Hence, if a $t \times n$ matrix is 1-disjunct, then $n \leq \binom{t}{\lfloor t/2 \rfloor}$. □

In the case $d = 2$, t is smaller than 6. Do notice that any $\overline{2}$-separable matrix is also a 1-disjunct matrix. By lemma 4.1.1, there is no $t \times (t+1)$ 1-disjunct matrix for $t = 1, 2, 3$ because $\binom{t}{\lfloor \frac{t}{2} \rfloor} = t < t + 1$ for $t = 1, 2, 3$.

For $t = 4$ and 5, it is found that the following concept and associated property are useful.

Definition 4.1.2. (*an isolated column*)

A column is called isolated if there exists a row incident to it but not to any other column.

Example 4.1.3. Suppose

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \tag{4.1}$$

Then, C_1 and C_2 are isolated.

Do notice that if a $t \times n$ $\overline{d}$-separable (d-disjunct) matrix having an isolated column C_1 and $d > 1$, then removing C_1 and the row incident to C_1 yields a $(t-1) \times (n-1)$ $\overline{d}$-separable (d-disjunct) matrix. Hence, we can assume that a $t \times n$ $\overline{d}$-separable (d-disjunct) matrix does not have an isolated column if all $(t-1) \times (n-1)$ matrix are not $\overline{d}$-separable (d-disjunct).

Lemma 4.1.4. *A nonisolated column in a d-disjunct matrix has weight at least $d + 1$.*

Proof. Suppose there exists a column C_1 with weight at most d. Do notice that for any position of 1 in C_1, there exists a column having 1 at the same position. Since C_1 is weight at most d, there exists at most such d columns and the union of these columns contains C_1, a contradiction. □

Due to the fact that there is no 3×4 $\overline{2}$-separable matrix, and if there is a 4×5 $\overline{2}$-separable matrix $\mathbf{M}$, we can assume that $\mathbf{M}$ does not have an isolated column and the weight of each column in $\mathbf{M}$ is at least 2. If the weight of each column in $\mathbf{M}$ is at most 1, then it shows that there is no 4×5 $\overline{2}$-separable matrix $\mathbf{M}$ and this sufficient condition is provided by the following lemma.

Lemma 4.1.5. *For a $t \times n$ $\overline{2}$-separable matrix $\mathbf{M}$, each column has weight at most $t - k - 1$ where $k = \max\left(\lfloor \log_2(n-1) \rfloor, 0\right)$.*

Proof. It is sufficient to suppose there is a column C with weight $t - k$. Then, without loss of generality, we assume that the position of 0 in C is in the last k position. For these k position, we have 2^k way to put 0 and 1. Because $n > 2^k$, by Pigeonhole principle, there exist two columns C^1 and C^2 such that the last k elements of them are the same.

In this case, we have $U(\{C^1, C\}) = U(\{C^2, C\})$. This contradicts to the definition of $\overline{2}$-separable. □

To investigate the case: $t = 5$, we also need the following lemma.

Lemma 4.1.6. *For a $t \times n$ $\overline{2}$-separable matrix $\mathbf{M}$, if the weight of each column in $\mathbf{M}$ is either 2 or 3, then $n \leq t(t+1)/6$.*

Proof. The proof of this lemma is the same as that of Theorem 3.7.4 in [1] and, for self-contained of this thesis, we prove this again.

Do notice that n is equal to the number of columns with weight 2 and 3. Suppose the number of columns with weight 2 is p. Then, these p columns generate $2p$ row indices (these indices could not be distinct). Due to the definition of 2-separable, if two pairs of indices $\{x, y\}$ and $\{x, z\}$ are columns of $\mathbf{M}$, there does not exist a column C' such that the y-th and z-th row of C' are 1. To find the number of columns with weight 3, the column with weight 3 should satisfy the above condition.

First, we find the number of pairs that share an index. If $2p - t \leq 0$, there exists at least 0 such pairs. This lower bound is tight since it is possible that these $2p$ row indices are all distinct. If $2p - t > 0$, we can put 1 in t rows and any other indices yields a pair that share an index with one column. Hence, there are at least $2p - t$ such pairs. If there is a row having zero in all columns, there are at least $2p - t + 1$ such pairs with the same reason. To sum up, the number of pairs that share an index is at least $\max(2p - t, 0)$.

Hence, the number of pairs that destroy the definition of 2-separable with the p columns having weight 2 is at least $p + \max(2p - t, 0)$. Therefore, the number of pairs that can be contained in a column with weight 3 is at most

$$\binom{t}{2} - p - \max(2p - t, 0) \leq \binom{t}{2} - 3p + t$$

It is clear that a column C' with weight 3 can contain at least 3 such pairs. Hence, the number of columns with weight 3 is at most $(\binom{t}{2} - 3p + t)/3$ and $n \leq p + (\binom{t}{2} - 3p + t)/3 = t(t-1)/6 + t/3 = t(t+1)/6$. □

According to lemma 4.1.4 we can show that if there exists a 5×6 $\overline{2}$-separable matrix $\mathbf{M}$, the weight of any column in $\mathbf{M}$ is 2. In addition, from lemma 4.1.5, the weight of $\mathbf{M}$ is at most 2. Therefore, the weight of any column in $\mathbf{M}$ is 2. From lemma 4.1.6, the maximum value of n is $5(5+1)/6 = 5 < 6 = t+1$. Hence, $\mathbf{M}$ is not a $\overline{2}$-separable matrix.

From the above discussions, we show that there exists no $t \times (t+1)$ $\overline{2}$-separable matrix for $t \leq 5$ and the proof is finished.

4.2 d=3

The following theorem shows that there exists no $t \times (t+1)$ 2-disjunct matrix if $t \leq 8$. Therefore, there exists no $t \times (t+1)$ $\overline{3}$-separable matrix if $t \leq 8$ by lemma lemma 2.2.4.

Theorem 4.2.1. *$t(2,n) = n$ if $n \leq 9$*

Proof. See [5]. ◻

Lemma 4.2.2. *For a $t \times n$ 2-disjunct matrix $\mathbf{M}$, each column has weight at most $t-k-1$ 1's if $n > \binom{k}{\lfloor \frac{k}{2} \rfloor} + 1$.*

Proof. We prove this by contrapositive. It is sufficient to suppose there is a column C_1 with weight $t-k$. Without loss of generality, we assume that the position of 0 in C_1 is in the last k position. Then, remove the C_1 and the first $t-k$ rows in $\mathbf{M}$, we have a $k \times (n-1)$ matrix $\mathbf{M}'$. Since $\mathbf{M}$ is 2-disjunct matrix, $\mathbf{M}'$ must be 1-disjunct matrix. Otherwise, there exists a column C_2 contained in another column C_3 in M' and C_2 is contained in $U(\{C_3, C_1\})$ in M, a contradiction. From lemma 4.1.1, it requires $n-1 \leq \binom{k}{\lfloor \frac{k}{2} \rfloor}$ and we are done. ◻

Lemma 4.2.3. *For a $\overline{d}$-separable matrix $\mathbf{M}$, removing a column C in $\mathbf{M}$ and all rows intersecting it yields a $\overline{(d-1)}$-separable matrix $\mathbf{M}'$.*

Proof. Suppose $\mathbf{M}'$ is not a $\overline{(d-1)}$-separable matrix $\mathbf{M}'$. Then there exists two set of columns S_1, S_2 such that $U(S_1) = U(S_2)$ and $1 \leq |S_1|, |S_2| \leq d-1$. However, $U(S_1 \bigcup \{C_1\}) = U(S_2 \bigcup \{C_1\})$ and $1 \leq |S_1 \bigcup \{C_1\}|, |S_2 \bigcup \{C_1\}| \leq d$, a contradiction. □

Lemma 4.2.4. *There is no* 9×10 $\overline{3}$*-separable matrix.*

Proof. Based on lemma 4.2.2, a $s \times (s+1)$ $\overline{3}$-separable matrix $\mathbf{M}$ has weight at most $s-k-1$ if $s > \binom{k}{\lfloor \frac{k}{2} \rfloor}$. For $s = 9$, the maximum k such that the inequality holds is 4. Hence, for a 9×10 $\overline{3}$-separable matrix, each column has weight at most four $(9-4-1 = 4)$ by lemma 2.2.4.

Moreover, because any 8×9 matrix is not $\overline{3}$-separable matrix, any 9×10 $\overline{3}$-separable matrix is not isolated. Therefore, from lemma 2.2.4 and 4.1.4, each column has weight at least 3.

If there is a column with weight 4, then removing this column and all rows intersecting it yields a 5×9 $\overline{2}$-separable matrix $\mathbf{M}'$ from lemma 4.2.3. However, it has been shown that there is no 5×6 $\overline{2}$-separable matrix, which shows that there is no column having weight 4 and any column in $\mathbf{M}$ has weight 3.

Up to now, we show that each column in $\mathbf{M}$ has weight 3. Suppose there are two columns C_1 and C_2 having weight 3 and and they are disjoint. Then, by lemma 4.2.3, C_1 and C_2 and all rows intersecting them yields a 3×8 $\overline{1}$-separable matrix $\mathbf{M}'$. However, it is impossible because $\binom{3}{1} + \binom{3}{2} + \binom{3}{3} = 3 + 3 + 1 = 7 < 8$ (the number of column).

Now, let $\mathbf{M}'$ be the 6×9 matrix obtained by removing the first column and all rows intersecting the first column. Then, each column of $\mathbf{M}'$ has weight at most 2. If a column of $\mathbf{M}'$ has weight 3, we have disjoint two columns columns and it has been shown that it is impossible. By lemma 4.2.3, $\mathbf{M}'$ is $\overline{2}$-separable.

Suppose $\mathbf{M}'$ has a column C_k having weight 1 in the ith row, for some integer i. If this column does not intersect with other columns, removing this column and kth row from yields a 5×8 $\overline{2}$-separable matrix, which is impossible since there is no 5×6 $\overline{2}$-separable matrix. If there exists a column C'_k intersecting with C_k, then C_k is contained in $U(\{C_1, C'_k\})$, which shows that $\mathbf{M}$ is not 2-disjunct and not a $\overline{3}$-separable. Hence, each column in $\mathbf{M}'$ has weight 2.

Since $\mathbf{M}'$ is $\overline{2}$-separable, the columns which we chosen have $\binom{9}{1} + \binom{9}{2} = 45$ conditions. On the other hand, there are 9 boolean sums consisting one column having weight 2 ($\mathbf{M}'$ is 6×9). Because the boolean sum of two columns has weight at most 4 and at least 3 ($\mathbf{M}'$ is $\overline{2}$-separable with constant weight 2), there are $\binom{6}{3} + \binom{6}{4} = 20 + 15 = 35$ boolean sums involving two columns. Therefore, we have $9 + 35 = 44$ results. Since the number of results is smaller than the number of conditions, $\mathbf{M}'$ must be not a $\overline{2}$-separable matrix.

From the discussion above, we are done. □

Lemma 4.2.5. *There is no* 10×11 $\overline{3}$*-separable matrix.*

***Proof*.** Based on lemma 4.2.2, a 10×11 $\overline{3}$-separable matrix $\mathbf{M}$ has weight at most $10 - k - 1$ if $10 > \binom{k}{\lfloor \frac{k}{2} \rfloor}$. Hence, for a 10×11 3-separable matrix, each column has weight at most 5. Moreover, from Lemma 4.1.4, each column has weight at least 3.

If there is a column having weight 5, then removing this column and all rows intersecting it yields a 5×10 $\overline{2}$-separable matrix $\mathbf{M}'$ from lemma 4.2.3. However, it has been shown that there is no 5×6 $\overline{2}$-separable matrix, which shows that there is no column having weight 5.

Suppose there are two columns C_1 and C_2 having weight 4 and they are disjoint. Then, by lemma 4.2.3, removing C_1 and C_2 and all rows intersecting them yields a 2×9 $\overline{1}$-separable matrix $\mathbf{M}'$. However, it is impossible because $\binom{2}{1} + \binom{2}{2} = 1 + 2 = 3 < 9$ (the number of column).

Similarly, suppose there are two columns C_1 and C_2 having weight 4 and they intersect in only one position (or $w(C_1) = 4$, $w(C_2) = 3$, and they are disjoint). Then, by lemma 4.2.3, removing C_1 and C_2 and all rows intersecting them yields a 3×9 $\overline{1}$-separable matrix $\mathbf{M}'$. However, it is impossible because $\binom{3}{1} + \binom{3}{2} + \binom{3}{3} = 3 + 3 + 1 = 7 < 9$ (the number of column).

Hence, if there is a column having weight 4, then removing this column and all rows intersecting it yields a 6×10 $\overline{2}$-separable matrix $\mathbf{M}'$ from lemma 4.2.3 and $\mathbf{M}'$ has weight at most 2. Recall that M' is not isolated and $\overline{2}$-separable matrix, implying 1-disjunct matrix. Hence, M' has weight at least 2. In fact, there is no such $\overline{2}$-separable matrix because there are $\binom{10}{1} + \binom{10}{2} = 55$ conditions and we have only $\binom{6}{3} + \binom{6}{4} + 10 = 20 + 15 + 10 = 45$ possible ways to be a boolean sum of two or one column. Up to now, we have prove that each column of $\mathbf{M}$ has weight 3. Without loss of generality, we assume that the first column C_1 has 1 in the first 3 rows. Then, removing the first column and the first 3 rows yields a 7×10 $\overline{2}$-separable matrix $\mathbf{M}'$. Notice that each column of $\mathbf{M}'$ has weight at most 3 and at least 1. If there is a column C_2 with weight 1 in the fourth row, then there does not exist a column C_3 having 1 in the fourth row because $\mathbf{M}'$ is 1-disjunct. This implies that C_2 is a isolated column in $\mathbf{M}$ (C_1 has zero in the fourth row) and this is a contradiction. Hence, each column of $\mathbf{M}'$ has weight 2 or 3. By lemma 4.1.6, we have $n = 7(7+1)/6 = 28/3 < 10$ and $\mathbf{M}'$ does not exist. Therefore, there exists no 10×11 $\overline{3}$-separable matrix with constant weight 3.

From the above discussions, this lemma is proved. □

Lemma 4.2.6. *There is no* 11×12 $\overline{3}$*-separable matrix* $\mathbf{M}$.

Proof. Based on lemma 4.2.2 and 2.2.4, a $t \times (t+1)$ $\overline{3}$-separable matrix $\mathbf{M}$ has weight at most $t - k - 1$ if $t > \binom{k}{\lfloor \frac{k}{2} \rfloor}$. For $t = 11$, the maximum k such that the inequality holds is 5. Hence, for a 11×12 $\overline{3}$-separable matrix, the weight of each column is at most five $(11 - 5 - 1 = 5)$.

In addition, any 10×11 matrix is not $\overline{3}$-separable matrix; therefore, any 11×12 $\overline{3}$-separable matrix is not isolated. From lemma 2.2.4 and 4.1.4, the weight of each column is at least *three*.

Suppose there are two columns C_1 and C_2 with weight 5. If they are disjoint, then removing C_1, C_2, and all rows intersecting them yields a 1×10 $\overline{1}$-separable matrix, which is impossible. Now, suppose they intersect in t positions only and following the same procedure, we have a $(1+t) \times 10$ $\overline{1}$-separable matrix. In this case, we require

$\sum_{k=1}^{\min(1+t,5)} \binom{(1+t)}{k} > 10$ (by lemma 4.1.1), which is hold when $t \geq 3$.

Now, removing C_1 and all row intersecting it yields a $\overline{2}$-separable matrix $\mathbf{M}'$. Since the original matrix is not isolated, $\mathbf{M}'$ is not isolated and, by lemma 2.2.4 and 4.1.4, each column has weight at least *two*. This implies that C_2 can intersect C_1 at most 3 positions ($t \leq 3$). Hence, we could assume that each column of $\mathbf{M}'$ has weight 2. Since $\mathbf{M}'$ is 6×11 $\overline{2}$-separable, we require $11 \leq 6(6+1)/6$ by lemma 4.1.6, which is impossible. Hence, there are no two or more columns with weight 5.

Suppose C_2 has weight 4. Then, it is equivalent to the case that C_2 has weight 5 and C_1 and C_2 intersect at least in a position. Following the same discussion, it can be proved that there is no such C_2. Similarly, we can also prove that there is no column with weight 3. In conclusion, there is no column with weight 5 in $\mathbf{M}$.

Now, suppose there is a column C_1 with weight 4. If there is another column C_2 with weight 4 and C_2 is disjoint with C_1, removing them and all the rows intersecting them gives a 3×10 $\overline{1}$-separable matrix $\mathbf{M}_2$. Since $\binom{3}{1} + \binom{3}{2} + \binom{3}{3} = 7 < 10$, there is no such $\mathbf{M}_2$. Let $\mathbf{M}_1$ be the matrix by removing C_1 and all rows intersect it. Then, each column of $\mathbf{M}_1$ is at least 2 because $\mathbf{M}_1$ is $\overline{2}$-separable matrix and $\mathbf{M}_1$ is not isolated. Moreover, since there are no columns with weight 4 and disjoint in $\mathbf{M}$, each column of $\mathbf{M}_1$ is at most 3. In this case, from lemma 4.1.6, we require $11 < 7(7+1)/6$, which is impossible. Hence, there is no column with weight 4 in $\mathbf{M}$.

Up to now, we show that if $\mathbf{M}$ is a 11×12 $\overline{3}$-separable matrix, it must be a matrix with constant weight 3. Notice that any pair of two columns C_1, C_2 must share at most one row. If they intersect in two rows, then removing C_1 and all rows intersect C_1 yields a $\overline{2}$-separable matrix $\mathbf{M}_1$ having a column with weight 1. However, we know that any column in $\mathbf{M}_1$ has at least weight 2, a contradiction.

Because any two columns share at most one row and $\mathbf{M}$ is constant weight 3, there are at most five 1s in a row. To show that $\mathbf{M}$ is impossible to be $\overline{3}$-separable matrix, we first consider a row with five 1s. Without loss of generality, we have the following structure.

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & & & & \\ 1 & & & & \\ & 1 & & & \\ & 1 & & & \\ & & 1 & & \\ & & 1 & & \\ & & & 1 & \\ & & & 1 & \\ & & & & 1 \\ & & & & 1 \end{pmatrix}$$

Up to permutation, there is only one way for the sixth column and we have the following structure.

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & \\ 1 & & & & & 1 \\ 1 & & & & & \\ & 1 & & & & 1 \\ & 1 & & & & \\ & & 1 & & & 1 \\ & & 1 & & & \\ & & & 1 & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & 1 & \end{pmatrix}$$

Case 1. C_7 has 1 in the second row.

There are three subcases in this case.

Subcase 1.

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & & \\ 1 & & & & & 1 & 1 \\ 1 & & & & & & \\ & 1 & & & & 1 & \\ & 1 & & & & & 1 \\ & & 1 & & & 1 & \\ & & 1 & & & & 1 \\ & & & 1 & & & \\ & & & 1 & & & \\ & & & & 1 & & \\ & & & & 1 & & \end{pmatrix}$$

It can be check that $U(C_1, C_2, C_3) = U(C_1, C_6, C_7)$.

Subcase 2.

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & & \\ 1 & & & & & 1 & 1 \\ 1 & & & & & & \\ & 1 & & & & 1 & \\ & 1 & & & & & 1 \\ & & 1 & & & 1 & \\ & & 1 & & & & \\ & & & 1 & & & 1 \\ & & & 1 & & & \\ & & & & 1 & & \\ & & & & 1 & & \end{pmatrix}$$

It can be check that $U(C_2, C_3, C_7) = U(C_3, C_6, C_7)$.

Subcase 3.

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & & \\ 1 & & & & & 1 & 1 \\ 1 & & & & & & \\ & 1 & & & & 1 & \\ & 1 & & & & & \\ & & 1 & & & 1 & \\ & & 1 & & & & \\ & & & 1 & & & 1 \\ & & & 1 & & & \\ & & & & 1 & & 1 \\ & & & & 1 & & \end{pmatrix}$$

It is $\overline{3}$-separable. Hence, we consider the all possibilities of C_8. Do notice that it is impossible that C_8 has 1 in the second row. If C_8 has 1 in the second row, it has the structure discussed in subcase 1 or subcase 2. Therefore, we start our discussion with 1 in the third row.

Case a.

$$\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & & & \\
1 & & & & & 1 & 1 & \\
1 & & & & & & & 1 \\
 & 1 & & & & 1 & & 1 \\
 & 1 & & & & & & \\
 & & 1 & & & 1 & & \\
 & & 1 & & & & & 1 \\
 & & & 1 & & & 1 & \\
 & & & 1 & & & & \\
 & & & & 1 & & 1 & \\
 & & & & 1 & & &
\end{pmatrix}$$

It can be check that $U(C_1, C_3, C_8) = U(C_1, C_6, C_8)$.

Case b.

$$\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & & & \\
1 & & & & & 1 & 1 & \\
1 & & & & & & & 1 \\
 & 1 & & & & 1 & & 1 \\
 & 1 & & & & & & \\
 & & 1 & & & 1 & & \\
 & & 1 & & & & & \\
 & & & 1 & & & 1 & 1 \\
 & & & 1 & & & & \\
 & & & & 1 & & 1 & \\
 & & & & 1 & & &
\end{pmatrix}$$

It can be check that $U(C_1, C_2, C_7) = U(C_2, C_7, C_8)$.

Case c.

$$\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & & & \\
1 & & & & & 1 & 1 & \\
1 & & & & & & & 1 \\
 & 1 & & & & 1 & & 1 \\
 & 1 & & & & & & \\
 & & 1 & & & 1 & & \\
 & & 1 & & & & & \\
 & & & 1 & & & 1 & \\
 & & & 1 & & & & 1 \\
 & & & & 1 & & 1 & \\
 & & & & 1 & & &
\end{pmatrix}$$

It can be check that $U(C_1, C_3, C_8) = U(C_3, C_6, C_8)$.

Case d.

$$\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & & & \\
1 & & & & & 1 & 1 & \\
1 & & & & & & & 1 \\
 & 1 & & & & 1 & & \\
 & 1 & & & & & & 1 \\
 & & 1 & & & 1 & & \\
 & & 1 & & & & & 1 \\
 & & & 1 & & & 1 & \\
 & & & 1 & & & & \\
 & & & & 1 & & 1 & \\
 & & & & 1 & & &
\end{pmatrix}$$

It can be check that $U(C_1, C_2, C_3) = U(C_1, C_6, C_8)$.

Case e.

$$\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & & & \\
1 & & & & & 1 & 1 & \\
1 & & & & & & & 1 \\
 & 1 & & & & 1 & & \\
 & 1 & & & & & & 1 \\
 & & 1 & & & 1 & & \\
 & & 1 & & & & & \\
 & & & 1 & & & 1 & \\
 & & & 1 & & & & 1 \\
 & & & & 1 & & 1 & \\
 & & & & 1 & & &
\end{pmatrix}$$

It can be check that $U(C_1, C_6, C_8) = U(C_2, C_6, C_8)$.

From the discussion of case a-e, it is shown that there is no 1 in the third row. It is impossible as the is no isolated column in $\mathbf{M}$. Hence, we finish the case 1.

Case 2. C_7 has 1 in the third row.

In this case, there are five subcases.

Subcase 1.

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & & \\ 1 & & & & & 1 & \\ 1 & & & & & & 1 \\ & 1 & & & & 1 & 1 \\ & 1 & & & & & \\ & & 1 & & & 1 & \\ & & 1 & & & & 1 \\ & & & 1 & & & \\ & & & 1 & & & \\ & & & & 1 & & \\ & & & & 1 & & \end{pmatrix}$$

It can be check that $U(C_1, C_3, C_7) = U(C_1, C_6, C_7)$.

Subcase 2.

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & & \\ 1 & & & & & 1 & \\ 1 & & & & & & 1 \\ & 1 & & & & 1 & 1 \\ & 1 & & & & & \\ & & 1 & & & 1 & \\ & & 1 & & & & \\ & & & 1 & & & 1 \\ & & & 1 & & & \\ & & & & 1 & & \\ & & & & 1 & & \end{pmatrix}$$

It can be check that $U(C_1, C_3, C_7) = U(C_3, C_6, C_7)$.

Subcase 3.

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & & \\ 1 & & & & & 1 & \\ 1 & & & & & & 1 \\ & 1 & & & & 1 & \\ & 1 & & & & & 1 \\ & & 1 & & & 1 & \\ & & 1 & & & & 1 \\ & & & 1 & & & \\ & & & 1 & & & \\ & & & & 1 & & \\ & & & & 1 & & \end{pmatrix}$$

It can be check that $U(C_1, C_2, C_3) = U(C_1, C_6, C_8)$.

Subcase 4.

$$\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & & \\
1 & & & & & 1 & \\
1 & & & & & & 1 \\
 & 1 & & & & 1 & \\
 & 1 & & & & & 1 \\
 & & 1 & & & 1 & \\
 & & 1 & & & & \\
 & & & 1 & & & 1 \\
 & & & 1 & & & \\
 & & & & 1 & & \\
 & & & & 1 & &
\end{pmatrix}$$

It can be check that $U(C_1, C_6, C_7) = U(C_2, C_6, C_7)$.

Subcase 4.

$$\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & & \\
1 & & & & & 1 & \\
1 & & & & & & 1 \\
 & 1 & & & & 1 & \\
 & 1 & & & & & \\
 & & 1 & & & 1 & \\
 & & 1 & & & & \\
 & & & 1 & & & 1 \\
 & & & 1 & & & \\
 & & & & 1 & & 1 \\
 & & & & 1 & &
\end{pmatrix}$$

It can be check that it is $\overline{3}$-separable.

Now, consider the structure of C_8.

Case a.

$$\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & & & \\
1 & & & & & 1 & & \\
1 & & & & & & 1 & \\
 & 1 & & & & 1 & & 1 \\
 & 1 & & & & & & \\
 & & 1 & & & 1 & & \\
 & & 1 & & & & & 1 \\
 & & & 1 & & & 1 & 1 \\
 & & & 1 & & & & \\
 & & & & 1 & & 1 & \\
 & & & & 1 & & &
\end{pmatrix}$$

It can be check that $U(C_1, C_3, C_8) = U(C_1, C_6, C_8)$.

Case b.

$$\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & & & \\
1 & & & & & 1 & & \\
1 & & & & & & 1 & \\
 & 1 & & & & 1 & & 1 \\
 & 1 & & & & & & \\
 & & 1 & & & 1 & & \\
 & & 1 & & & & & 1 \\
 & & & 1 & & & 1 & \\
 & & & 1 & & & & 1 \\
 & & & & 1 & & 1 & \\
 & & & & 1 & & &
\end{pmatrix}$$

It can be check that $U(C_1, C_3, C_8) = U(C_1, C_6, C_8)$.

Case c.

$$\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & & & \\
1 & & & & & 1 & & \\
1 & & & & & & 1 & \\
 & 1 & & & & 1 & & \\
 & 1 & & & & & & 1 \\
 & & 1 & & & 1 & & \\
 & & 1 & & & & & 1 \\
 & & & 1 & & & 1 & 1 \\
 & & & 1 & & & & \\
 & & & & 1 & & 1 & \\
 & & & & 1 & & &
\end{pmatrix}$$

It can be check that $U(C_2, C_6, C_8) = U(C_3, C_6, C_8)$.

Case d.

$$\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & & & \\
1 & & & & & 1 & & \\
1 & & & & & & 1 & \\
 & 1 & & & & 1 & & \\
 & 1 & & & & & & 1 \\
 & & 1 & & & 1 & & \\
 & & 1 & & & & & 1 \\
 & & & 1 & & & 1 & \\
 & & & 1 & & & & 1 \\
 & & & & 1 & & 1 & \\
 & & & & 1 & & &
\end{pmatrix}$$

It can be check that $U(C_2, C_6, C_8) = U(C_3, C_6, C_8)$ and we finish the case 2.

From the above discussions, we conclude that there is no row with five 1s. In addition, there is no case involving C_5 in the above discussions. This implies that there is no row with four 1s. Hence, any row has at most three 1s. Yet, it is impossible for $\mathbf{M}$ because $\mathbf{M}$ is a constant weight 3 matrix. We have to put $3 \times 12 = 36$ 1s in $\mathbf{M}$ but there are at most $3 \times 11 = 33$ positions for us to choose.

From the discussion above, we are done. □

From lemma 4.2.4-4.2.6, we have the following main theorem.

Theorem 4.2.7. *The minimum number of t for the existence of a $t \times (t+1)$ $\overline{d}$-separable matrix is $d^2 + d$ when $d = 3$.*

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