國立交通大學應用數學系

博士論文

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Lit-only sigma-game from the view of algebra

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摘要

亮點西格瑪遊戲是一個在有限簡圖上的單人益智遊戲。已知亮點西格瑪遊戲可視為群作用。在這篇論文裡，我們展示此遊戲和考斯特群的關係。我們並由代數的技巧推廣一些此遊戲已知的成果。

Lit-only sigma-game from the view of algebra

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Abstract

The lit-only σ-game is a one-player game played on a finite simple graph. It is known that this game can be view as a group action. In this thesis we show how this game is related to Coxeter groups. Moreover we use algebraic techniques to generalize some known results on the game.

誌　　謝

完成此論文，我想感謝很多人，最感謝是我的阿媽。從小把我帶大，雖然阿媽不識半字，但阿媽懂得加減法、看時鐘。我相信我的數學細胞都是遺傳自於阿媽。謝謝我兩位姐姐、姑姑、雙親等人給予我生活上的幫助。感謝我的女友及其家人，謝謝他們的支持和生活上的幫忙。

在我求學過程中，國中時期的數學老師鄭瑞欽是我數學上的啟蒙教師，一堂又一堂有趣且縝密的數學課，讓我開始對數學有了初步了解。大學時期於清華大學數學系所修讀的課程，使我對數學有進一層的認識，在此感謝所有教導過我的任課教授。謝謝我碩博班的指導教授翁志文，謝謝他協助我撰寫此論文、給予研究費讓我經濟無虞、以及協助我前往麥迪遜威斯康辛大學數學系接觸不同的研究主題。

最後，謝謝同研究室的學長李信儀、楊川和及學弟陳德軒平常的照顧。謝謝以前碩班同學張澍仁、李張圳、陳柏澍、卜文強、張雁婷等人平常的照顧。

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Chapter 1

Introduction

My object of this thesis is to use algebraic techniques to study a combinatorial game called the lit-only σ-game. The game is a one-player game played on a finite graph. Let Γ denote a finite graph. A configuration of the lit-only σ-game on Γ is an assignment of one of two states, *on* or *off*, to each vertex of Γ. Given a configuration, a move of the lit-only σ-game on Γ allows the player to choose one *on* vertex s of Γ and change the states of all neighbors of s. Given a starting configuration, the goal is usually to minimize the number of *on* vertices of Γ or to reach an assigned configuration by a finite sequence of moves. In the thesis, we are only concerned with the lit-only σ-game on a finite simple graph and always assume that Γ is a finite simple graph.

The game implicitly appeared in the classification of simple Lie algebras over real number field. See [2, 8] for details. In 2005 International and Third Cross-strait Conference on Graph Theory and Combinatorics, Gerard J. Chang's talk "Graph Painting and Lie Algebra" promoted the birth of this game. Later Yaokun Wu and Xinmao Wang [26] realized this game is a variation of σ-game and named it lit-only σ-game. They also found that the game appeared as early as 2001 in the paper [12].

As far as we know, the first result on this topic is from [2], which claimed that if Γ is a simply-laced Dynkin diagram then given any configuration one can reduce the number of *on* vertices to at most one. Some results of [8] can be viewed as a description of the orbits of this game on simply-laced Dynkin diagrams. Gerard J. Chang, on his talk, gave a conjecture: if Γ is a tree with ℓ leaves then for any configuration one can reduce the number of *on* vertices to at most $\lceil \frac{\ell}{2} \rceil$. Later Yaokun Wu and Xinmao Wang [26] proved this conjecture. Also they [26] found that a subgroup of the general linear group over the two-element field of which the natural action can be viewed as the lit-only σ-game. Later in the paper [29], Yaokun Wu named this group the *lit-only group* and proved that it is isomorphic to the symmetric group on n letters when the underlying graph is the line graph of a tree of order $n \geq 3$. In 2007 the author independently found this group, and in 2008 the author named it the *flipping group.* In this dissertation we will adopt the latter name. For the study of the difference between the lit-only σ-game and σ-game, please refer to [14, 15, 27].

The organization of this dissertation is as follows. In Chapter 2 we show how the flipping groups are related to the simply-laced Coxeter groups, and from the view of the flipping groups we give an alternative description of the orbits of the game on simply-laced Dynkin diagrams. In Chapter 3 we consider the game on an n-vertex graph with an induced path of $n-1$ vertices, which generalizes the study of the latter part of Chapter 2. Motivated by the first result [2], Chapter 4 is devoted to finding more trees for which given any configuration one can reach a configuration with at most one *on* vertex by a finite sequence of moves. The topic of Chapter 5 is to study the edge-version of lit-only σ-game. We may view this variation as the lit-only σ-game on a line graph. We find that the structure of the flipping group of a line graph, which only depends on its order and size not on its graph structure.

Chapter 2

Lit-only sigma-game and simply-laced Coxeter groups

The *lit-only σ-game* is a one-player game played on a finite simple graph. Let Γ denote a finite simple graph. A configuration of the lit-only σ-game on Γ is an assignment of one of two states, *on* or *off*, to all vertices of Γ. Given a configuration, a move of the lit-only σ-game on Γ consisting of choosing one *on* vertex s of Γ and changing the states of all neighbors of s. Given a starting configuration, the goal is usually to minimize the number of *on* vertices of Γ or to reach an assigned configuration by a finite sequence of moves. In this chapter, we show how the lit-only σ-game is related to simply-laced Coxeter groups and study the game on simply-laced Dynkin diagrams.

2.1 The flipping group of a graph

An ordered pair $\Gamma = (S, R)$ is called a *finite simple graph* whenever S is a finite set and R is a set of some two-element subsets of S. The elements of S are called *vertices* of Γ and the elements of R are called *edges* of Γ. For any $s, t \in S$ we say s and t are *neighbors* whenever $\{s, t\} \in R$. For convenience we usually write $st \in R$ or $ts \in R$ for $\{s, t\} \in R$. We say that a finite simple graph $\Gamma = (S, R)$ is *connected* whenever for any two distinct vertices s, t of Γ there exists a subset $\{s_0s_1, s_1s_2, \ldots, s_{k-1}s_k\}$ of R with $s_0 = s$ and $s_k = t$.

Throughout this dissertation let $\Gamma = (S, R)$ denote a finite simple graph. Moreover we assume that S is nonempty and that Γ is connected. Let $\mathbb{F}_2$ denote the two-element field $\{0, 1\}$. Let $\mathrm{Mat}_S(\mathbb{F}_2)$ denote the set consisting of square matrices over $\mathbb{F}_2$ with rows and columns indexed by S. Let $\mathrm{GL}_S(\mathbb{F}_2)$ denote the group consisting of all invertible matrices in $\mathrm{Mat}_S(\mathbb{F}_2)$. The group operation of $\mathrm{GL}_S(\mathbb{F}_2)$ is ordinary matrix multiplication. We use I to denote the identity in $\mathrm{GL}_S(\mathbb{F}_2)$. Let $\mathbb{F}_2^S$ denote the vector space consisting of column vectors over $\mathbb{F}_2$ indexed by S. For $s \in S$ let e_s denote the characteristic vector of s in $\mathbb{F}_2^S$; i.e. $e_s = (0, 0, \ldots, 0, 1, 0, \ldots, 0)^t$, where 1 is in the position corresponding to s. Here a^t means the transpose of a.

We interpret each configuration a of the lit-only σ-game on Γ as the vector

$$\sum_s e_s \tag{2.1}$$

of $\mathbb{F}_2^S$, where the sum is over all vertices s of Γ that are assigned the *on* state by a; if all vertices of Γ are assigned the *off* state by a, then (2.1) is interpreted as zero vector. We may view a move of the lit-only σ-game as choosing any vertex s of Γ and changing the states of all neighbors of s if the state of s is *on.*

Definition 2.1.1. For $s \in S$ define a matrix $\kappa_s \in \mathrm{Mat}_S(\mathbb{F}_2)$ by

$$(\kappa_s)_{uv} = \begin{cases} 1 & \text{if } u = v, \text{ or } v = s \text{ and } uv \in R, \\ 0 & \text{else} \end{cases}$$

for all $u, v \in S$.

The following is a reformulating of Definition 2.1.1.

Lemma 2.1.2. *For $s, v \in S$ we have*

$$\kappa_s\, e_v = \begin{cases} e_v + \sum\limits_{uv \in R} e_u & \text{if } v = s, \\ e_v & \text{if } v \neq s. \end{cases}$$

Let $a \in \mathbb{F}_2^S$. By Lemma 2.1.2, if the state of s is *on* then $\kappa_s a$ is obtained from a by changing the states of all neighbors of s; if the state of s is *off* then $\kappa_s a = a$. Therefore we may view κ_s as the move of the lit-only σ-game on Γ for which we choose the vertex s and change the states of all neighbors of s if the state of s is *on.*

Lemma 2.1.3. *For $s \in S$ we have $\kappa_s^2 = I$. In particular $\kappa_s \in \mathrm{GL}_S(\mathbb{F}_2)$.*

Proof. Use Lemma 2.1.2. □

Definition 2.1.4. Let $\mathbf{W}$ denote the subgroup of $\mathrm{GL}_S(\mathbb{F}_2)$ generated by κ_s for all $s \in S$. We call $\mathbf{W}$ the *flipping group of* Γ.

As far as we know the flipping group of Γ was first mentioned in [26, Introduction].

Observe that for any $a, b \in \mathbb{F}_2^S$, b is obtained from a by a finite sequence of moves of the lit-only σ-game on Γ if and only if $b = Ga$ for some $G \in \mathbf{W}$. We now define the $\mathbf{W}$-orbits of $\mathbb{F}_2^S$, which are exactly the orbits of the lit-only σ-game on Γ.

Definition 2.1.5. Let $a \in \mathbb{F}_2^S$. By the $\mathbf{W}$-*orbit of* a we mean the set $\mathbf{W}a = \{Ga \mid G \in \mathbf{W}\}$. By a $\mathbf{W}$-*orbit of* $\mathbb{F}_2^S$ we mean a $\mathbf{W}$-orbit of a for some $a \in \mathbb{F}_2^S$.

We finish this section with a property about the flipping group $\mathbf{W}$ of Γ. To see this we establish a lemma.

Lemma 2.1.6. *For $s \in S$ define $E_s \in \mathrm{Mat}_S(\mathbb{F}_2)$ by*

$$E_s\, e_v = \begin{cases} 0 & \text{if } v \neq s, \\ \sum\limits_{uv \in R} e_u & \text{if } v = s. \end{cases} \tag{2.2}$$

for all $v \in S$. Then the following (i)–(iii) *hold.*

(i) $\kappa_s = I + E_s$ *for all $s \in S$.*

(ii) $E_sE_t = 0$ *if* $st \notin R$.

(iii) *If* $s_is_{i-1} \in R$ *for* $i = 1, 2, \ldots, k$ *then*

$$E_{s_k}E_{s_{k-1}}\cdots E_{s_0} = \begin{cases} E_{s_0} & \text{if } s_k = s_0, \\ E_{s_k}E_{s_0} & \text{if } s_ks_0 \in R. \end{cases}$$

Proof. (i) is immediate from Lemma 2.1.2. Using (2.2) we find $E_sE_te_v = 0$ for any $v, s, t \in S$ with $st \notin R$. Hence we have (ii). (iii) follows from the same reason as in (ii) by applying the product of matrices in either side of the equation to e_v and obtaining the desired equality in each case. □

Proposition 2.1.7. *For* $s, t \in S$ *we have* $(\kappa_s\kappa_t)^2 = I$ *if* $st \notin R$ *and* $(\kappa_s\kappa_t)^3 = I$ *if* $st \in R$.

Proof. By Lemma 2.1.6(i)

$$\begin{aligned}\kappa_s\kappa_t &= (I + E_s)(I + E_t)\\ &= I + E_s + E_t + E_sE_t.\end{aligned}$$

In the case $s \neq t$ and $st \notin R$,

$$\begin{aligned}(\kappa_s\kappa_t)^2 &= (I + E_s + E_t)(I + E_s + E_t)\\ &= I + 2E_s + 2E_t\\ &= I\end{aligned}$$

by Lemma 2.1.6(ii). In the case $st \in R$,

$$\begin{aligned}(\kappa_s\kappa_t)^2 &= (I + E_s + E_t + E_sE_t)(I + E_s + E_t + E_sE_t)\\ &= I + 3E_s + 3E_t + 4E_sE_t + E_tE_s\\ &= I + E_s + E_t + E_tE_s\end{aligned}$$

and

$$\begin{aligned}(\kappa_s\kappa_t)^3 &= (\kappa_s\kappa_t)^2(\kappa_s\kappa_t)\\ &= (I + E_s + E_t + E_tE_s)(I + E_s + E_t + E_sE_t)\\ &= I + 2E_s + 4E_t + 2E_sE_t + 2E_tE_s\\ &= I\end{aligned}$$

by Lemma 2.1.6(iii). □

2.2 A representation of the Coxeter group of type Γ

A *Coxeter group* is a group generated by a set T subject to relations of the form

$$(st)^{m(s,t)} = 1 \qquad \text{for all } s, t \in T,$$

where $m(s, s) = 1$ and $m(s, t) = m(t, s) \in \{2, 3, \ldots, \infty\}$ for $s \neq t$ in T. If $m(s, t) \in \{2, 3\}$ for all $s \neq t$ in T, the Coxeter group is said to be *simply-laced*. Proposition 2.1.7 motivates us to consider a certain (simply-laced) Coxeter group as follows.

Definition 2.2.1. Let W denote the group generated by all elements of S subject to the following relations

$$s^2 = 1, \qquad (st)^2 = 1 \quad \text{if } st \notin R, \qquad (st)^3 = 1 \quad \text{if } st \in R$$

for all $s, t \in S$. We call W the (*simply-laced*) *Coxeter group of type* Γ.

We now establish a connection between the Coxeter group of type Γ and the lit-only σ-game on Γ.

Theorem 2.2.2. *There exists a unique representation* $\kappa : W \to \mathrm{GL}_S(\mathbb{F}_2)$ *such that* $\kappa(s) = \kappa_s$ *for all* $s \in S$. *In particular* $\kappa(W) = \mathbf{W}$.

Proof. Immediate from Proposition 2.1.7 and Definition 2.2.1. $\square$

For the rest of this dissertation let κ denote as in Theorem 2.2.2.

For the rest of this chapter we shall give a new description of $\mathbf{W}$-orbits of $\mathbb{F}_2^S$ when Γ is a simply-laced Dynkin diagram, which is different than the description from [8].

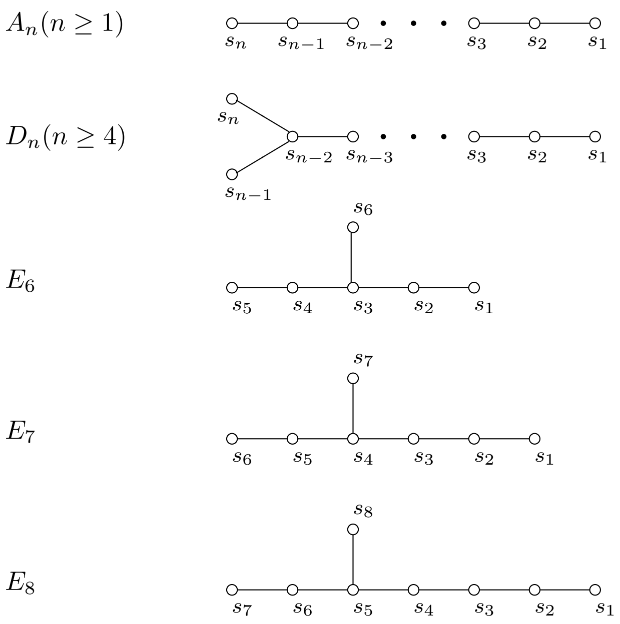

Figure 1.1: simply-laced Dynkin diagrams.

2.3 The center of the flipping group $\mathbf{W}$ of type Γ

Proposition 2.3.1. *Let* $Z(\mathbf{W})$ *denote the center of* $\mathbf{W}$. *Then* $Z(\mathbf{W}) = \{I\}$.

Proof. Let G denote any element in $Z(\mathbf{W})$ and let u, v denote two distinct elements in S. We show that the (v, u)-entry G_{vu} of G is zero to conclude $G = I$. Proceed by contradiction. Suppose $G_{vu} = 1$. On the one hand $\kappa_v G e_u \neq G e_u$ since Ge_u has 1 in the vth position. On the other hand, $\kappa_v G e_u = G \kappa_v e_u = G e_u$ since $\kappa_v e_u = e_u$. Hence we have a contradiction. $\square$

Corollary 2.3.2. *Let $Z(W)$ denote the center of W. Then $Z(W)$ is contained in the kernel of κ.*

Proof. Immediate from Proposition 2.3.1. $\square$

Since the generator $s \in S$ have order 2 in W, each $w \neq 1$ in W can be written in the form $w = s_1 s_2 \cdots s_r$ for some s_i in S. If r is as small as possible, call it the *length* of w. If W has finite order, it is well-known that there exists a unique longest element in W (for example see [21, p. 115]). We shall denote this by $w_\circ$. It is well-known that $Z(W) = \{1, w_\circ\}$ or $\{1\}$ (for example see [21, p. 132]).

2.4 Lit-only σ-game on the Dynkin diagram of type A_n

In this section we assume that Γ is the (simply-laced) Dynkin diagram of type A_n ($n \geq 1$). The goal of this section is to show $\operatorname{Ker}\kappa = Z(W)$ and to determine when κ is irreducible. We also find a description of the $\mathbf{W}$-orbits of $\mathbb{F}_2^S$. We start with the smallest case $n = 1$.

Proposition 2.4.1. *Assume $n = 1$. Then the following* (i)–(iii) *hold.*

(i) *The $\mathbf{W}$-orbits of $\mathbb{F}_2^S$ are* $\{0\}$, $\{1\}$.

(ii) $\operatorname{Ker}\kappa$ *and $Z(W)$ are equal to* $\{1, w_\circ\}$.

(iii) *The representation κ is irreducible.*

Proof. In this case $W = \{1, s_1\}$ and $\mathbf{W} = \{I\}$. By these (i)–(iii) follow. $\square$

For the rest of this section we assume $n \geq 2$. Let

$$\overline{1} = e_{s_1}, \quad \overline{i+1} = \kappa_{s_i}\kappa_{s_{i-1}} \cdots \kappa_{s_1}\overline{1} \qquad (1 \leq i \leq n). \tag{2.3}$$

Note that

$$\overline{i} = e_{s_{i-1}} + e_{s_i} \qquad (2 \leq i \leq n), \tag{2.4}$$

$$\overline{n+1} = e_{s_n} = \overline{1} + \overline{2} + \cdots + \overline{n}. \tag{2.5}$$

Let $\Delta = \Delta(A_n) := \{\overline{1}, \overline{2}, \ldots, \overline{n}\}$. Using (2.4) we find that Δ is a basis of $\mathbb{F}_2^S$. We refer Δ to the *simple basis of* $\mathbb{F}_2^S$. For $a \in \mathbb{F}_2^S$, let $\Delta(a)$ denote the subset of Δ consisting of all the elements appeared in the expression of a as a linear combination of elements in Δ. For $a \in \mathbb{F}_2^S$ let $||a||_s := |\Delta(a)|$ and we call $||a||_s$ the *simple weight of a*. For example $\Delta(\overline{n+1}) = \Delta$ and $||\overline{n+1}||_s = n$.

Lemma 2.4.2. *For $1 \leq i \leq n$, $\kappa_{s_i}\overline{i} = \overline{i+1}$, $\kappa_{s_i}\overline{i+1} = \overline{i}$ and κ_{s_i} fixes other vectors in* $\{\overline{1}, \overline{2}, \ldots, \overline{n+1}\} \setminus \{\overline{i}, \overline{i+1}\}$.

Proof. Use Lemma 2.1.2, (2.3), (2.4) to check. $\square$

For the rest of this section let S_{n+1} denote the symmetric group on $\{\overline{1}, \overline{2}, \ldots, \overline{n+1}\}$. By Lemma 2.4.2 we may make the following definition.

Definition 2.4.3. Let $\alpha : \mathbf{W} \to S_{n+1}$ denote the homomorphism defined by

$$\alpha(G)\overline{j} := G\overline{j} \qquad (1 \le j \le n+1)$$

for $G \in \mathbf{W}$.

Note that $\alpha(\kappa_{s_i})$ is the transposition $(\overline{i}, \overline{i+1})$ in S_{n+1} for each $1 \le i \le n$.

Lemma 2.4.4. *α is an isomorphism from $\mathbf{W}$ to S_{n+1}.*

Proof. α is surjective since the transpositions $\alpha(\kappa_{s_1})$, $\alpha(\kappa_{s_2})$,…, $\alpha(\kappa_{s_n})$ generate S_{n+1}. Since $\Delta \cup \{\overline{n+1}\}$ spans $\mathbb{F}_2^S$, α is injective. The result follows. □

Proposition 2.4.5. *The $\mathbf{W}$-orbits of $\mathbb{F}_2^S$ are*

$$O_i = \{a \in \mathbb{F}_2^S \mid ||a||_s = i \text{ or } n+1-i\} \qquad (0 \le i \le \lfloor \tfrac{n+1}{2} \rfloor),$$

where $\lfloor t \rfloor$ is the largest integer less than or equal to t.

Proof. Suppose $a \in \mathbb{F}_2^S$ with $||a||_s = i$. Observe that from Lemma 2.4.4 and (2.5),

$$\Delta(Ga) = \begin{cases} \alpha(G)\Delta(a) & \text{if } \overline{n+1} \notin \alpha(G)\Delta(a), \\ \Delta \setminus \alpha(G)\Delta(a) & \text{if } \overline{n+1} \in \alpha(G)\Delta(a) \end{cases}$$

for $G \in \mathbf{W}$. The proposition follows from this observation because the subgroup of $\alpha(\mathbf{W}) = S_{n+1}$ generated by the transpositions $\alpha(\kappa_{s_1})$, $\alpha(\kappa_{s_2})$,…, $\alpha(\kappa_{s_{n-1}})$ acts transitively on the fixed size subsets of Δ, and $\kappa_{s_n}\overline{n} = \overline{1} + \overline{2} + \cdots + \overline{n}$ by Lemma 2.4.2 and (2.5). □

Proposition 2.4.6. *The representation κ is irreducible if and only if n is even.*

Proof. Let V denote a nontrivial proper subspace of $\mathbb{F}_2^S$ such that $\kappa(W)V \subseteq V$. Referring to Proposition 2.4.5, note that

$$V = \bigcup_{i \in J} O_i \tag{2.6}$$

for some proper subset $J \subseteq \{0, 1, \ldots, \lfloor \frac{n+1}{2} \rfloor\}$ with $J \ne \{0\}$. Note that the set in the right-hand side of (2.6) to be closed under addition is when it is the set of even weight vectors, and this occurs if and only if n is odd. □

Proposition 2.4.7. *The representation κ is faithful.*

Proof. Immediate from Lemma 2.4.4 and the fact that W is isomorphic to S_{n+1} (for example see [21, p. 41]). □

Proposition 2.4.8. $\operatorname{Ker} \kappa = Z(W)$ *is the trivial group.*

Proof. By Proposition 2.4.7 $\operatorname{Ker} \kappa = \{1\}$. By this and Corollary 2.3.2 $\operatorname{Ker} \kappa = Z(W)$. The result follows. □

2.5 Lit-only σ-game on the Dynkin diagram of type D_n

In this section we assume that Γ is the (simply-laced) Dynkin diagram of type D_n ($n \geq 4$). We shall do the same things as Section 2.4 for this case.

Let

$$\overline{1} = e_{s_1}, \qquad \overline{i+1} = \kappa_{s_i}\kappa_{s_{i-1}}\cdots\kappa_{s_1}\overline{1} \quad (1 \leq i \leq n-1), \qquad \overline{n+1} = e_{s_n}. \tag{2.7}$$

Note that

$$\overline{i} = e_{s_{i-1}} + e_{s_i} \qquad\qquad (2 \leq i \leq n-2), \tag{2.8}$$
$$\overline{n-1} = e_{s_{n-2}} + e_{s_{n-1}} + e_{s_n}, \tag{2.9}$$
$$\overline{n} = e_{s_{n-1}} + e_{s_n} = \overline{1} + \overline{2} + \cdots + \overline{n-1}. \tag{2.10}$$

Set $\Delta = \Delta(D_n) := \{\overline{1}, \overline{2}, \ldots, \overline{n-1}, \overline{n+1}\}$ to be the simple basis of $\mathbb{F}_2^S$ (in the case of type D_n). For $a \in \mathbb{F}_2^S$ set $\Delta(a)$ and $||a||_s$ as Section 2.4. For example $\Delta(\overline{n}) = \Delta \setminus \{\overline{n+1}\}$ by (2.10), and $||\overline{n}||_s = n-1$.

Lemma 2.5.1. *The following* (i), (ii) *hold.*

(i) *For* $1 \leq i \leq n-1$, $\kappa_{s_i}\overline{i} = \overline{i+1}$, $\kappa_{s_i}\overline{i+1} = \overline{i}$, *and*

$$\kappa_{s_i}\overline{j} = \overline{j} \qquad \text{for } \; j \in \{\overline{1}, \overline{2}, \ldots, \overline{n+1}\} \setminus \{\overline{i}, \overline{i+1}\}.$$

(ii) $\kappa_{s_n}\overline{n-1} = \overline{n}$, $\kappa_{s_n}\overline{n} = \overline{n-1}$, $\kappa_{s_n}\overline{n+1} = \overline{n-1} + \overline{n} + \overline{n+1}$, *and*

$$\kappa_{s_n}\overline{j} = \overline{j} \qquad \text{for } \; j \in \{\overline{1}, \overline{2}, \ldots, \overline{n-2}\}.$$

In particular $\overline{n+1} \in \Delta(G\overline{n+1})$ *and* $G(\{\overline{1}, \overline{2}, \ldots, \overline{n}\}) \subseteq \{\overline{1}, \overline{2}, \ldots, \overline{n}\}$ *for all* $G \in \mathbf{W}$.

Proof. Use Lemma 2.1.2, (2.7)–(2.9) to check. □

For the rest of this section let S_n denote the group of permutations on $\{\overline{1}, \overline{2}, \ldots, \overline{n}\}$. By Lemma 2.5.1 we may make the following definition.

Definition 2.5.2. Let $\beta : \mathbf{W} \to S_n$ denote the homomorphism defined by

$$\beta(G)(\overline{j}) = G\overline{j} \qquad (1 \leq j \leq n)$$

for $G \in \mathbf{W}$.

Lemma 2.5.3. $\beta : \mathbf{W} \to S_n$ *is an epimorphism.*

Proof. It follows that the $n-1$ transpositions $\beta(\kappa_{s_1}), \beta(\kappa_{s_2}), \ldots, \beta(\kappa_{s_{n-1}})$ generate S_n. □

Let O denote a subset of $\mathbb{F}_2^S$. We say that O is *closed under* $\mathbf{W}$ whenever $\mathbf{W}O \subseteq O$.

Proposition 2.5.4. *Let Z denote the subspace of* $\mathbb{F}_2^S$ *spanned by the set* $\{\overline{1}, \overline{2}, \ldots, \overline{n-1}\}$. *Then Z is closed under* $\mathbf{W}$.

Proof. Note that $a \in Z$ if and only if $\overline{n+1} \notin \Delta(a)$ for $a \in \mathbb{F}_2^S$. By Lemma 2.5.1 and (2.10), Z is closed under $\mathbf{W}$. $\square$

Corollary 2.5.5. *The representation κ is not irreducible.*

Proof. Immediate from Proposition 2.5.4 $\square$

For the rest of this section let Z denote as in Proposition 2.5.4. By Proposition 2.5.4, Z is a disjoint union of some $\mathbf{W}$-orbits of $\mathbb{F}_2^S$. It follows that $\mathbb{F}_2^S \setminus Z$ is also a disjoint union of some $\mathbf{W}$-orbits of $\mathbb{F}_2^S$. To find the $\mathbf{W}$-orbits of $\mathbb{F}_2^S$, we may divide this into the two cases: (i) the $\mathbf{W}$-orbits of $\mathbb{F}_2^S$ in Z; (ii) the $\mathbf{W}$-orbits of $\mathbb{F}_2^S$ in $\mathbb{F}_2^S \setminus Z$.

Proposition 2.5.6. *The $\mathbf{W}$-orbits of $\mathbb{F}_2^S$ are*

$$\begin{aligned} O_i &= \{a \in Z \mid ||a||_s = i \text{ or } n-i\} \qquad (0 \le i \le \lfloor \tfrac{n}{2} \rfloor), \\ \Omega_o &= \{a \in \mathbb{F}_2^S \setminus Z \mid ||a||_s \equiv 1 \text{ or } n-1 \pmod 2\}, \\ \Omega_e &= \{a \in \mathbb{F}_2^S \setminus Z \mid ||a||_s \equiv 0 \text{ or } n \pmod 2\}. \end{aligned}$$

In particular $\Omega_o = \Omega_e = \mathbb{F}_2^S \setminus Z$ when n is odd.

Proof. The proof is similar to the proof of Proposition 2.4.5. The reason that O_i is a $\mathbf{W}$-orbit of $\mathbb{F}_2^S$ follows from two facts: (i) $\beta(\kappa_{s_1})$, $\beta(\kappa_{s_2})$, …, $\beta(\kappa_{s_{n-2}})$ generate the subgroup S_{n-1} of S_n consisting of permutations on $\Delta \setminus \{\overline{n+1}\}$ and S_{n-1} acts transitively on fixed size subsets of $\Delta \setminus \{\overline{n+1}\}$; (ii)

$$\kappa_{s_{n-1}}\overline{n-1} = \kappa_{s_n}\overline{n-1} = \overline{n} = \overline{1} + \overline{2} + \cdots + \overline{n-1}$$

by Lemma 2.5.1(i), (ii) and (2.10). The reason that Ω_o and Ω_e are orbits follows from an additional fact that $||\kappa_{s_n}\overline{n+1}||_s = ||\overline{1} + \overline{2} + \cdots + \overline{n-2} + \overline{n+1}||_s = n-1$. $\square$

From now on we view Z as an additive group. Let $\mathrm{Aut}(Z)$ denote the group consisting of all automorphisms of Z. We now study the structure of $\mathbf{W}$.

Definition 2.5.7. Let $\gamma : \mathbf{W} \to \mathrm{Aut}(Z)$ denote the homomorphism defined by

$$\gamma(G)(u) = Gu$$

for $u \in Z$ and $G \in \mathbf{W}$.

Lemma 2.5.8. *There exists a unique homomorphism $\theta : S_n \to \mathrm{Aut}(Z)$ such that $\gamma = \theta \circ \beta$.*

Proof. Since β is surjective, it suffices to show that the kernel of β is contained in the kernel of γ. Suppose $G \in \mathrm{Ker}\,\beta$. Then $G\overline{i} = \overline{i}$ for $1 \le i \le n$. It follows that G fixes each element of Z. Therefore $G \in \mathrm{Ker}\,\gamma$. The result follows. $\square$

In view of Lemma 2.5.8 we can define the (external) semidirect product of Z and S_n with respect to θ (for example see [23, p.155]). We denote this group by $Z \rtimes_\theta S_n$. This group is the set $Z \times S_n$ with the group operation defined by

$$(u, \sigma)(v, \kappa) = (u + \theta(\sigma)(v), \sigma\kappa),$$

where $u, v \in Z$ and $\sigma, \kappa \in S_n$. Note that $\overline{n+1} + G\overline{n+1} \in Z$ for any $G \in \mathbf{W}$ by Lemma 2.5.1. By the above comment we can define a map as follows.

Definition 2.5.9. Let $\delta : \mathbf{W} \to Z \rtimes_\theta S_n$ denote the map defined by

$$\delta(G) = (\overline{n+1} + G\overline{n+1}, \beta(G))$$

for $G \in \mathbf{W}$.

Lemma 2.5.10. *The map $\delta : \mathbf{W} \to Z \rtimes_\theta S_n$ is a group monomorphism.*

Proof. For $G, H \in \mathbf{W}$,

$$\begin{aligned}
\delta(G)\delta(H) &= (\overline{n+1} + G\overline{n+1}, \beta(G))(\overline{n+1} + H\overline{n+1}, \beta(H)) \\
&= (\overline{n+1} + G\overline{n+1} + \theta(\beta(G))(\overline{n+1} + H\overline{n+1}),\ \beta(G)\beta(H)) \\
&= (\overline{n+1} + G\overline{n+1} + G(\overline{n+1} + H\overline{n+1}),\ \beta(G)\beta(H)) \\
&= (\overline{n+1} + GH\overline{n+1},\ \beta(GH)) \\
&= \delta(GH).
\end{aligned}$$

This shows that δ is a homomorphism. Let $G \in \operatorname{Ker}\delta$. Since $G\overline{n+1} = \overline{n+1}$ and $G \in \operatorname{Ker}\beta$, G fixes all vectors in Δ and so $G = I$. This shows that δ is injective. The result follows. $\square$

Note that $Z = \overline{n+1} + \Omega_o$ if n is odd, and $Z = (\overline{n+1} + \Omega_o) \cup (\overline{n+1} + \Omega_e)$ if n is even.

Lemma 2.5.11. $\delta(\mathbf{W}) = (\overline{n+1} + \Omega_o) \rtimes_\theta S_n$. *Moreover $\delta(\mathbf{W}) = Z \rtimes_\theta S_n$ if n is odd, and $\delta(\mathbf{W})$ has index 2 in $Z \rtimes_\theta S_n$ if n is even.*

Proof. Note that $\delta(\kappa_{s_1}), \delta(\kappa_{s_2}), \ldots, \delta(\kappa_{s_{n-1}})$ generate $\{0\} \rtimes_\theta S_n$. By this and since Ω_o is an orbit containing $\overline{n+1}$, it follows that $\delta(\mathbf{W}) = (\overline{n+1} + \Omega_o) \rtimes_\theta S_n$. The second part follows from Proposition 2.5.6. $\square$

Proposition 2.5.12. *The representation κ is faithful when n is odd;* $\operatorname{Ker}\kappa$ *has order 2 when n is even. Moreover* $\operatorname{Ker}\kappa = Z(W)$.

Proof. Note that W is isomorphic to the semidirect product $Z \rtimes S_n$ of Z and S_n (for example see [21, p.42]). By Lemma 2.5.11, κ is faithful when n is odd, and $\operatorname{Ker}\kappa$ has order 2 when n is even. From Corollary 2.3.2, $Z(W) \subseteq \operatorname{Ker}\kappa$, and from the fact that a normal subgroup of order 2 is contained in the center, we have $\operatorname{Ker}\kappa \subseteq Z(W)$. $\square$

2.6 Lit-only σ-game on Γ and its induced subgraph

To help us study $\operatorname{Ker}\kappa$ in the case E_8, we now discuss some relations between the lit-only σ-game on Γ and an induced subgraph of Γ.

Let $J \subseteq S$. Let $\mathbf{W}_J$ denote the subgroup of $\mathbf{W}$ generated by the κ_s for all $s \in J$. Let W_J denote the subgroup of W generated by $s \in J$. It is well known that W_J is isomorphic to the Coxeter group of type $\Gamma[J]$ (For example see [21, Section 5.5]). Therefore we will use the same symbol W_J to express these two isomorphic groups. For $G \in \operatorname{Mat}_S(\mathbb{F}_2)$ let $G[J]$ denote the submatrix of G with rows and columns indexed by J.

Lemma 2.6.1. *Let the notation be as above. Let $\Gamma[J]$ denote the subgraph of Γ induced by J. Let $\mathbf{W}_J[J]$ denote the set of those $G[J] \in \operatorname{GL}_J(\mathbb{F}_2)$ where $G \in \mathbf{W}_J$. Then the following* (i), (ii) *hold.*

(i) $\mathbf{W}_J[J]$ *is the flipping group of* $\Gamma[J]$.

(ii) *The map* $\psi : \mathbf{W}_J \to \mathbf{W}_J[J]$ *defined by*

$$\psi(G) = G[J] \qquad \textit{for } G \in \mathbf{W}_J$$

is a surjective homomorphism.

Proof. By Definition 2.1.1, $(\kappa_s)_{uv} = 0$ for $s, u \in J$ and $v \in S \setminus J$. By this, each matrix $G \in \mathbf{W}_J$ has the form

$$G = \begin{pmatrix} A & \mathbf{0} \\ B & C \end{pmatrix}$$

if indices in J are placed in the beginning of rows and columns, where A is a $|J| \times |J|$ matrix, B is an $(n - |J|) \times |J|$ matrix, C is an $(n - |J|) \times (n - |J|)$ matrix, and $\mathbf{0}$ is a $|J| \times (n - |J|)$ zero matrix. Then (i), (ii) follows from the following matrix product rule in block form:

$$\begin{pmatrix} A & \mathbf{0} \\ B & C \end{pmatrix} \begin{pmatrix} A' & \mathbf{0} \\ B' & C' \end{pmatrix} = \begin{pmatrix} AA' & \mathbf{0} \\ BA' + CB' & CC' \end{pmatrix}.$$

□

By Theorem 2.2.2 there exists a unique representation $\kappa' : W_J \to \mathrm{GL}_J(\mathbb{F}_2)$ such that $\kappa'(s) = \kappa_s[J]$ for all $s \in J$.

Lemma 2.6.2. *Let the notation be as above. Then the following* (i), (ii) *hold.*

(i) $\kappa' = \psi \circ \kappa \restriction W_J$.

(ii) $\mathrm{Ker}\,\kappa \restriction W_J \subseteq \mathrm{Ker}\,\kappa'$.

Proof. Since $(\psi \circ \kappa)(s) = \kappa_s[J] = \kappa'(s)$ for all $s \in J$, it follows that $\kappa' = \psi \circ \kappa \restriction W_J$. This shows (i). (ii) immediate from Lemma 2.6.1(i) and (i). □

2.7 Lit-only σ-game on the Dynkin diagram of type E_n

In this section we assume that Γ is the graph in Figure 1.2. We shall give a description of $\mathbf{W}$-orbits of $\mathbb{F}_2^S$. Restricting to the case $n = 6, 7, 8$, we shall show that $\mathrm{Ker}\,\kappa = Z(W)$.

$E_n (n \geq 6)$

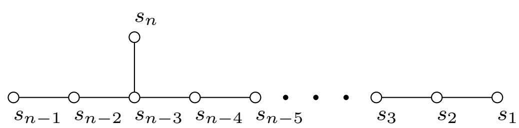

Figure 1.2: a finite simple graph E_n

Let $\overline{1} = e_{s_1}$, $\overline{i+1} = \kappa_{s_i} \kappa_{s_{i-1}} \cdots \kappa_{s_1} \overline{1}$ for $1 \leq i \leq n-1$ and $\overline{n+1} = e_{s_n}$. Note that

$$\begin{aligned} \overline{i} &= e_{s_i} + e_{s_{i-1}} \qquad (2 \leq i \leq n-3), \\ \overline{n-2} &= e_{s_{n-3}} + e_{s_{n-2}} + e_{s_n}, \\ \overline{n-1} &= e_{s_{n-2}} + e_{s_{n-1}} + e_{s_n}, \\ \overline{n} &= e_{s_{n-1}} + e_{s_n}. \end{aligned} \tag{2.11}$$

Set $\Delta = \Delta(E_n) := \{\overline{1}, \overline{2}, \dots, \overline{n}\}$ to be the simple basis of $\mathbb{F}_2^S$ in this case. Observe that

$$\overline{n+1} = \overline{1} + \overline{2} + \cdots + \overline{n}. \tag{2.12}$$

Set $\Delta(a)$ and $||a||_s = |\Delta(a)|$ as before for $a \in \mathbb{F}_2^S$. For example $\Delta(\overline{n+1}) = \Delta$ and $||\overline{n+1}||_s = n$.

Lemma 2.7.1. *The following* (i), (ii) *hold.*

(i) *For each* $1 \le i \le n-1$, $\kappa_{s_i}\overline{i} = \overline{i+1}$, $\kappa_{s_i}\overline{i+1} = \overline{i}$, *and*

$$\kappa_{s_i}\overline{j} = \overline{j} \qquad \text{for } \ \overline{j} \in \{\overline{1}, \overline{2}, \dots, \overline{n+1}\} \setminus \{\overline{i}, \overline{i+1}\}.$$

(ii) $\kappa_{s_n}\overline{n+1} = \overline{n-2} + \overline{n-1} + \overline{n}$, $\kappa_{s_n}\overline{n} = \overline{n-2} + \overline{n-1} + \overline{n+1}$, $\kappa_{s_n}\overline{n-1} = \overline{n-2} + \overline{n} + \overline{n+1}$, $\kappa_{s_n}\overline{n-2} = \overline{n-1} + \overline{n} + \overline{n+1}$ *and*

$$\kappa_{s_n}\overline{j} = \overline{j} \qquad \text{for } \ 1 \le j \le n-3.$$

Proof. Use Lemma 2.1.2 and (2.11) to check. □

For the rest of this section, let S_n denote the group of permutations on $\Delta = \{\overline{1}, \overline{2}, \dots, \overline{n}\}$ and let

$$T := \{s_1, s_2, \dots, s_{n-1}\}.$$

Recall that $\mathbf{W}_T$ is the subgroup of $\mathbf{W}$ generated by $\{\kappa_s \mid s \in T\}$. In view of Lemma 2.7.1 we may make a definition.

Definition 2.7.2. Let $\epsilon : \mathbf{W}_T \to S_n$ denote the homomorphism defined by

$$\epsilon(G)(\overline{j}) = G\overline{j} \qquad (1 \le j \le n)$$

for $G \in \mathbf{W}_T$.

Lemma 2.7.3. $\epsilon : \mathbf{W}_T \to S_n$ *is an isomorphism.*

Proof. It follows from that Δ is a spanning set and that the $n-1$ transpositions $\epsilon(\kappa_{s_1})$, $\epsilon(\kappa_{s_2}), \dots, \epsilon(\kappa_{s_{n-1}})$ generate S_n. □

Proposition 2.7.4. *The* $\mathbf{W}$*-orbits of* $\mathbb{F}_2^S$ *are*

$$\begin{aligned}
O_0 &= \{0\}, \\
O_1 &= \{a \in \mathbb{F}_2^S \mid a \ne 0, ||a||_s \equiv 1 \text{ or } n-2 \pmod 4\}, \\
O_2 &= \{a \in \mathbb{F}_2^S \mid a \ne 0, ||a||_s \equiv 2 \text{ or } n-3 \pmod 4\}, \\
O_3 &= \{a \in \mathbb{F}_2^S \mid a \ne 0, ||a||_s \equiv 3 \text{ or } n \pmod 4\}, \\
O_4 &= \{a \in \mathbb{F}_2^S \mid a \ne 0, ||a||_s \equiv 0 \text{ or } n-1 \pmod 4\}.
\end{aligned} \tag{2.13}$$

In particular $O_1 = O_3$ *when* $n \equiv 1 \pmod 4$, $O_1 = O_4$ *and* $O_2 = O_3$ *when* $n \equiv 2 \pmod 4$, $O_2 = O_4$ *when* $n \equiv 3 \pmod 4$, *and* $O_1 = O_2$ *and* $O_3 = O_4$ *when* $n \equiv 0 \pmod 4$.

Proof. It is clear that O_0 is a $\mathbf{W}$-orbit of $\mathbb{F}_2^S$. There are four cases to put nonzero vectors a, b in an orbit. (a) $||a||_s = ||b||_s$: this is because $\epsilon(\mathbf{W}_T) = S_n$ acts transitively on the fixed size subsets of Δ; (b) $||b||_s = n+3-||a||_s$ or $n-1-||a||_s$: this is from (a) and the observation that

$$\kappa_{s_n}||a||_s = \begin{cases} n+3-||a||_s & \text{if } |\Delta(a)\cap\{\overline{n},\overline{n-1},\overline{n-2}\}| = 3, \\ n-1-||a||_s & \text{if } |\Delta(a)\cap\{\overline{n},\overline{n-1},\overline{n-2}\}| = 1, \\ w(a) & \text{else} \end{cases} \tag{2.14}$$

by Lemma 2.7.1(ii) and (2.12); (c) $||a||_s = ||b||_s - 4$: this is by applying the first case of (2.14) and then applying the second case of (2.14); and (d) $||a||_s = ||b||_s + 4$: this is by applying the second case of (2.14) and then the first case of (2.14). The proposition follows from the above cases (a)–(d). □

For the rest of this section let O_i $(0 \le i \le 4)$ denote the sets from Proposition 2.7.4.

Proposition 2.7.5. *The representation κ is irreducible if and only if n is even.*

Proof. Immediate from Proposition 2.7.4. □

Corollary 2.7.6. *We have*

$$|O_1| = \begin{cases} 2^{n-1} - (-1)^{\frac{n}{4}} 2^{\frac{n-2}{2}} & \textit{if } n \equiv 0 \pmod 4, \\ 2^{n-1} & \textit{if } n \equiv 1 \pmod 4, \\ 2^{n-1} + (-1)^{\frac{n-2}{4}} 2^{\frac{n-2}{2}} - 1 & \textit{if } n \equiv 2 \pmod 4, \\ 2^{n-2} + (-1)^{\frac{n-3}{4}} 2^{\frac{n-3}{2}} & \textit{if } n \equiv 3 \pmod 4. \end{cases} \tag{2.15}$$

Proof. By (2.13) we have

$$|O_1| = \begin{cases} \displaystyle\sum_{\substack{k\equiv 1,2 (\bmod 4) \\ 1\le k\le n}} \binom{n}{k} & \text{if } n \equiv 0 \pmod 4, \\ \displaystyle\sum_{\substack{k\equiv 1 (\bmod 2) \\ 1\le k\le n}} \binom{n}{k} & \text{if } n \equiv 1 \pmod 4, \\ \displaystyle\sum_{\substack{k\equiv 0,1 (\bmod 4) \\ 1\le k\le n}} \binom{n}{k} & \text{if } n \equiv 2 \pmod 4, \\ \displaystyle\sum_{\substack{k\equiv 1 (\bmod 4) \\ 1\le k\le n}} \binom{n}{k} & \text{if } n \equiv 3 \pmod 4, \end{cases}$$

where $\binom{n}{k}$ is the binomial coefficient. From this we routinely prove (2.15) by induction on n. □

Let $a \in \mathbb{F}_2^S$. Recall that the *isotropy group of a in* $\mathbf{W}$ is $\{G \in \mathbf{W} \mid Ga = a\}$. By the elementary knowledge of group theory, the cardinality of the $\mathbf{W}$-orbit of a is equal to the index of the isotropy group of a in $\mathbf{W}$. For the rest of this section let

$$J := \{s_2, s_3, \ldots, s_n\}.$$

Observe that $\mathbf{W}_J$ is a subgroup of the isotropy group of e_{s_1} in $\mathbf{W}$ and that the $\mathbf{W}$-orbit of e_{s_1} is O_1. Therefore $|\mathbf{W}_J||O_1|$ divides $|\mathbf{W}|$.

Proposition 2.7.7. *Assume Γ is the Dynkin diagram of type E_6. Then* $\operatorname{Ker}\kappa = Z(W)$. *Moreover κ is faithful.*

Proof. By Corollary 2.7.6 we have $|O_1| = 27$. By Lemma 2.6.2(ii) and Proposition 2.5.12 (the case D_5), we know $|\mathbf{W}_J| = 2^4 5!$. Since $|\mathbf{W}_J||O_1|$ divides $|\mathbf{W}|$ we have $|\mathbf{W}| \geq 2^7 3^4 5$. By this and since $|W| = 2^7 3^4 5$ (for example see [21, p.44]), W is isomorphic to $\mathbf{W}$ and so $\operatorname{Ker}\kappa$ is trivial. By this and Corollary 2.3.2, $Z(W)$ is trivial. ☐

In order to show $\operatorname{Ker}\kappa = Z(W)$ in the cases E_7 and E_8, we cite [6, Lemma 10.2.11].

Lemma 2.7.8. ([6, Lemma 10.2.11]). *Assume that Γ is one of simply-laced Dynkin diagram of type E_7 or E_8. Then $Z(W) = \{1, w_\circ\}$.*

Proposition 2.7.9. *Assume Γ is the Dynkin diagram of type E_7. Then $\operatorname{Ker}\kappa = Z(W)$. Moreover $\operatorname{Ker}\kappa = \{1, w_\circ\}$.*

Proof. By Corollary 2.3.2 and Lemma 2.7.8, $|\operatorname{Ker}\kappa| \geq 2$. By this and since $|W| = 2^{10}3^4 5 \cdot 7$ (for example see [21, p.44]) we have $|\mathbf{W}| \leq 2^9 3^4 5 \cdot 7$. By Corollary 2.7.6 we have $|O_1| = 28$ and by Proposition 2.7.7 we have $|\mathbf{W}_J| = 2^7 3^4 5$. Since $|\mathbf{W}_J||O_1|$ divides $|\mathbf{W}|$ it follows that $|\mathbf{W}| \geq 2^9 3^4 5 \cdot 7$. Therefore $|\mathbf{W}| = 2^9 3^4 5 \cdot 7$ and this forces $|Z(W)| = |\operatorname{Ker}\kappa| = 2$. ☐

For the rest of this section we assume that Γ is the Dynkin diagram of type E_8. Let $u_\circ$ denote the longest element of W_J.

Lemma 2.7.10. $\kappa(u_\circ)\overline{8} = \overline{1} + \overline{8}$.

Proof. By Lemma 2.7.8, $u_\circ \in Z(W_J)$. Note that $T \cap J = \{s_2, s_3, \ldots, s_7\}$, and that $\kappa \restriction W_{T\cap J}$ is an isomorphism of $W_{T\cap J}$ onto $\mathbf{W}_{T\cap J}$ by Lemma 2.6.2(ii) and Proposition 2.4.7. Also $\epsilon \restriction \mathbf{W}_{T\cap J} : \mathbf{W}_{T\cap J} \to S_7$ is an isomorphism , where ϵ is from Definition 2.7.2 and S_7 is the group of permutations on $\{\overline{2}, \overline{3}, \ldots, \overline{8}\}$. Let

$$\begin{aligned} u'_\circ = {} & \kappa^{-1}(\epsilon^{-1}((\overline{2},\overline{8},\overline{3},\overline{7},\overline{4},\overline{6},\overline{5})))s_8\kappa^{-1}(\epsilon^{-1}((\overline{5},\overline{8})(\overline{4},\overline{7})(\overline{3},\overline{6})))s_8 \\ & \kappa^{-1}(\epsilon^{-1}((\overline{4},\overline{8})(\overline{3},\overline{7})(\overline{2},\overline{6})))s_8\kappa^{-1}(\epsilon^{-1}((\overline{5},\overline{8})(\overline{4},\overline{7})))s_8 \\ & \kappa^{-1}(\epsilon^{-1}((\overline{3},\overline{7})(\overline{2},\overline{6})))s_8. \end{aligned}$$

It is routine to check that the above $u'_\circ$ maps to $-I$ by the faithful representation defined in [11, p. 291] to conclude $u'_\circ = u_\circ$. Therefore $\kappa(u_\circ)$ equals

$$\begin{aligned} & \epsilon^{-1}((\overline{2},\overline{8},\overline{3},\overline{7},\overline{4},\overline{6},\overline{5}))\kappa_{s_8}\epsilon^{-1}((\overline{5},\overline{8})(\overline{4},\overline{7})(\overline{3},\overline{6}))\kappa_{s_8}\epsilon^{-1}((\overline{4},\overline{8})(\overline{3},\overline{7})(\overline{2},\overline{6}))\kappa_{s_8} \\ & \epsilon^{-1}((\overline{5},\overline{8})(\overline{4},\overline{7}))\kappa_{s_8}\epsilon^{-1}((\overline{3},\overline{7})(\overline{2},\overline{6}))\kappa_{s_8}. \end{aligned} \tag{2.16}$$

Applying (2.16) to $\overline{8}$ and using Lemma 2.7.1 and (2.12) for $n = 8$, the result follows. ☐

Lemma 2.7.11. *The restriction $\kappa \restriction W_J$ of κ to J is injective.*

Proof. Let κ' denote the corresponding representation from W_J into $\mathrm{GL}_J(\mathbb{F}_2)$. From Lemma 2.6.2(ii) and Proposition 2.7.7, we see that $\operatorname{Ker}\kappa \restriction W_J \subseteq \operatorname{Ker}\kappa' = \{1, u_\circ\}$. By Lemma 2.7.10, $u_\circ$ is not in $\operatorname{Ker}\kappa \restriction W_J$. Therefore $\operatorname{Ker}\kappa \restriction W_J$ is trivial and the result follows. ☐

We now can show $\operatorname{Ker}\kappa = Z(W)$ in the case E_8.

Proposition 2.7.12. *Assume that Γ is the Dynkin diagram of type E_8 then $\operatorname{Ker}\kappa = Z(W)$. Moreover $\operatorname{Ker}\kappa = \{1, w_\circ\}$.*

Proof. We have $|O_1| = 2^3 \cdot 3 \cdot 5$ from Corollary 2.7.6 and $|\mathbf{W}_J| = |W_J| = 2^{10}3^4 5 \cdot 7$ from Lemma 2.7.11. Note that $|W| = 2^{14}3^5 5^2 7$ (for example see [21, p.44]). It follows that $|\operatorname{Ker}\kappa| = 2$. By Corollary 2.3.2 and Lemma 2.7.8, $\operatorname{Ker}\kappa$ and $Z(W)$ are equal to $\{1, w_\circ\}$. ☐

2.8 Summary

We now summarize the main results of this chapter.

Theorem 2.8.1. *Let* Γ *denote a finite simple graph. Let* W *denote the Coxeter group of type* Γ. *Let* $\kappa : W \to \mathrm{GL}_S(\mathbb{F}_2)$ *denote the representation from Theorem 2.2.2. Then the following* (i), (ii) *are equivalent.*

(i) $\operatorname{Ker}\kappa = Z(W)$.

(ii) Γ *is a simply-laced Dynkin diagram.*

Proof. (i) $\Rightarrow$ (ii): Recall that $Z(W)$ has finite order, from below Corollary 2.3.2. By this and since $W/Z(W) \cong \mathbf{W}$ is finite, W has finite order. It is well-known that Γ is a simply-laced Dynkin diagram if and only if the Coxeter group W of type Γ is finite, for example see [21, p. 133]. Therefore (ii) follows.

(ii) $\Rightarrow$ (i): Immediate from Propositions 2.4.1, 2.4.8, 2.5.12, 2.7.7, 2.7.9, 2.7.12. $\square$

Remark 2.8.2. Theorem 2.8.1 is probably known to some experts on Lie algebras [3, 4, 5, 22].

simply-laced Dynkin diagrams	reducibility of κ	$\operatorname{Ker}\kappa$
A_n $(n \geq 1)$	κ is irr. iff $n = 1$ or n is even.	$\begin{cases} \{1, w_\circ\} & \text{if } n = 1, \\ \{1\} & \text{else.} \end{cases}$
D_n $(n \geq 4)$	κ is not irr.	$\begin{cases} \{1, w_\circ\} & \text{if } n \text{ is even,} \\ \{1\} & \text{else.} \end{cases}$
E_6	κ is irr.	$\{1\}$
E_7	κ is not irr.	$\{1, w_\circ\}$
E_8	κ is irr.	$\{1, w_\circ\}$

Table 1: the reducibility and the kernel of κ.

Γ	$\mathbf{W}$-orbits of $\mathbb{F}_2^S$
A_n $(n \geq 1)$	$O_i = \{a \in \mathbb{F}_2^S \mid \lvert\lvert a \rvert\rvert_s = i \text{ or } n+1-i\} \quad (0 \leq i \leq \lfloor \frac{n+1}{2} \rfloor)$.
D_n $(n \geq 4)$	$O_i = \{a \in Z \mid \lvert\lvert a \rvert\rvert_s = i \text{ or } n-i\} \quad (0 \leq i \leq \lfloor \frac{n}{2} \rfloor)$, $\Omega_o = \{a \in \mathbb{F}_2^S \setminus Z \mid \lvert\lvert a \rvert\rvert_s \equiv 1 \text{ or } n-1 \pmod 2\}$, $\Omega_e = \{a \in \mathbb{F}_2^S \setminus Z \mid \lvert\lvert a \rvert\rvert_s \equiv 0 \text{ or } n \pmod 2\}$, $\Omega_o = \Omega_e = \mathbb{F}_2^S \setminus Z$ when n is odd.
E_n $(n \geq 6)$	$O_0 = \{0\}$, $O_1 = \{a \in \mathbb{F}_2^S \mid a \neq 0,\ \lvert\lvert a \rvert\rvert_s \equiv 1 \text{ or } n-2 \pmod 4\}$, $O_2 = \{a \in \mathbb{F}_2^S \mid a \neq 0,\ \lvert\lvert a \rvert\rvert_s \equiv 2 \text{ or } n-3 \pmod 4\}$, $O_3 = \{a \in \mathbb{F}_2^S \mid a \neq 0,\ \lvert\lvert a \rvert\rvert_s \equiv 3 \text{ or } n \pmod 4\}$, $O_4 = \{a \in \mathbb{F}_2^S \mid a \neq 0,\ \lvert\lvert a \rvert\rvert_s \equiv 0 \text{ or } n-1 \pmod 4\}$. $O_1 = O_3$ when $n \equiv 1 \pmod 4$, $O_1 = O_4$ and $O_2 = O_3$ when $n \equiv 2 \pmod 4$, $O_2 = O_4$ when $n \equiv 3 \pmod 4$, $O_1 = O_2$ and $O_3 = O_4$ when $n \equiv 0 \pmod 4$.

Table 2: the $\mathbf{W}$-orbits of $\mathbb{F}_2^S$.

Chapter 3

Lit-only sigma-game on a graph with a long induced path

For $a \in \mathbb{F}_2^S$ let $||a||$ denote the number of *on* vertices of Γ that are assigned by a, and we call $||a||$ the *weight of* a. For a subset O of $\mathbb{F}_2^S$ define $||O||$ to be

$$\min_{a \in O} ||a||.$$

Motivated by a goal of lit-only σ-game, we consider the following numbers.

Definition 3.0.3. Let $k \geq 1$ denote an integer. We say that Γ is *k-lit for lit-only σ-game* whenever $||O|| \leq k$ for any W-orbit O of $\mathbb{F}_2^S$.

Definition 3.0.4. ([26]) Let $\mu(\Gamma)$ denote the minimum number k such that Γ is k-lit for lit-only σ-game. We call $\mu(\Gamma)$ the *minimum light number for lit-only σ-game on* Γ.

There are three known results about $\mu(\Gamma)$. If Γ is a simply-laced Dynkin diagram then $\mu(\Gamma) = 1$ (see [2] or [8]). If Γ is the graph E_n ($n \geq 6$) shown in Figure 1.2 then one can use Proposition 2.7.4 to check $\mu(\Gamma) = 1$. If Γ is a tree with ℓ leaves X. Wang and Y. Wu [26] prove $\mu(\Gamma) \leq \lceil \ell/2 \rceil$. In this chapter we consider an extension of simply-laced Dynkin diagrams: an n-vertex graph with an induced path of $n-1$ vertices. In Chapter 2 we studied the lit-only σ-game on a simply-laced Dynkin diagram with the help of a specific basis for $\mathbb{F}_2^S$. We extend the idea to this case. We shall find a criterion of $\mu(\Gamma)$ and give a description of $\mathbf{W}$-orbits of $\mathbb{F}_2^S$ for this case.

For the rest of this chapter we adopt the following assumption.

Assumption 3.0.5. Assume that $\Gamma = (S, R)$ is a simple connected graph whose vertex set $S = \{s_1, s_2, \ldots, s_n\}$ ($n \geq 2$). Suppose the sequence $s_1, s_2, \ldots, s_{n-1}$ forms an induced path in Γ. Let $j_1, j_2, \ldots, j_m$ ($m \geq 1$) denote a subsequence of $1, 2, \ldots, n-1$ such that $s_{j_1}, s_{j_2}, \ldots, s_{j_m}$ are all neighbors of s_n in Γ. See Figure 2.1.

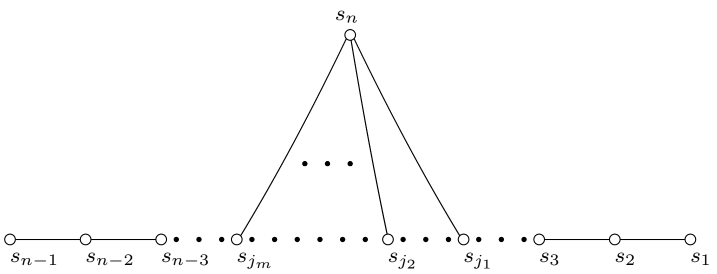

Figure 2.1: an n-vertex graph with an induced path of $n-1$ vertices.

3.1 The sets Π, Π_0 and Π_1

In this chapter let

$$\overline{1} = e_{s_1}, \quad \overline{i+1} = \kappa_{s_i}\kappa_{s_{i-1}}\cdots\kappa_{s_1}\overline{1} \quad (1 \le i \le n-1), \quad \overline{n+1} = e_{s_n}. \tag{3.1}$$

Let

$$\Pi \ = \{\overline{1}, \overline{2}, \ldots, \overline{n}\}, \tag{3.2}$$
$$\Pi_0 = \{\overline{i} \in \Pi \mid \overline{i}^t\, \overline{n+1} = 0\}, \tag{3.3}$$
$$\Pi_1 = \Pi \setminus \Pi_0. \tag{3.4}$$

For convenience let $e_{s_0} = 0$. From (3.1) and the construction,

$$\Pi_0 = \{\overline{i} \mid \overline{i} = e_{s_{i-1}} + e_{s_i},\ 1 \le i \le n-1 \text{ or } \overline{i} = e_{s_{n-1}}\},$$
$$\Pi_1 = \{\overline{i} \mid \overline{i} = e_{s_{i-1}} + e_{s_i} + e_{s_n},\ 1 \le i \le n-1 \text{ or } \overline{i} = e_{s_{n-1}} + e_{s_n}\}.$$

Note that $1 \le |\Pi_0|, |\Pi_1| \le n-1$ and $|\Pi_0| + |\Pi_1| = n$. For convenience let $j_{m+1} = n$ and $j_{m+2} = n$. Observe that

$$\Pi_0 = \{\overline{i} \in \Pi \mid i \in (0, j_1] \cup (j_2, j_3] \cup \cdots \cup (j_{2k}, j_{2k+1}]\}, \tag{3.5}$$
$$\Pi_1 = \{\overline{i} \in \Pi \mid i \in (j_1, j_2] \cup (j_3, j_4] \cup \cdots \cup (j_{2k-1}, j_{2k}]\}, \tag{3.6}$$

where $k = \lceil \frac{m}{2} \rceil$ and $(a, b] = \{x \mid x \in \mathbb{Z},\ a < x \le b\}$. We now establish some lemmas for later use.

Proposition 3.1.1.

$$|\Pi_1| = \sum_{k=1}^{\lceil \frac{m}{2} \rceil} j_{2k} - j_{2k-1}.$$

Proof. Immediate from (3.6). $\square$

For the rest of this chapter let

$$[\overline{i}] := \{\overline{1}, \overline{2}, \ldots, \overline{i}\} \qquad \text{for } i = 1, 2, \ldots, n.$$

Lemma 3.1.2. *For* $1 \le i \le n-1$ *we have*

$$\overline{1} + \overline{2} + \cdots + \overline{i} = \begin{cases} e_{s_i} + e_{s_n} & \text{if } |[\overline{i}] \cap \Pi_1| \text{ is odd,} \\ e_{s_i} & \text{if } |[\overline{i}] \cap \Pi_1| \text{ is even,} \end{cases}$$

and

$$\overline{1} + \overline{2} + \cdots + \overline{n} = \begin{cases} e_{s_n} & \text{if } |\Pi_1| \text{ is odd,} \\ 0 & \text{if } |\Pi_1| \text{ is even.} \end{cases}$$

Proof. Use (3.1). □

Lemma 3.1.3. $$\sum_{\overline{i} \in \Pi_0} \overline{i} = \sum_{k=1}^{m} e_{s_{j_k}}.$$

Proof. Use Lemma 3.1.2 and (3.5) to verify this. □

Lemma 3.1.4. $\kappa_{s_i}\overline{i} = \overline{i+1}$, $\kappa_{s_i}\overline{i+1} = \overline{i}$ *and* κ_{s_i} *fixes other vectors in* $\Pi \setminus \{\overline{i}, \overline{i+1}\}$ *for* $1 \le i \le n-1$.

Proof. Immediate from (3.1). □

For the rest of this chapter let S_n denote the symmetric group on Π. From Lemma 3.1.4, κ_{s_i} acts on Π as the transposition $(\overline{i}, \overline{i+1})$ in S_n for $1 \le i \le n-1$.

Corollary 3.1.5. *Let* U *denote the subspace of* $\mathbb{F}_2^S$ *spanned by the vectors in* Π. *Then* U *is closed under* $\mathbf{W}$.

Proof. By Lemma 3.1.4, U is closed under the action of $\kappa_{s_1}, \kappa_{s_2}, \ldots, \kappa_{s_{n-1}}$. For $\overline{i} \in \Pi$ we have

$$\kappa_{s_n}\overline{i} = \begin{cases} \overline{i} & \text{if } \overline{i} \in \Pi_0, \\ \overline{i} + \displaystyle\sum_{\overline{j} \in \Pi_0} \overline{j} & \text{if } \overline{i} \in \Pi_1 \end{cases}$$

by Lemma 3.1.3. It follows that $\kappa_{s_n}\overline{i}$ lies in U. The result follows. □

For the rest of this chapter let U denote the subspace of $\mathbb{F}_2^S$ from Corollary 3.1.5.

Proposition 3.1.6. *If* $|\Pi_1|$ *is odd then* Π *is a basis for* U; *if* $|\Pi_1|$ *is even then for any* $\overline{j} \in \Pi$, $\Pi \setminus \{\overline{j}\}$ *is a basis for* U. *Moreover* $e_{s_n} \notin U$ *if* $|\Pi_1|$ *is even.*

Proof. By Lemma 3.1.2, $\overline{1}, \overline{2}, \ldots, \overline{n-1}$ are linearly independent and hence U has dimension at least $n-1$. Since $e_{s_n} \notin \mathrm{Span}\{\overline{1}, \overline{2}, \ldots, \overline{n-1}\}$, the proposition follows from the second case of Lemma 3.1.2. □

For the rest of this chapter let P denote the subset of S consisting of $s_1, s_2, \ldots, s_{n-1}$. Recall that $\mathbf{W}_P$ denotes the subgroup of $\mathbf{W}$ generated by $\kappa_{s_1}, \kappa_{s_2}, \ldots, \kappa_{s_{n-1}}$.

Corollary 3.1.7. *The subgroup* $\mathbf{W}_P$ *of* $\mathbf{W}$ *is isomorphic to the symmetric group* S_n *on* Π.

Proof. Use Lemma 3.1.4, Proposition 3.1.6 and the fact $Ge_{s_n} = e_{s_n}$ for $G \in \mathbf{W}_P$. □

3.2 The simple basis Δ of $\mathbb{F}_2^S$

To better describe the $\mathbf{W}$-orbits of $\mathbb{F}_2^S$ we choose a specific basis of $\mathbb{F}_2^S$. Let

$$\Delta := \begin{cases} \Pi & \text{if } |\Pi_1| \text{ is odd,} \\ \Pi \cup \{\overline{n+1}\} \setminus \{\overline{n}\} & \text{if } |\Pi_1| \text{ is even.} \end{cases}$$

By Proposition 3.1.6, Δ is a basis of $\mathbb{F}_2^S$. We call Δ the *simple basis of* $\mathbb{F}_2^S$. For each $u \in \mathbb{F}_2^S$, u can be uniquely written as a linear combination of elements in Δ, so let $\Delta(u)$ denote the subset of Δ such that

$$u = \sum_{\bar{i} \in \Delta(u)} \bar{i}.$$

Let $||u||_s := |\Delta(u)|$. We refer to $||u||_s$ as the *simple weight of* u. Note that for $1 \le i \le n-1$, the vector $\overline{1} + \overline{2} + \cdots + \overline{i}$ has simple weight i but has weight

$$||\overline{1} + \overline{2} + \cdots + \overline{i}|| = \begin{cases} 1 & \text{if } |[\overline{i}] \cap \Pi_1| \text{ is even,} \\ 2 & \text{if } |[\overline{i}] \cap \Pi_1| \text{ is odd} \end{cases}$$

by Lemma 3.1.2.

In the next two sections we shall give a description of $\mathbf{W}$-orbits of $\mathbb{F}_2^S$. For convenience we adopt the following notation. For $V \subseteq \mathbb{F}_2^S$ and $T \subseteq \{0, 1, \ldots, n\}$ define

$$V_T := \{u \in V \mid ||u||_s \in T\}.$$

For shortness $V_{t_1, t_2, \ldots, t_i} := V_{\{t_1, t_2, \ldots, t_i\}}$ where $t_1, t_2, \ldots, t_i \in \{0, 1, \ldots, n\}$. Let *odd* denote the set of all odd integers among $\{0, 1, \ldots, n\}$.

3.3 The case $|\Pi_1|$ is odd

In this section we assume $|\Pi_1|$ to be odd and the counter part is treated in the next section. In this case $U = \mathbb{F}_2^S$ and so $\Delta = \{\overline{1}, \overline{2}, \ldots, \overline{n}\}$ is a basis of $\mathbb{F}_2^S$. By Lemma 3.1.2 we have

$$e_{s_i} = \begin{cases} \overline{1} + \overline{2} + \cdots + \overline{i} & \text{if } |[\overline{i}] \cap \Pi_1| \text{ is even,} \\ \overline{i+1} + \overline{i+2} + \cdots + \overline{n} & \text{if } |[\overline{i}] \cap \Pi_1| \text{ is odd,} \end{cases} \qquad (1 \le i \le n-1),$$

and

$$e_{s_n} = \overline{1} + \overline{2} + \cdots + \overline{n}.$$

Hence we have

$$||e_{s_i}||_s = \begin{cases} i & \text{if } |[\overline{i}] \cap \Pi_1| \text{ is even,} \\ n-i & \text{if } |[\overline{i}] \cap \Pi_1| \text{ is odd,} \end{cases} \qquad (1 \le i \le n-1)$$

and $||e_{s_n}||_s = n$. Therefore there exists a vector with simple weight i and weight 1 if and only if $|[\overline{i}] \cap \Pi_1|$ is even, $i = n$ or $|[\overline{n-i}] \cap \Pi_1|$ is odd. Let $[i] := \{1, 2, \ldots, i\}$ for $i = 1, 2, \ldots, n$. Let

$$K := \{i \in [n] \mid |[\overline{i}] \cap \Pi_1| \text{ is even, } i = n \text{ or } |[\overline{n-i}] \cap \Pi_1| \text{ is odd}\}. \qquad (3.7)$$

By Lemma 3.1.2, $||U_i|| \le 2$ for $1 \le i \le n$. Note that

$$||U_i|| = 1 \qquad \text{if and only if} \qquad i \in K. \qquad (3.8)$$

Lemma 3.3.1. *For $u \in \mathbb{F}_2^S$ we have*

$$\kappa_{s_n} u = \begin{cases} u & \text{if } |\Delta(u) \cap \Pi_1| \text{ is even,} \\ u + \sum_{\bar{i} \in \Pi_0} \bar{i} & \text{else.} \end{cases}$$

Moreover

$$||\kappa_{s_n} u||_s = \begin{cases} ||u||_s & \text{if } |\Delta(u) \cap \Pi_1| \text{ is even,} \\ n + 2k - |\Pi_1| - ||u||_s & \text{else,} \end{cases}$$

where $k = |\Pi_1 \cap \Delta(u)|$.

Proof. If $|\Delta(u) \cap \Pi_1|$ is even then $u^t e_{s_n} = 0$ and $\kappa_{s_n} u = u$ by construction. If $|\Delta(u) \cap \Pi_1|$ is odd, then

$$\begin{aligned} \kappa_{s_n} u &= u + \sum_{k=1}^{m} e_{s_{j_k}} \\ &= u + \sum_{\bar{i} \in \Pi_0} \bar{i} \end{aligned}$$

by Lemma 3.1.3, and $||\kappa_{s_n} u||_s = |\Delta(u) \cap \Pi_1| + (|\Pi_0| - |\Delta(u) \cap \Pi_0|) = n + 2k - |\Pi_1| - ||u||_s$. The result follows. □

Lemma 3.3.2. *The $\mathbf{W}_P$-orbits of $\mathbb{F}_2^S$ are $\{0\}$ and U_i for $1 \leq i \leq n$.*

Proof. Immediate from Corollary 3.1.7 and $\Delta = \Pi$. □

We now give a description of $\mathbf{W}$-orbits of $\mathbb{F}_2^S$ and characterize $\mu(\Gamma)$ in the case $3 \leq |\Pi_1| \leq n - 3$.

Theorem 3.3.3. *Assume that $3 \leq |\Pi_1| \leq n - 3$. Then the $\mathbf{W}$-orbits of $\mathbb{F}_2^S$ are $\{0\}$, U_{A_1}, U_{A_2}, U_{A_3}, U_{A_4}, where*

$$A_i := \{j \in [n] \mid j \equiv i, n + |\Pi_1| - i \pmod 4\}.$$

Moreover the number of $\mathbf{W}$-orbits of $\mathbb{F}_2^S$ is 3 if n is even and 4 if n is odd.

Proof. Fix an integer $1 \leq i \leq n$. By Lemma 3.3.2, U_i is a $\mathbf{W}$-orbit of $\mathbb{F}_2^S$. Note that $\mathbf{W}$ is the subgroup of $\mathrm{GL}_S(\mathbb{F}_2)$ generated by $\mathbf{W}_P$ and κ_{s_n}. By the above comments and by Lemma 3.3.1, the union of those $U_{i,n+2k-|\Pi_1|-i}$ forms a $\mathbf{W}$-orbit of $\mathbb{F}_2^S$, where k runs through possible odd integers $|\Pi_1 \cap \Delta(u)|$ for $u \in U_i$. In fact k is any odd number that satisfies $k \leq |\Pi_1|$ and $0 \leq i - k \leq |\Pi_0|$; equivalently

$$\max\{1, i + |\Pi_1| - n\} \leq k \leq \min\{|\Pi_1|, i\}. \tag{3.9}$$

Such an odd integer k exists for any $1 \leq i \leq n$, and note that

$$n + 2k - |\Pi_1| - i \equiv n + |\Pi_1| - i \pmod 4$$

since k and $|\Pi_1|$ are odd integers. To see the $\mathbf{W}$-orbits of $\mathbb{F}_2^S$ as stated in the theorem, it remains to show that $U_{i,i+4}$ is contained in a $\mathbf{W}$-orbit of $\mathbb{F}_2^S$ for $1 \leq i \leq n-4$. Set k to be the least odd integer greater than or equal to $\max\{1, i+|\Pi_1|-n+2\}$. For this k, (3.9) holds and then $U_{i,n+2k-|\Pi_1|-i}$ is contained in a $\mathbf{W}$-orbit of $\mathbb{F}_2^S$. Here we use the assumption $|\Pi_1| \leq n-3$ to guarantee the existence of such k. Replacing (i,k) by $(n+2k-|\Pi_1|-i, k+2)$ in (3.9) we have

$$\max\{1, 2k-i\} \leq k+2 \leq \min\{|\Pi_1|, n+2k-|\Pi_1|-i\}. \tag{3.10}$$

The above k and the assumption $3 \leq |\Pi_1|$ guarantee the equation (3.10). Since $n+2(k+2)-|\Pi_1|-(n+2k-|\Pi_1|-i) = i+4$ we have $U_{i+4,n+2k-|\Pi_1|-i}$ is contained in a $\mathbf{W}$-orbit of $\mathbb{F}_2^S$. Putting these together, $U_{i,i+4}$ is in a $\mathbf{W}$-orbit of $\mathbb{F}_2^S$. The result follows. □

Corollary 3.3.4. *Assume that* $3 \leq |\Pi_1| \leq n-3$. *Then*

$$\mu(\Gamma) = \begin{cases} 1 & \text{if } A_i \cap K \neq \emptyset \text{ for all } i, \\ 2 & \text{else,} \end{cases}$$

where K *is defined as* (3.7).

Proof. Use (3.8) and Theorem 3.3.3. □

We now consider the cases $|\Pi_1| = 1, n-2, n-1$.

Theorem 3.3.5. *Assume that* $|\Pi_1| = 1$, $n-2$ *or* $n-1$. *Then the* $\mathbf{W}$*-orbits of* $\mathbb{F}_2^S$ *are* $\{0\}$ *and*

$$\begin{cases} U_{i,n+1-i} & \text{if } |\Pi_1| = 1, \\ U_{odd}, U_{2j} & \text{if } |\Pi_1| = n-2, \\ U_{2i-1,2i} & \text{if } |\Pi_1| = n-1 \end{cases}$$

for $1 \leq i \leq \lceil \frac{n}{2} \rceil$ *and* $1 \leq j \leq (n-1)/2$. *Moreover the number of* $\mathbf{W}$*-orbits of* $\mathbb{F}_2^S$ *is*

$$\begin{cases} \lceil (n+2)/2 \rceil & \text{if } |\Pi_1| = 1, \\ (n+3)/2 & \text{if } |\Pi_1| = n-2, \\ (n+2)/2 & \text{if } |\Pi_1| = n-1. \end{cases}$$

Proof. As the proof in Theorem 3.3.3, $U_{i,n+2k-|\Pi_1|-i}$ is contained in a $\mathbf{W}$-orbit of $\mathbb{F}_2^S$, where k needs to satisfy (3.9). In the case $|\Pi_1| = 1$, $k=1$ is the only possible choice and hence $U_{i,n+1-i}$ is a $\mathbf{W}$-orbit of $\mathbb{F}_2^S$. In the case $|\Pi_1| = n-2$, we have $k = i-2$ or i if i is odd; $k = i-1$ if i is even. In the case $|\Pi_1| = n-1$, we have $k = i$ if i is odd; $k = i-1$ if i is even. In each of the remaining the proof follows similarly. □

Corollary 3.3.6. *Assume that* $|\Pi_1| = 1$, $n-2$ *or* $n-1$. *Then* $\mu(\Gamma) \leq 2$. *Moreover* $\mu(\Gamma) = 1$ *if and only if*

$$\begin{cases} \{i, n+1-i\} \cap K \neq \emptyset \quad \text{for } 1 \leq i \leq \lceil \frac{n}{2} \rceil & \text{if } |\Pi_1| = 1, \\ odd \cap K \neq \emptyset,\ U_{2j} \cap K \neq \emptyset \quad \text{for } 1 \leq j \leq \lfloor \frac{n}{2} \rfloor & \text{if } |\Pi_1| = n-2, \\ \{2i-1, 2i\} \cap K \neq \emptyset \quad \text{for } 1 \leq i \leq \lceil \frac{n}{2} \rceil & \text{if } |\Pi_1| = n-1, \end{cases}$$

where K *is defined as* (3.7).

Proof. Use (3.8) and Theorem 3.3.5. □

We end this section with an example.

Example 3.3.7. Let Γ be an odd cycle of length n; i.e. n is odd, $m=2$, $j_1 = 1$ and $j_2 = n-1$. Then $\Pi_0 = \{\overline{1}, \overline{n}\}$ and $\Pi_1 = \{\overline{2}, \overline{3}, \ldots, \overline{n-1}\}$. Note that $|\Pi_1| = n-2$ is odd, and $K = \{1, 3, \ldots, n\}$. By Theorem 3.3.5 we have the $\mathbf{W}$-orbits of $\mathbb{F}_2^S$ are

$$\{0\}, U_{odd}, U_0, U_2, U_4, \ldots, U_{n-1}.$$

By Corollary 3.3.6, $\mu(\Gamma) = 2$.

3.4 The case $|\Pi_1|$ is even

In this section we assume that $|\Pi_1|$ is even. In this case $\Delta = \Pi \cup \{\overline{n+1}\} \setminus \{\overline{n}\}$ is a basis for $\mathbb{F}_2^S$ and $\Delta \setminus \{\overline{n+1}\}$ is a basis for U. Recall that

$$\overline{1} + \overline{2} + \cdots + \overline{n} = 0 \tag{3.11}$$

Let $\overline{U} := \mathbb{F}_2^S \setminus U$. Note that $U_n = \emptyset$, $\overline{U} = \overline{n+1} + U$ and $\overline{U}_1 = \{\overline{n+1}\}$. By Lemma 3.1.2 we have

$$e_{s_i} = \begin{cases} \overline{1} + \overline{2} + \cdots + \overline{i} \in U & \text{if } |[\overline{i}] \cap \Pi_1| \text{ is even,} \\ \overline{1} + \overline{2} + \cdots + \overline{i} + \overline{n+1} \in \overline{U} & \text{if } |[\overline{i}] \cap \Pi_1| \text{ is odd,} \end{cases} \qquad (1 \leq i \leq n-1),$$

and

$$e_{s_n} = \overline{n+1} \in \overline{U}.$$

It follows that

$$||e_{s_i}||_s = \begin{cases} i & \text{if } |[\overline{i}] \cap \Pi_1| \text{ is even,} \\ i+1 & \text{if } |[\overline{i}] \cap \Pi_1| \text{ is odd,} \end{cases} \qquad (1 \leq i \leq n-1),$$

and $||e_{s_n}||_s = 1$. Therefore there exists a vector in U with simple weight i and weight 1 if and only if $|[\overline{i}] \cap \Pi_1|$ is even; there exists a vector in $\overline{U}$ with simple weight i and weight 1 if and only if $|[\overline{i-1}] \cap \Pi_1|$ is odd or $i = 1$. For the rest of this section let

$$K := \{i \in [n-1] \mid |[\overline{i}] \cap \Pi_1| \text{ is even}\}, \tag{3.12}$$
$$L := \{i \in [n] \mid |[\overline{i-1}] \cap \Pi_1| \text{ is odd or } i = 1\}. \tag{3.13}$$

Note that $||U_i||, ||\overline{U}_j|| \leq 2$ and that

$$\begin{aligned} ||U_i|| = 1 \qquad & \text{if and only if} \qquad i \in K, \\ ||\overline{U}_j|| = 1 \qquad & \text{if and only if} \qquad j \in L \end{aligned}$$

for $1 \leq i \leq n-1$ and $1 \leq j \leq n$.

Lemma 3.4.1. *For $u \in \mathbb{F}_2^S$ let $k = |\Pi_1 \cap \Delta(u)|$. Then the following* (i), (ii) *hold.*

(i) *For $u \in U$ we have*

$$\kappa_{s_n} u = \begin{cases} u & \text{if } |\Delta(u) \cap \Pi_1| \text{ is even,} \\ u + \sum\limits_{\overline{i} \in \Pi_0} \overline{i} & \text{else.} \end{cases}$$

Moreover

$$||\kappa_{s_n} u||_s = \begin{cases} ||u||_s & \text{if } |\Delta(u) \cap \Pi_1| \text{ is even,} \\ n + 2k - |\Pi_1| - ||u||_s & \text{if } |\Delta(u) \cap \Pi_1| \text{ is odd and } \overline{n} \in \Pi_1, \\ ||u||_s + |\Pi_1| - 2k & \text{else.} \end{cases}$$

(ii) *For $u \in \overline{U}$ we have*

$$\kappa_{s_n} u = \begin{cases} u & \text{if } |\Delta(u) \cap \Pi_1| \text{ is odd,} \\ u + \sum\limits_{\overline{i} \in \Pi_0} \overline{i} & \text{else.} \end{cases}$$

Moreover

$$||\kappa_{s_n} u||_s = \begin{cases} ||u||_s & \text{if } |\Delta(u) \cap \Pi_1| \text{ is odd,} \\ n + 2k + 2 - |\Pi_1| - ||u||_s & \text{if } |\Delta(u) \cap \Pi_1| \text{ is even and } \overline{n} \in \Pi_1, \\ ||u||_s + |\Pi_1| - 2k & \text{else.} \end{cases}$$

Proof. The proof is similar to the proof of Lemma 3.3.1, except that at this time since the choice of simple basis Δ is different, the action of κ_{s_n} on a vector is a little different, and we need to use (3.11) to adjust the simple weight of a vector. □

In view of Corollary 3.1.5 we discuss the $\mathbf{W}$-orbits (resp. $\mathbf{W}_P$-orbits) of $\mathbb{F}_2^S$ into the two parts, one in U and the other in $\overline{U}$.

Lemma 3.4.2. *The $\mathbf{W}_P$-orbits of $\mathbb{F}_2^S$ are $\{0\}$, $\overline{U}_1$, $\overline{U}_{i+1,n+1-i}$ and $U_{i,n-i}$ for $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$.*

Proof. By construction $\overline{U}_1 = \{e_{s_n}\}$ is a $\mathbf{W}_P$-orbit of $\mathbb{F}_2^S$. By Corollary 3.1.5 and Corollary 3.1.7, U_i is contained in a $\mathbf{W}_P$-orbit of U and $\overline{U}_{i+1}$ is in a $\mathbf{W}_P$-orbit of $\overline{U}$ for $1 \leq i \leq n-1$. By (3.11), $U_{i,n-i}$ is contained in a $\mathbf{W}_P$-orbit of $\mathbb{F}_2^S$ and $\overline{U}_{i+1,n+1-i}$ is in a $\mathbf{W}_P$-orbit of $\overline{U}$ for $1 \leq i \leq n-1$. Since no other ways to put these sets together the result follows. □

Theorem 3.4.3. *Assume that $4 \leq |\Pi_1| \leq n-3$. Then the $\mathbf{W}$-orbits of $\mathbb{F}_2^S$ are $\{0\}$, U_{B_1}, U_{B_2}, U_{B_3}, U_{B_4}, $\overline{U}_{C_1}$, $\overline{U}_{C_2}$, $\overline{U}_{C_3}$, $\overline{U}_{C_4}$, where*

$$B_i = \{j \in [n-1] \mid j \equiv i, i+|\Pi_1|-2, n-i, n-i+|\Pi_1|-2 \pmod 4\},$$
$$C_i = \{j \in [n] \mid j \equiv i, i+|\Pi_1|, n+2-i, n+2-i+|\Pi_1| \pmod 4\}.$$

Moreover the number of $\mathbf{W}$-orbits of $\mathbb{F}_2^S$ is 6 if n is even and 4 if n is odd.

Proof. We first determine the $\mathbf{W}$-orbits of U. By Lemma 3.4.2, $U_{i,n-i}$ is contained in a $\mathbf{W}$-orbit of U for $1 \leq i \leq n-1$. Suppose $\overline{n} \in \Pi_0$ and the case $\overline{n} \in \Pi_1$ is left to the reader. In this case $U_{i,i+|\Pi_1|-2k}$ is contained in a $\mathbf{W}$-orbit of U by Lemma 3.4.1(i), where $1 \leq i + |\Pi_1| - 2k \leq n-1$ and k runs through possible odd integers $|\Pi_1 \cap \Delta(u)|$ for $u \in U_i$. In fact k is any odd number that satisfies $k \leq |\Pi_1| - 1$ and $0 \leq i - k \leq |\Pi_0| - 1$; equivalently

$$\max\{1, i + |\Pi_1| - n + 1\} \leq k \leq \min\{|\Pi_1| - 1, i\}. \tag{3.14}$$

Such an odd k exists for any $1 \leq i \leq n-3$, and note that

$$i + |\Pi_1| - 2k \equiv i + |\Pi_1| - 2 \pmod 4.$$

To determine the $\mathbf{W}$-orbits of U, it remains to show that $U_{i,i+4}$ is contained in a $\mathbf{W}$-orbit of U for $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$. Suppose $4 \leq |\Pi_1| \leq 6$. Set $k = 1$ to conclude that $U_{i,i+2}$ in a $\mathbf{W}$-orbit of U if $|\Pi_1| = 4$; $U_{i,i+4}$ in a $\mathbf{W}$-orbit of U if $|\Pi_1| = 6$. Thus we suppose that $|\Pi_1| \geq 8$. Then $n \geq 11$ and $\lfloor \frac{n}{2} \rfloor \leq n - 6$. Set k to be the least odd integer greater than or equal to $\max\{1, i+|\Pi_1|-n+3\}$. For this k, (3.14) holds and then $U_{i,i+|\Pi_1|-2k}$ is contained in a $\mathbf{W}$-orbit of U. Here we use the assumption $|\Pi_1| \leq n-3$. Replacing (i,k) by $(i+|\Pi_1|-2k, |\Pi_1|-k-2)$ in (3.14) we have

$$\max\{1, i+2|\Pi_1|-2k-n+1\} \leq |\Pi_1|-k-2 \leq \min\{|\Pi_1|-1, i+|\Pi_1|-2k\}. \quad (3.15)$$

The above k, the assumption $4 \leq |\Pi_1|$ and $i \leq n-6$ guarantee the equation (3.15). Since $(i+|\Pi_1|-2k)+|\Pi_1|-2(|\Pi_1|-k-2) = i+4$ we have $U_{i+4,i+|\Pi_1|-2k}$ in a $\mathbf{W}$-orbit of U. Putting these together, $U_{i,i+4}$ is contained in a $\mathbf{W}$-orbit of U. Therefore the $\mathbf{W}$-orbits of U are U_{B_1}, U_{B_2}, U_{B_3}, U_{B_4}.

We next determine the $\mathbf{W}$-orbits of $\overline{U}$. Since the proof is similar to the above case, we only give a sketch. By Lemma 3.4.2, $\overline{U}_{i,n+2-i}$ is contained in a $\mathbf{W}$-orbit of $\overline{U}$ for $2 \leq i \leq n$. We suppose $\overline{n} \in \Pi_1$ and leave the case $\overline{n} \in \Pi_0$ to the reader. By Lemma 3.4.1(ii) we have $\overline{U}_{i,n+2k+2-|\Pi_1|-i}$ is contained in a $\mathbf{W}$-orbit of $\overline{U}$, where $k = |\Delta(u) \cap \Pi_1|$ is an even number for some $u \in \overline{U}_i$ and $1 \leq i \leq n-4$. By the same argument with replacing k by $k+2$ we find $\overline{U}_{i+4,n+2k+2-|\Pi_1|-i}$ is contained in a $\mathbf{W}$-orbit of $\overline{U}$. Therefore $\overline{U}_{i,i+4}$ is contained in a $\mathbf{W}$-orbit of $\overline{U}$. We have determined the $\mathbf{W}$-orbits of $\mathbb{F}_2^S$. The result follows. $\square$

Corollary 3.4.4. *Assume that $4 \leq |\Pi_1| \leq n-3$. Then*

$$\mu(\Gamma) = \begin{cases} 1 & \text{if } B_i \cap K \neq \emptyset \text{ and } C_i \cap L \neq \emptyset \text{ for all } i, \\ 2 & \text{else,} \end{cases}$$

where K and L are defined as (3.12) and (3.13), respectively.

Proof. Use (3.12), (3.13) and Theorem 3.4.3. $\square$

We now consider the cases $|\Pi_1| = 2$, $n-2$, $n-1$.

Theorem 3.4.5. *Assume that $|\Pi_1| = 2$, $n-2$ or $n-1$. Let the sets C_1, C_2 be as in Theorem 3.4.3. Then the $\mathbf{W}$-orbits of $\mathbb{F}_2^S$ are $\{0\}$ and*

$$\begin{cases} U_{i,n-i}, \overline{U}_{C_1}, \overline{U}_{C_2} & \text{if } |\Pi_1| = 2, \\ U_{odd}, U_{2j,n-2j}, \overline{U}_{odd}, \overline{U}_{2t,n+2-2t} & \text{if } |\Pi_1| = n-2, \\ U_{2j-1,2j,n-2j,n+1-2j}, \overline{U}_{2t-1,2t,n+2-2t,n+3-2t}, & \text{if } |\Pi_1| = n-1 \end{cases}$$

for $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$, $1 \leq j \leq \lceil \frac{n-2}{4} \rceil$ and $1 \leq t \leq \lceil \frac{n}{4} \rceil$. Moreover the number of $\mathbf{W}$-orbits of $\mathbb{F}_2^S$ is

$$\begin{cases} (n+6)/2 & \text{if } |\Pi_1| = 2 \text{ and } n \text{ is even, or } |\Pi_1| = n-2, \\ (n+3)/2 & \text{if } |\Pi_1| = 2 \text{ and } n \text{ is odd, or } |\Pi_1| = n-1. \end{cases}$$

Proof. The proof is similar to the proof of Theorem 3.3.5 that follows from the proof of Theorem 3.3.3. At this time, to determine the $\mathbf{W}$-orbits of U we check what values of odd k occur in (3.14) in each case of $|\Pi_1| \in \{2, n-2, n-1\}$. To determine the $\mathbf{W}$-orbits of $\overline{U}$, we do similarly as in the second part of the proof of Theorem 3.4.3. $\square$

Corollary 3.4.6. *Assume that $|\Pi_1| = 2$, $n-2$ or $n-1$. Then $\mu(\Gamma) \leq 2$. Moreover $\mu(\Gamma) = 1$ if and only if*

$$\{i, n-i\} \cap K \neq \emptyset \quad C_1 \cap L \neq \emptyset, \quad C_2 \cap L \neq \emptyset \qquad \text{for } 1 \leq i \leq \lfloor \tfrac{n}{2} \rfloor \qquad \text{if } |\Pi_1| = 2,$$

$$\left\{ \begin{array}{ll} odd \cap K \neq \emptyset, \ \{2j, n-2j\} \cap K \neq \emptyset & \text{for } 1 \leq j \leq \lceil \frac{n-2}{4} \rceil \\ odd \cap L \neq \emptyset, \ \{2t, n+2-2t\} \cap L \neq \emptyset & \text{for } 1 \leq t \leq \lceil \frac{n}{4} \rceil \end{array} \right. \qquad \text{if } |\Pi_1| = n-2,$$

$$\left\{ \begin{array}{ll} \{2j-1, 2j, n-2j, n+1-2j\} \cap K \neq \emptyset & \text{for } 1 \leq j \leq \lceil \frac{n-2}{4} \rceil \\ \{2t-1, 2t, n+2-2t, n+3-2t\} \cap L \neq \emptyset & \text{for } 1 \leq t \leq \lceil \frac{n}{4} \rceil \end{array} \right. \qquad \text{if } |\Pi_1| = n-1,$$

where K and L are defined as (3.12) and (3.13), respectively.

Proof. Use (3.12), (3.13) and Theorem 3.4.5. $\square$

We end this section with an example.

Example 3.4.7. Let Γ be an even cycle of length n; i.e. n is even, $m = 2$, $j_1 = 1$ and $j_2 = n-1$. Then $\Pi_0 = \{\overline{1}, \overline{n}\}$ and $\Pi_1 = \{\overline{2}, \overline{3}, \ldots, \overline{n-1}\}$. Note that $|\Pi_1| = n-2$ is even and $K = L = \{1, 3, \ldots, n-1\}$. By Theorem 3.4.5 we have the $\mathbf{W}$-orbits of $\mathbb{F}_2^S$ are

$$\{0\}, U_{odd}, U_{2,n-2}, U_{4,n-4}, \ldots, U_{2j,n-2j}, \overline{U}_{odd}, \overline{U}_{2,n}, \overline{U}_{4,n-2}, \ldots, \overline{U}_{2t,n-2t+2},$$

where $j = \lceil \frac{n-2}{4} \rceil$ and $t = \lceil \frac{n}{4} \rceil$. By Corollary 3.4.6, $\mu(\Gamma) = 2$.

3.5 Summary

In this section we list the main results of this chapter. Assume that $\Gamma = (S, R)$ is a simple connected graph whose vertex set $S = \{s_1, s_2, \ldots, s_n\}$ $(n \geq 2)$. Suppose the sequence $s_1, s_2, \ldots, s_{n-1}$ forms an induced path in Γ. Let $j_1, j_2, \ldots, j_m$ $(m \geq 1)$ denote a subsequence of $1, 2, \ldots, n-1$ such that $s_{j_1}, s_{j_2}, \ldots, s_{j_m}$ are all neighbors of s_n in Γ.

Let

$$\overline{1} = e_{s_1}, \quad \overline{i+1} = \kappa_{s_i}\kappa_{s_{i-1}} \cdots \kappa_{s_1}\overline{1} \quad (1 \leq i \leq n-1), \quad \overline{n+1} = e_{s_n}.$$

Let

$$\begin{aligned} \Pi &= \{\overline{1}, \overline{2}, \ldots, \overline{n}\}, \\ \Pi_0 &= \{\overline{i} \in \Pi \mid \overline{i}^t\, \overline{n+1} = 0\}, \\ \Pi_1 &= \Pi \setminus \Pi_0. \end{aligned}$$

For convenience let $j_{m+1} = n$. Recall from Proposition 3.1.1 that

$$|\Pi_1| = \sum_{k=1}^{\lceil \frac{m}{2} \rceil} j_{2k} - j_{2k-1}.$$

In particular $1 \leq |\Pi_1| \leq n-1$. Let

$$\Delta := \begin{cases} \Pi & \text{if } |\Pi_1| \text{ is odd,} \\ \Pi \cup \{\overline{n+1}\} \setminus \{\overline{n}\} & \text{if } |\Pi_1| \text{ is even.} \end{cases}$$

The set Δ is a basis for $\mathbb{F}_2^S$, and we call Δ the simple basis of $\mathbb{F}_2^S$. For $u \in \mathbb{F}_2^S$ let $||u||_s$ denote the simple weight of u; i.e. the number nonzero terms in writing u as a linear combination of elements in Δ. Let U denote the subspace spanned by the vectors in Π. For $V \subseteq \mathbb{F}_2^S$ and $T \subseteq \{0, 1, \ldots, n\}$, let $V_T := \{u \in V \mid ||u||_s \in T\}$. For shortness $V_{t_1,t_2,\ldots,t_i} := V_{\{t_1,t_2,\ldots,t_i\}}$. Let *odd* denote the set of all odd integers among $\{1, 2, \ldots, n\}$. For $1 \leq i \leq 4$ let

$$\begin{aligned} A_i &= \{j \in [n] \mid j \equiv i, n + |\Pi_1| - i \pmod 4\}, \\ B_i &= \{j \in [n-1] \mid j \equiv i, i + |\Pi_1| - 2, n - i, n - i + |\Pi_1| - 2 \pmod 4\}, \\ C_i &= \{j \in [n] \mid j \equiv i, i + |\Pi_1|, n + 2 - i, n + 2 - i + |\Pi_1| \pmod 4\}. \end{aligned}$$

Let $\mathbf{W}$ denote the flipping group of Γ. The $\mathbf{W}$-orbits of $\mathbb{F}_2^S$ are given in the following table according to all possible values of the pair $(|\Pi_1|, n)$.

$\lvert\Pi_1\rvert$	n	The $\mathbf{W}$-orbits of $\mathbb{F}_2^S$ (might be repeated)	The number of $\mathbf{W}$-orbits of $\mathbb{F}_2^S$
$3 \leq \lvert\Pi_1\rvert \leq n-3$, $\lvert\Pi_1\rvert$ is odd	even	$\{0\}, U_{A_j}$	3
$3 \leq \lvert\Pi_1\rvert \leq n-3$, $\lvert\Pi_1\rvert$ is odd	odd	$\{0\}, U_{A_j}$	4
$4 \leq \lvert\Pi_1\rvert \leq n-3$, $\lvert\Pi_1\rvert$ is even	even	$\{0\}, U_{B_j}, \overline{U}_{C_j}$	6
$4 \leq \lvert\Pi_1\rvert \leq n-3$, $\lvert\Pi_1\rvert$ is even	odd	$\{0\}, U_{B_j}, \overline{U}_{C_j}$	4
$\lvert\Pi_1\rvert = 1$		$\{0\}, U_{t,n+1-t}$	$\lceil (n+2)/2 \rceil$
$\lvert\Pi_1\rvert = 2$	even	$\{0\}, U_{i,n-i}, \overline{U}_{C_1}, \overline{U}_{C_2}$	$(n+6)/2$
$\lvert\Pi_1\rvert = 2$	odd	$\{0\}, U_{i,n-i}, \overline{U}_{C_1}, \overline{U}_{C_2}$	$(n+3)/2$
$\lvert\Pi_1\rvert = n-2$, $\lvert\Pi_1\rvert$ is odd	odd	$\{0\}, U_{odd}, U_{2i}$	$\frac{n+3}{2}$
$\lvert\Pi_1\rvert = n-2$, $\lvert\Pi_1\rvert$ is even	even	$\{0\}, U_{odd}, U_{2h,n-2h}$, $\overline{U}_{odd}, \overline{U}_{2g,n+2-2g}$	$\frac{n+6}{2}$
$\lvert\Pi_1\rvert = n-1$, $\lvert\Pi_1\rvert$ is odd	even	$\{0\}, U_{2t-1,2t}$	$\frac{n+2}{2}$
$\lvert\Pi_1\rvert = n-1$, $\lvert\Pi_1\rvert$ is even	odd	$\{0\}, U_{2h-1,2h,n-2h,,n+1-2h}$, $\overline{U}_{2g-1,2gn+2-2g,n+3-2g}$	$\frac{n+3}{2}$

where $1 \leq j \leq 4$, $1 \leq t \leq \lceil \frac{n}{2} \rceil$, $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$, $1 \leq h \leq \lfloor \frac{n-2}{4} \rfloor$, $1 \leq g \leq \lceil \frac{n}{4} \rceil$.

3.6 Remarks

In this final section we make a comment about the number of flipping groups of those Γ that satisfy Assumption 3.0.5.

Theorem 3.6.1. *The flipping group* $\mathbf{W}$ *of* Γ *is unique up to isomorphism among all the graphs that satisfy Assumption 3.0.5 with a given cardinality* $|\Pi_1|$ *computed from* (*3.1.1*).

Proof. Let $\Gamma' = (S', R')$ denote another graph satisfying Assumption 3.0.5. Let $S' = \{s'_1, \ldots, s'_n\}$. Let $\kappa_{s'_i}$ for all $s'_i \in S'$ denote the corresponding matrices in Definition 2.1.1. Let $\mathbf{W}'$ denote the flipping group of Γ'. Let $\overline{i'}$ $(1 \leq i \leq n+1)$, Π', Π'_0, Π'_1 denote the corresponding vectors and sets in (3.1)–(3.4). Assume $|\Pi_1| = |\Pi'_1|$. Define a linear isomorphism $\phi : \mathbb{F}_2^S \to \mathbb{F}_2^{S'}$ such that

$$\begin{array}{ll} \phi(\Pi_0) = \Pi'_0, \quad \phi(\Pi_1) = \Pi'_1 & \text{if } |\Pi_1| \text{ is odd,} \\ \phi(\Pi_0) = \Pi'_0, \quad \phi(\Pi_1) = \Pi'_1, \quad \phi\overline{n+1} = \overline{(n+1)'} & \text{if } |\Pi_1| \text{ is even.} \end{array}$$

Observe that $\phi^{-1}\kappa_{s'_n}\phi = \kappa_{s_n}$. By Corollary 3.1.7 $\kappa_{s'_i}$ for all $s'_i \in S'$ generate the symmetric group on Π'. It follows that $\phi^{-1}\kappa_{s'_i}\phi$ for all $s'_i \in S'$ generate the symmetric group on Π. By the above comments $\phi^{-1}\mathbf{W}'\phi = \mathbf{W}$ and the result follows. □

Corollary 3.6.2. *The number of flipping groups of those* Γ *that satisfy Assumption 3.0.5 is less than or equal to* $n-1$, *up to isomorphism.*

Proof. Immediate from Theorem 3.6.1. □

Chapter 4

One-lit trees for lit-only sigma-game

Motivated by the first result on the lit-only σ-game which is mentioned in Chapter 1, we are specially interested in the 1-lit trees. In general it is difficult to determine whether a tree is 1-lit for lit-only σ-game. In this chapter we will contribute two new classes of 1-lit trees.

4.1 The degenerate and nondegenerate graphs

Definition 4.1.1. Define a bilinear form $B : \mathbb{F}_2^S \times \mathbb{F}_2^S \to \mathbb{F}_2$ by

$$B(e_s, e_t) := \begin{cases} 1 & \text{if } st \in R, \\ 0 & \text{else} \end{cases} \tag{4.1}$$

for all $s, t \in S$.

For $a, b \in \mathbb{F}_2^S$ we say that a is *orthogonal to* b (*with respect to* B) whenever $B(a, b) = 0$. Let $\mathrm{rad}\,\mathbb{F}_2^S$ denote the subspace of $\mathbb{F}_2^S$ consisting of the vectors a that are orthogonal to all vectors. This subspace of $\mathbb{F}_2^S$ is called the *radical of* $\mathbb{F}_2^S$ (*relative to* B). The form B is said to be *degenerate* whenever $\mathrm{rad}\,\mathbb{F}_2^S \neq \{0\}$ and *nondegenerate* otherwise.

We distinguish finite simple graphs into two classes.

Definition 4.1.2. We say that Γ is *degenerate* whenever the form B is degenerate, and *nondegenerate* otherwise.

Definition 4.1.3. Let $\widehat{B}$ denote the matrix in $\mathrm{Mat}_S(\mathbb{F}_2)$ whose (s, t)-entry is $B(e_s, e_t)$.

Observe that $B(a, b) = a^t \widehat{B} b$ for all $a, b \in \mathbb{F}_2^S$ and that B is nondegenerate if and only if $\widehat{B}$ is nonsingular.

We now mention a graph-theoretical characterization of nondegenerate graphs. By a *matching in* $\Gamma = (S, R)$ we mean a subset of R in which no two edges share a vertex. By *a perfect matching in* $\Gamma = (S, R)$ we mean a matching in Γ that covers S.

Lemma 4.1.4. *The following* (i), (ii) *are equivalent.*

(i) Γ *is a nondegenerate graph.*

(ii) *The number of perfect matchings in* Γ *is odd.*

Proof. For a square matrix C let $\det C$ denote the determinant of C. Note that $\det \widehat{B} = 1$ if and only if B is nondegenerate. Let A denote the adjacency matrix of Γ (over the ring of integers $\mathbb{Z}$). Using the canonical map from $\mathbb{Z}$ to $\mathbb{F}_2$, we obtain $\det \widehat{B} = \det A \pmod 2$. By [13, Section 2.1], $\det A$ and the number of perfect matchings in Γ have the same parity. By the above comments the result follows. □

4.2 Some combinatorial properties of nondegenerate trees

In this section we mention some combinatorial properties of nondegenerate trees.

Proposition 4.2.1. *The following* (i), (ii) *are equivalent.*

(i) Γ *is a nondegenerate tree.*

(ii) Γ *is a tree with a perfect matching.*

Proof. Use Lemma 4.1.4 and observe that a tree contains at most one perfect matching. □

Example 4.2.2. The only nondegenerate tree of order at most 2 is a path of order 2.

Proof. It is routine to verify. □

Proposition 4.2.3. *If* $\Gamma = (S, R)$ *is a nondegenerate tree of order at least* 3, *then there exists a vertex of* Γ *with degree* 2.

Proof. Fix a leaf u of Γ. Let s denote a vertex of Γ farthest away from u in Γ. Observe s is a leaf of Γ. Let t denote the neighbor of s. We proceed by contradiction to show that t has degree 2 in Γ. Since the order of Γ is at least 3 the degree of t is at least 2. Suppose the degree of t is greater than 2. By our choice of s, at least one other neighbor of t is a leaf besides s. Thus there is no prefect matching in Γ, a contradiction to Proposition 4.2.1. □

4.3 The Reeder's game

In this section we mention another combinatorial game and introduce some related material. We call this game the *Reeder's game* because as far as we know, this game first appeared in one of Reeder's papers [24]. We start with the description of the Reeder's game.

The Reeder's game is a one-player game played on a graph. A *configuration of the Reeder's game on* Γ is an assignment of one of two states, *on* or *off*, to all vertices of Γ. Given a configuration, a *move of the Reeder's game on* Γ consists of choosing a vertex s and changing the state of s if the number of *on* neighbors of s is odd. Given a starting configuration, the goal is to minimize the number of *on* vertices of Γ by a finite sequence of moves of the Reeder's game on Γ.

For the rest of this chapter we interpret each configuration a of the Reeder's game on Γ as the vector

$$\sum_s e_s \tag{4.2}$$

of $\mathbb{F}_2^S$, where the sum is over all vertices s of Γ that are assigned the *on* state by a; if all vertices of Γ are assigned the *off* state by a then (5.6) is interpreted as zero vector. Observe that for any configuration $a \in \mathbb{F}_2^S$ of the Reeder's game on Γ, $e_s^t a = 1$ (resp. 0) means that the vertex s is assigned the *on* (resp. *off*) state by a.

Definition 4.3.1. For each $s \in S$ define a matrix $\tau_s \in \mathrm{Mat}_S(\mathbb{F}_2)$ by

$$\tau_s a := a + B(a, e_s)e_s \qquad \text{for all } a \in \mathbb{F}_2^S.$$

Observe that $\tau_s^2 = I$ and so $\tau_s \in \mathrm{GL}_S(\mathbb{F}_2)$ for all $s \in S$.

Lemma 4.3.2. *For each $s \in S$ we have*

$$B(a, e_s) = \sum_{st \in R} e_t^t a \qquad \textit{for all } a \in \mathbb{F}_2^S.$$

Proof. It is routine to verify this using (4.1). □

Fix a vertex s of Γ. Observe given any configuration $a \in \mathbb{F}_2^S$ of the Reeder's game on Γ, if the number of *on* neighbors of s is odd then $\tau_s a$ is obtained from a by changing the state of s; if the number of *on* neighbors of s is even then $\tau_s a = a$. Therefore we may view τ_s as the move of the Reeder's game on Γ for which we choose the vertex s and change the state of s if the number of *on* neighbors of s is odd.

The following theorem establishes a connection between the Reeder's game on Γ and the simply-laced Coxeter group W of type Γ.

Theorem 4.3.3. ([24, p.41]). *There exists a unique representation $\tau : W \to \mathrm{GL}_S(\mathbb{F}_2)$ such that $\tau(s) = \tau_s$ for all $s \in S$.*

For the rest of this chapter let τ denote as in Theorem 4.3.3.

We now give a dual relationship between the Reeder's game and the lit-only σ-game.

Proposition 4.3.4. *The representations $\kappa : W \to \mathrm{GL}_S(\mathbb{F}_2)$ and $\tau : W \to \mathrm{GL}_S(\mathbb{F}_2)$ are dual; i.e. $\kappa(w) = \tau(w^{-1})^t$ for all $w \in W$.*

Proof. Since S is a generating set of W and $s^{-1} = s$ in W for all $s \in S$, it suffices to show $\kappa_s = \tau_s^t$ for all $s \in S$. Let $u, v \in S$. Using Lemma 4.3.2 we find

$$\tau_s e_v = e_v + (\sum_{st \in R} e_t^t e_v)e_s. \tag{4.3}$$

Using (4.3), we find $(\tau_s)_{uv}$ equals 1 if and only if $u = v$, or $u = s$ and $uv \in R$. Comparing this with Definition 2.1.1, we have $\tau_s^t = \kappa_s$. The result follows. □

By Proposition 4.3.4 the image of W under τ is exactly the transpose $\mathbf{W}^t$ of $\mathbf{W}$. Observe for any $a, b \in \mathbb{F}_2^S$, b is obtained from a by a finite sequence of moves of the Reeder's game on Γ if and only if $b = Ga$ for some $G \in \mathbf{W}^t$. We now define the $\mathbf{W}^t$-orbits of $\mathbb{F}_2^S$, which are exactly the orbits of the Reeder's game on Γ.

Definition 4.3.5. Let $a \in \mathbb{F}_2^S$. By the $\mathbf{W}^t$-*orbit of* a we mean the set $\mathbf{W}^t a = \{Ga \mid G \in \mathbf{W}^t\}$. By a $\mathbf{W}^t$-*orbit of* $\mathbb{F}_2^S$ we mean a $\mathbf{W}^t$-orbit of a for some $a \in \mathbb{F}_2^S$.

There is a characterization for a $\mathbf{W}^t$-orbit of $\mathbb{F}_2^S$ which contains exactly one vector.

Lemma 4.3.6. *Let $a \in \mathbb{F}_2^S$. Then $\{a\}$ is a $\mathbf{W}^t$-orbit of $\mathbb{F}_2^S$ if and only if $a \in \operatorname{rad}\mathbb{F}_2^S$.*

Proof. By Definition 4.3.1, a is fixed by τ_s for all $s \in S$ if and only if $B(e_s, a) = 0$ for all $s \in S$. The latter condition is equivalent to $a \in \operatorname{rad}\mathbb{F}_2^S$. The result follows. □

4.4 Reeder's game on a nondegenerate tree

In this section we use [24, Theorem 7.3] to realize the $\mathbf{W}^t$-orbits of $\mathbb{F}_2^S$ for the case Γ is a nondegenerate tree and not a path. We begin with a quadratic form on $\mathbb{F}_2^S$.

Definition 4.4.1. Define a quadratic form $Q : \mathbb{F}_2^S \to \mathbb{F}_2$ by

$$Q(e_s) := 1 \qquad \text{for all } s \in S, \tag{4.4}$$
$$Q(a+b) := Q(a) + Q(b) + B(a,b) \qquad \text{for all } a, b \in \mathbb{F}_2^S. \tag{4.5}$$

We now recall a combinatorial interpretation for the form Q. For each $a \in \mathbb{F}_2^S$ define $\Gamma[a]$ to be the subgraph of Γ induced by the vertices s of Γ that assigned the *on* states by a in the Reeder's game.

Lemma 4.4.2. ([24, Section 1]). *Let $a \in \mathbb{F}_2^S$. Then $Q(a) = 1$ whenever the number of vertices in $\Gamma[a]$ plus the number of edges in $\Gamma[a]$ is odd, and $Q(a) = 0$ otherwise.*

Definition 4.4.3. Let $O(\mathbb{F}_2)$ denote the group consisting of all $\sigma \in \mathrm{GL}_S(\mathbb{F}_2)$ that satisfy $Q(\sigma a) = Q(a)$ for all $a \in \mathbb{F}_2^S$. This group is called the *orthogonal group* of $\mathbb{F}_2^S$ (relative to Q).

Definition 4.4.4. Let $\operatorname{Ker} Q$ denote the subspace of $\operatorname{rad}\mathbb{F}_2^S$ consisting of all $a \in \operatorname{rad}\mathbb{F}_2^S$ that satisfy $Q(a) = 0$. This is called the *kernel* of Q. The form Q is said to be *regular* whenever $\operatorname{Ker} Q = \{0\}$.

We now explain the roles of the two forms B and Q in the Reeder's game on Γ.

Proposition 4.4.5. *The following* (i), (ii) *hold.*

(i) $Q(\tau(w)a) = Q(a)$ *for all* $w \in W$ *and* $a \in \mathbb{F}_2^S$.

(ii) $B(\tau(w)a, \tau(w)b) = B(a,b)$ *for all* $w \in W$ *and* $a, b \in \mathbb{F}_2^S$.

Proof. (i) Since S is a generating set of W, it suffices to show $Q(\tau_s a) = Q(a)$ for all $s \in S$ and $a \in \mathbb{F}_2^S$. Let $s \in S$ and $a \in \mathbb{F}_2^S$ be given. Using Definition 4.3.1 and (4.5) we find

$$Q(\tau_s a) = Q(a) + Q(B(a, e_s)e_s) + B(a, e_s)^2. \tag{4.6}$$

By (4.4) and since $Q(0) = 0$ we find $Q(B(a, e_s)e_s) = B(a, e_s)^2$ whether $B(a, e_s)$ equals 0 or 1. It follows that the right-hand side of (4.6) is equal to $Q(a)$. The result follows.

(ii) In (4.5) we replace a and b by $\tau(w)a$ and $\tau(w)b$ respectively and simplify the resulting equation using (i) and (4.5). □

Corollary 4.4.6. *$\tau(W) = \mathbf{W}^t$ is a subgroup of $O(\mathbb{F}_2^S)$.*

Proof. Immediate from Proposition 4.4.5(i). □

Definition 4.4.7. Let $\mathcal{C}_0 := \{a \in \mathbb{F}_2^S \setminus \operatorname{rad}\mathbb{F}_2^S \mid Q(a) = 0\}$ and let $\mathcal{C}_1 := \{a \in \mathbb{F}_2^S \setminus \operatorname{rad}\mathbb{F}_2^S \mid Q(a) = 1\}$.

We now give sufficient conditions for $\mathcal{C}_0$ and $\mathcal{C}_1$ to be nonempty.

Lemma 4.4.8. *The following* (i), (ii) *hold.*

(i) *If Γ is a nondegenerate graph of order at least* 3 *then $\mathcal{C}_0$ is nonempty.*

(ii) *If Γ contains at least one edge then $\mathcal{C}_1$ is nonempty.*

Proof. (i) If there exist two vertices s, t of Γ with $st \notin R$, then we find $e_s + e_t \in \mathcal{C}_0$ using (4.4) and (4.5). Now suppose that any two vertices of Γ are neighbors. Pick any three vertices s, t, u of Γ. Using (4.4), (4.5) we find $e_s + e_t + e_u \in \mathcal{C}_0$. The result follows.

(ii) Let $s \in S$ for which there is $t \in S$ such that $st \in R$. By (4.1), $e_s \notin \operatorname{rad}\mathbb{F}_2^S$. By this and (4.4), $\alpha_s \in \mathcal{C}_1$. □

We now explain the roles of $\mathcal{C}_0$, $\mathcal{C}_1$ in the Reeder's game on Γ.

Lemma 4.4.9. *The sets $\mathcal{C}_0$ and $\mathcal{C}_1$ are closed under $\mathbf{W}^t$.*

Proof. Immediate from Lemma 4.3.6 and Proposition 4.4.5(i). □

Lemma 4.4.10. ([24, Theorem 7.3]). *Assume that $\Gamma = (S, R)$ is a tree and not a path, and that the quadratic form Q is regular. Then $\tau(W) = O(\mathbb{F}_2)$.*

Corollary 4.4.11. *Assume that $\Gamma = (S, R)$ is a nondegenerate tree and not a path. Then the $\mathbf{W}^t$-orbits of $\mathbb{F}_2^S$ are $\{0\}$, $\mathcal{C}_0$ and $\mathcal{C}_1$.*

Proof. Since Γ is nondegenerate, $\operatorname{rad}\mathbb{F}_2^S = \{0\}$ and so $\operatorname{Ker} Q = \{0\}$. Therefore $\tau(W) = O(\mathbb{F}_2^S)$ by Lemma 4.4.10. By this and applying Witt's extension theorem (for example, see [16, Theorem 12.10]), we find that for any $\alpha, \beta \in \mathcal{C}_0$ (resp. $\mathcal{C}_1$) there exists $w \in W$ such that $\tau(w)\alpha = \beta$. Since Γ is a nondegenerate tree and not a path, it follows from Example 4.2.2 that the order of Γ is at least 3. Therefore $\mathcal{C}_0$ and $\mathcal{C}_1$ are nonempty by Lemma 4.4.8. Combining the above comments with Lemma 4.4.9, we find the $\mathbf{W}^t$-orbits of $\mathbb{F}_2^S$ are $\{0\}$, $\mathcal{C}_0$ and $\mathcal{C}_1$. □

4.5 Lit-only σ-game on a nondegenerate tree

In this section we show that nondegenerate trees are 1-lit for lit-only σ-game. We begin with some lemmas.

Lemma 4.5.1. *For each $s \in S$ we have*

$$\widehat{B}\, e_s = \sum_{st \in R} e_t.$$

Proof. Immediate from Lemma 4.3.2 and Definition 4.1.3. □

Lemma 4.5.2. $\kappa(w)\widehat{B} = \widehat{B}\tau(w)$ *for all* $w \in W$.

Proof. Replacing b by $\tau(w^{-1})b$ in Proposition 4.4.5(ii), in terms of matrices we obtain

$$b^t \widehat{B}\tau(w)a = b^t \tau(w^{-1})^t \widehat{B}a \tag{4.7}$$

for all $a, b \in \mathbb{F}_2^S$. Therefore $\widehat{B}\tau(w) = \tau(w^{-1})^t\widehat{B}$. By Proposition 4.3.4 the result follows. □

Lemma 4.5.3. *Assume that Γ is nondegenerate. Let $w \in W$ and $a, b \in \mathbb{F}_2^S$. Then the following* (i), (ii) *are equivalent.*

(i) $b = \tau(w)a$.

(ii) $\widehat{B}b = \kappa(w)\widehat{B}a$.

Proof. Using Lemma 4.5.2, (ii) becomes

$$\widehat{B}b = \widehat{B}\tau(w)a.$$

Hence (i) implies (ii). Since Γ is nondegenerate $\widehat{B}$ is nonsingular. It follows that (ii) implies (i). □

Corollary 4.5.4. *Assume that Γ is a nondegenerate graph. Then the map from the $\mathbf{W}^t$-orbits of $\mathbb{F}_2^S$ to the $\mathbf{W}$-orbits of $\mathbb{F}_2^S$ defined by*

$$O \mapsto \widehat{B}O \qquad \textit{for any } \mathbf{W}^t\textit{-orbit } O \textit{ of } \mathbb{F}_2^S$$

is a bijection.

Proof. Use Lemma 4.5.3. □

Corollary 4.5.5. *Assume that Γ is a nondegenerate tree and not a path. Then the $\mathbf{W}$-orbits of $\mathbb{F}_2^S$ are $\{0\}$, $\widehat{B}\mathcal{C}_0$, $\widehat{B}\mathcal{C}_1$.*

Proof. Immediate from Corollary 4.4.11 and Corollary 4.5.4. □

Our last tool for proving the first result is [15, Theorem 6]. Here we offer a short proof.

Lemma 4.5.6. ([15, Theorem 6]). *Assume that Γ is a nondegenerate graph. Let $s \in S$ and let $a \in \mathbb{F}_2^S$ such that $e_s^t a = 0$. Then a and $a + \sum_{st\in R} e_t$ lie in distinct $\mathbf{W}$-orbits of $\mathbb{F}_2^S$.*

Proof. We proceed by contradiction. Suppose that there exists $G \in \mathbf{W}$ such that

$$Ga = a + \sum_{st \in R} e_t. \tag{4.8}$$

Let $w \in W$ such that $\kappa(w) = G$. So

$$\kappa(w)a = a + \sum_{st \in R} e_t. \tag{4.9}$$

Since Γ is nondegenerate, $\widehat{B}$ is nonsingular. Hence there exists a unique $b \in \mathbb{F}_2^S$ such that $\widehat{B}b = a$. Using this and Lemma 4.5.1 we find

$$a + \sum_{st \in R} e_t = \widehat{B}(b + e_s). \tag{4.10}$$

Substituting $a = \widehat{B}b$ and (4.10) into (4.9) and by Lemma 4.5.3 we find

$$\tau(w)b = b + e_s. \tag{4.11}$$

We now consider the Q-value on either side of (4.11). By Proposition 4.4.5(i) we find $Q(\tau(w)b)$ equals $Q(b)$. Since $\widehat{B}b = a$ and $e_s^t a = 0$ It follows that $B(e_s, b) = 0$. Using this and (4.4), (4.5) we find $Q(b + e_s)$ equals $Q(b) + 1$, a contradiction. $\square$

It is now a simple matter to prove that nondegenerate trees are 1-lit.

Theorem 4.5.7. *Assume that Γ is a nondegenerate tree. Then Γ is 1-lit for lit-only σ-game.*

Proof. Recall that all paths are 1-lit for lit-only σ-game. Thus we suppose that Γ is a nondegenerate tree and not a path; otherwise there is nothing to prove. By Corollary 4.5.5 there are exactly two nonzero $\mathbf{W}$-orbits of $\mathbb{F}_2^S$; i.e. $\widehat{B}\mathcal{C}_0$ and $\widehat{B}\mathcal{C}_1$. Therefore it suffices to show that there exist $u, v \in S$ such that e_u, e_v lie in distinct $\mathbf{W}$-orbits of $\mathbb{F}_2^S$. By Proposition 4.2.3 there exists a vertex s of Γ with degree 2. Let u, v denote the neighbors of s. Note that $e_s^t e_u = 0$ and

$$e_u + \sum_{st \in R} e_t = e_u + (e_u + e_v) = e_v.$$

Thus e_u and e_v are in distinct $\mathbf{W}$-orbits of $\mathbb{F}_2^S$ by applying Lemma 4.5.6 to e_s and e_u. The result follows. $\square$

We end this section with two examples. They give a degenerate tree and a nondegenerate graph which are not 1-lit for lit-only σ-game.

Example 4.5.8. The tree $\Gamma = (S, R)$ shown in Figure 3.2 is degenerate and not 1-lit for lit-only σ-game.

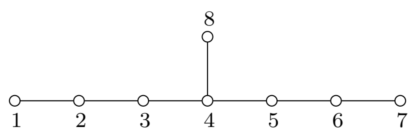

Figure 3.2: a degenerate tree is not 1-lit for lit-only σ-game.

Proof. There is no perfect matching in Γ. By Proposition 4.2.1, Γ is a degenerate tree. Using Theorem 3.4.3 we find that the $\mathbf{W}$-orbit of $e_1 + e_7$ doesn't contain $e_1, e_2, \ldots, e_8$. Therefore Γ is not 1-lit for lit-only σ-game. $\square$

Example 4.5.9. The graph $\Gamma = (S, R)$ shown in Figure 3.3 is nondegenerate and not 1-lit for lit-only σ-game.

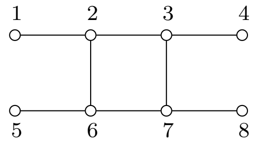

Figure 3.3: a nondegenerate graph is not 1-lit for lit-only σ-game.

Proof. $\{\{1,2\},\{3,4\},\{5,6\},\{7,8\}\}$ is the only perfect matching in Γ. By Lemma 4.1.4, Γ is nondegenerate. We now show Γ is not 1-lit for lit-only σ-game. To do this let $a = e_2 + e_3 + e_6 + e_7$ and let O denote the $\mathbf{W}$-orbit of a. It suffices to show $e_s \notin O$ for all $s = 1, 2, \ldots, 8$. Using Lemma 4.5.1 we find $b = e_1 + e_4 + e_5 + e_8$, $b_1 = e_2 + e_4 + e_5$, $b_2 = e_1$ such that $\widehat{B}b = a$, $\widehat{B}b_1 = e_1$, $\widehat{B}b_2 = e_2$. Using (4.4), (4.5) we find $b \in \mathcal{C}_0$ and $b_1, b_2 \in \mathcal{C}_1$. By Lemma 4.4.9(ii) b_1 and b_2 are not in the $\mathbf{W}$-orbit of b. By the above comments and Corollary 4.5.4 we find $e_1, e_2 \notin O$. By symmetry we obtain $e_s \notin O$ for all $s = 3, 4, \ldots, 8$. The result follows. $\square$

4.6 A homomorphism between simply-laced Coxeter groups

Before launching into the proof of the next result, we need a lemma about a homomorphism between simply-laced Coxeter groups.

For the rest of this chapter we adopt the following convention.

Definition 4.6.1. We assume that $\Gamma = (S, R)$ contains at least one edge. Fix $x, y \in S$ with $xy \in R$. We define $\Gamma' = (S', R')$ to be the simple graph obtained from Γ by inserting a new vertex z on the edge xy of Γ; i.e. z is a new symbol not in S, and the vertex and edge sets of Γ' are $S' = S \cup \{z\}$ and $R' = R \cup \{xz, yz\} \setminus \{xy\}$ respectively. Let W' denote the simply-laced Coxeter group of type Γ'; i.e. W' is the group generated by all elements of S' subject to the following relations

$$\begin{aligned} s^2 &= 1, && && (4.12)\\ (st)^2 &= 1 && \text{if } st \notin R', && (4.13)\\ (st)^3 &= 1 && \text{if } st \in R' && (4.14) \end{aligned}$$

for all $s, t \in S'$.

Lemma 4.6.2. *For each $u \in \{x, y\}$ there exists a unique homomorphism $\rho_u : W \to W'$ such that $\rho_u(u) = zuz$ and $\rho_u(s) = s$ for all $s \in S \setminus \{u\}$.*

Proof. Without loss of generality it suffices to show the uniqueness and existence of ρ_x. Since S is a generating set of W, if ρ_x exists then it is obviously unique. We now show the existence of ρ_x. By Definition 2.2.1 it suffices to check that for all $s, t \in S \setminus \{x\}$

$$
\begin{aligned}
s^2 &= 1, && && (4.15)\\
(st)^2 &= 1 && \text{if } st \notin R, && (4.16)\\
(st)^3 &= 1 && \text{if } st \in R, && (4.17)\\
(zxz)^2 &= 1, && && (4.18)\\
(szxz)^2 &= 1 && \text{if } sx \notin R, && (4.19)\\
(szxz)^3 &= 1 && \text{if } sx \in R && (4.20)
\end{aligned}
$$

hold in W'. It is clear that (4.15)–(4.17) is immediate from (4.12)–(4.14) respectively. To obtain (4.18), evaluate the left-hand side of (4.18) using (4.12). It remains to verify (4.19), (4.20). Observe that for $s \in S \setminus \{x, y\}$

$$
\begin{aligned}
(sz)^2 &= 1, && && (4.21)\\
(sx)^2 &= 1 && \text{if } sx \notin R, && (4.22)\\
(sx)^3 &= 1 && \text{if } sx \in R, && (4.23)
\end{aligned}
$$

and

$$
\begin{aligned}
(yx)^2 &= 1, && (4.24)\\
(xz)^3 &= 1, && (4.25)\\
(yz)^3 &= 1 && (4.26)
\end{aligned}
$$

hold in W' by (4.13) and (4.14). In what follows, the relation (4.12) will henceforth be used tacitly in order to keep the argument concise. Concerning (4.19), let $s \in S \setminus \{x\}$ with $sx \notin R$ be given. It is clear that $s \neq y$ in S since $yx \in R$. Hence (4.21) and (4.22) hold. From these we find s commutes with z and x in W', respectively. It follows that the left-hand side of (4.19) equals $(zxz)^2$. Now (4.19) follows from (4.18). To verify (4.20) we divide the argument into the following two cases. (I) $s \in S \setminus \{x, y\}$ and $sx \in R$; (II) $s = y$ in S.

Case I: $s \in S \setminus \{x, y\}$ and $sx \in R$.
Observe (4.21) and (4.23) can be rewritten as $zsz = s$ and $xsxsx = s$, respectively. By the above two relations, we may simplify the left-hand side of (4.20) by replacing zsz with s twice and then replacing $xsxsx$ with s. This yields

$$(szxz)^3 = (sz)^2$$

in W'. Now it is immediate from (4.21). This completes the argument for Case I.

Case II: $s = y$ in S.
We shall show $(yzxz)^3 = 1$ in W'. Observe first that $zxz = xzx$ in W' by (4.25). By this it suffices to show

$$(yxzx)^3 = 1 \qquad (4.27)$$

in W'. By a similar argument to Case I one can show (4.27). We display the details as follows. Observe (4.24) and (4.26) can be rewritten as $xyx = y$ and $zyzyz = y$, respectively. By the above two relations, we may simplify the left-hand side of (4.27) by replacing xyx with y twice and then replacing $zyzyz$ with y. This yields $(yxzx)^3 = (yx)^2$ in W'. Now it is immediate from (4.24). We have shown (4.20) and the proof is complete. □

For the rest of this chapter let ρ_u $(u \in \{x, y\})$ denote as in Lemma 4.6.2.

4.7 More one-lit trees for lit-only σ-game

In this section we contribute more 1-lit trees for lit-only σ-game.

Definition 4.7.1. For $s \in S$ let e'_s denote the characteristic vector of s in $\mathbb{F}_2^{S'}$. For $s \in S'$ define a matrix $\kappa'_s \in \mathrm{Mat}_S(\mathbb{F}_2)$ as

$$(\kappa'_s)_{uv} = \begin{cases} 1 & \text{if } u = v, \text{ or } v = s \text{ and } uv \in R', \\ 0 & \text{else} \end{cases} \tag{4.28}$$

for all $u, v \in S'$.

Applying Theorem 2.2.2 to Γ' there exists a unique representation $\kappa' : W' \to \mathrm{GL}_{S'}(\mathbb{F}_2)$ such that $\kappa'(s) = \kappa'_s$ for all $s \in S'$. The image of W' under κ', denoted by $\mathbf{W}'$, is called the flipping group of Γ'.

Definition 4.7.2. Let $\alpha \in \mathbb{F}_2^{S'}$. By the $\mathbf{W}'$-*orbit of* α we mean the set $\mathbf{W}'\alpha = \{G\alpha \mid G \in \mathbf{W}'\}$. By a $\mathbf{W}'$-*orbit of* $\mathbb{F}_2^{S'}$ we mean a $\mathbf{W}'$-orbit of α for some $\alpha \in \mathbb{F}_2^{S'}$.

Definition 4.7.3. For each $u \in \{x, y\}$ define a matrix δ_u with rows indexed by S and column indexed by S' such that

$$(\delta_u)_{uz} = 1, \qquad (\delta_u)_{ss} = 1 \qquad \text{for all } s \in S$$

and other entries are 0.

Lemma 4.7.4. *For each $u \in \{x, y\}$ the null space of δ_u is $\{0, e'_u + e'_z\}$.*

Proof. From Definition 4.7.3 we find $\{0, e'_u + e'_z\}$ is contained in the null space of δ_u and the rank of δ_u is $|S|$. By rank-nullity theorem the result follows. □

Lemma 4.7.5. *For $u \in \{x, y\}$ and $s \in S$ we have*

$$\delta_u\Big(\sum_{st \in R'} e'_t\Big) = \begin{cases} e_x + e_y + \sum\limits_{ut \in R} e_t & \textit{if } s = u, \\ \sum\limits_{st \in R} e_t & \textit{if } s \neq u. \end{cases} \tag{4.29}$$

Proof. Observe that

$$\begin{aligned} \{t \in S' \mid xt \in R'\} &= \{t \in S \mid xt \in R\} \cup \{z\} \setminus \{y\}, \\ \{t \in S' \mid yt \in R'\} &= \{t \in S \mid yt \in R\} \cup \{z\} \setminus \{x\}, \\ \{t \in S' \mid st \in R'\} &= \{t \in S \mid st \in R\} \qquad \text{if } s \in S \setminus \{x, y\}. \end{aligned} \tag{4.30}$$

To get (4.29), evaluate the left-hand side of (4.29) using Definition 4.7.3 and (4.40). □

Lemma 4.7.6. *For any $u \in \{x, y\}$ and $w \in W$ we have*

$$\kappa(w)\ \delta_u = \delta_u\ \kappa'(\rho_u(w)).$$

Proof. Let $u \in \{x, y\}$ be given. Recall from Lemma 4.6.2 that $\rho_u(u) = zuz$ and $\rho_u(s) = s$ for all $s \in S \setminus \{u\}$. By this and since S is a generating set of W, it suffices to show

$$\kappa_u \delta_u = \delta_u \kappa'_z \kappa'_u \kappa'_z, \tag{4.31}$$

$$\kappa_s \delta_u = \delta_u \kappa'_s \qquad \text{for all } s \in S \setminus \{u\}. \tag{4.32}$$

We first verify (4.31). It suffices to show that for all $s \in S'$

$$(\kappa_u \delta_u) e'_s = (\delta_u \kappa'_z \kappa'_u \kappa'_z) e'_s. \tag{4.33}$$

To do this we divide the argument into the following two cases. (I) $s \in \{u, z\}$; (II) $s \in S' \setminus \{u, z\}$.

Case I: $s \in \{u, z\}$.
Using (4.28) we find $(\kappa'_z \kappa'_u \kappa'_z) e'_s$ equals

$$e'_s + e'_x + e'_y + \sum_{ut \in R'} e'_t.$$

By this and using Definition 4.7.3 and (4.29), we find the right-hand side of (4.33) equals

$$e_u + \sum_{ut \in R} e_t. \tag{4.34}$$

On the other hand, using Definitions 2.1.1 and 4.7.3 we find the left-hand side of (4.33) also equals (4.34). Hence (4.33) holds in this case.

Case II: $s \in S' \setminus \{u, z\}$.
Observe $\kappa_u\, e_s = e_s$, $\delta_u e'_s = e_s$, and $\kappa'_u e'_s = \kappa'_z e'_s = e'_s$ by Definitions 2.1.1, 4.7.3, and (4.28) respectively. Using these we find either side of (4.33) equals e_s, so (4.33) holds in this case. Thus we have shown (4.31).

Concerning (4.32), let $s \in S \setminus \{u\}$ be given. It suffices to show that for all $t \in S'$

$$(\kappa_s \delta_u) e'_t = (\delta_u \kappa'_s) e'_t. \tag{4.35}$$

Similar to above we consider the two cases. (III) $t \in \{u, z\}$; (IV) $t \in S' \setminus \{u, z\}$.

Case III: $t \in \{u, z\}$.
Observe $\kappa_s e_u = e_u$, $\delta_u e'_t = e_u$, and $\kappa'_s e'_t = e'_t$ by Definitions 2.1.1, 4.7.3, and (4.28) respectively. Using these we find either side of (4.35) equals e_u, so (4.35) holds in this case.

Case IV: $t \in S' \setminus \{u, z\}$.
Using (4.28) and (4.29), we find the right-hand side of (4.35) equals

$$e_t + e'^t_t e'_s \sum_{sv \in R} e_v. \tag{4.36}$$

Using Definitions 2.1.1 and 4.7.3, we find the left-hand side of (4.35) equals

$$e_t + e^t_t e_s \sum_{sv \in R} e_v. \tag{4.37}$$

Since $e^t_t e_s = e'^t_t e'_s$ and comparing (4.36) with (4.37), we find (4.35) holds in this case. Thus we have shown (4.32) and the result follows. $\square$

We are now ready to prove our second result.

Theorem 4.7.7. *Assume that $\Gamma = (S, R)$ is a nondegenerate tree and that $x, y \in S$ such that $xy \in R$ and $e_y \notin \mathbf{W}e_x$. Let Γ' denote the tree obtained from Γ by inserting a new vertex on the edge xy of Γ. Then Γ' is 1-lit for lit-only σ-game.*

Proof. Use the notation as in Sections 3.6 and 3.7. If Γ is a path then Γ' is also a path and we have mentioned all paths are 1-lit for lit-only σ-game. Thus we suppose Γ is a nondegenerate tree and not a path; otherwise there is nothing to prove. Let O denote any nonzero $\mathbf{W}'$-orbit of $\mathbb{F}_2^{S'}$. To see that Γ' is 1-lit for lit-only σ-game, it suffices to show that there exists $s \in S'$ such that $e'_s \in O$. We first claim that $\delta_x(O) \neq \{0\}$. We show this by contradiction. Suppose $\delta_x(O) = \{0\}$. By Lemma 4.7.4 and since $0 \notin O$, we find $e'_x + e'_z \in O$ and hence $\kappa'_z(e'_x + e'_z) = e'_y + e'_z \in O$. It follows that

$$\delta_x(e'_y + e'_z) = e_y + e_x \in \delta_x(O),$$

a contradiction to $\delta_x(O) = \{0\}$. We have shown $\delta_x O \neq \{0\}$. Thus there exists $\alpha \in O$ such that $\delta_x \alpha \neq 0$. Recall from Corollary 4.5.5 that there are exactly two nonzero $\mathbf{W}$-orbits of $\mathbb{F}_2^S$. By this and since e_x and e_y are in distinct $\mathbf{W}$-orbits of $\mathbb{F}_2^S$, there exists $s \in \{x, y\}$ such that $\delta_x(\alpha)$ and e_s are in the same $\mathbf{W}$-orbit of $\mathbb{F}_2^S$; i.e. there exists $w \in W$ such that

$$\kappa(w)\delta_x\alpha = e_s. \tag{4.38}$$

By Lemma 4.7.6 we find the left-hand side of (4.38) equals $\delta_x\kappa'(\rho_x(w))\alpha$. By this and using Definition 4.7.3 and Lemma 4.7.4, we find

$$\kappa'(\rho_x(w))\alpha = \begin{cases} e'_x \text{ or } e'_z & \text{if } s = x, \\ e'_y \text{ or } e'_x + e'_y + e'_z & \text{if } s = y. \end{cases} \tag{4.39}$$

If $s = x$, then it follows from (4.39) that e'_x or e'_z lies in O, and we are done. If $s = y$, then e'_y or e'_z lies in O by (4.39) and since $\kappa'_z(e'_x + e'_y + e'_z) = e'_z$. The proof is complete. $\square$

We end this section with an example of a tree obtained from a nondegenerate tree by inserting a new vertex on some edge which is not 1-lit for lit-only σ-game.

Example 4.7.8. Assume that $\Gamma = (S, R)$ is the tree shown in Figure 3.4. Then the following (i)–(iii) hold.

(i) Γ is a nondegenerate tree.

(ii) e_3 and e_6 are in the same $\mathbf{W}$-orbit of $\mathbb{F}_2^S$.

(iii) The tree Γ' shown in Figure 4 is not 1-lit for lit-only σ-game.

Figure 3.4: Γ is a nondegenerate tree and Γ' is not 1-lit for lit-only σ-game.

Proof. (i) The set $\{\{1,2\},\{3,6\},\{4,5\}\}$ is a perfect matching in Γ. By Proposition 4.2.1, Γ is a nondegenerate tree.

(ii) Using Lemma 4.5.1 we find $b_1 = e_6$ and $b_2 = e_1 + e_3 + e_5$ such that $\widehat{B}\, b_1 = e_3$ and $\widehat{B}\, b_2 = e_6$. Using (4.4), (4.5) we find $b_1, b_2 \in \mathcal{C}_1$. By Corollary 4.5.5, e_3 and e_6 are in the $\mathbf{W}$-orbit $\widehat{B}\mathcal{C}_1$ of $\mathbb{F}_2^S$, as desired.

(iii) By [9, Proposition 3.2], the $\mathbf{W}'$-orbit of $e_1' + e_5'$ doesn't contain $e_1', e_2', \ldots, e_7'$. Therefore Γ' is not 1-lit for lit-only σ-game. $\square$

4.8 Combinatorial statements of Theorems 4.5.7 and 4.7.7

In order to easily execute Theorems 4.5.7 and 4.7.7, the goal of this section is to state the combinatorial versions of those results.

By Proposition 4.2.1 we restate Theorem 4.5.7 as follows.

Theorem 4.8.1. *Assume that Γ is a tree with a perfect matching. Then Γ is 1-lit.*

Assume that $\Gamma = (S, R)$ is a tree with a perfect matching $\mathcal{P}$. By an *alternating path in Γ (with respect to $\mathcal{P}$)*, we mean a path in which the edges belong alternatively to $\mathcal{P}$ and not to $\mathcal{P}$.

Definition 4.8.2. Assume $\Gamma = (S, R)$ is a tree with a perfect matching $\mathcal{P}$. For each $s \in S$ define A_s to be the set consisting of all $t \in S \setminus \{s\}$ such that the path between s and t is an alternating path which starts from and ends on edges in $\mathcal{P}$. For each $s \in S$ we say that A_s has *even parity* whenever the cardinality of A_s is even and *odd parity* otherwise.

Lemma 4.8.3. *Assume $\Gamma = (S, R)$ is a tree with a perfect matching $\mathcal{P}$. Let A_s $(s \in S)$ be as in Definition 4.8.2. Let $s \in S$. Then $e_s \in \widehat{B}\mathcal{C}_0$ whenever A_s has even parity, and $e_s \in \widehat{B}\mathcal{C}_1$ whenever A_s has odd parity*

Proof. For each $s \in S$ let

$$b_s = \sum_{t \in A_s} e_t.$$

Since no edges between any two vertices in A_s and using (4.4), (4.5) we find $b_s \in \mathcal{C}_0$ (resp. $\mathcal{C}_1$) if A_s has even (resp. odd) parity. Let $s \in S$ be given. Let $t \in S$ for which $st \in \mathcal{P}$. Observe that A_s equals the disjoint union of $\{t\}$ and these sets A_u for all $u \in S \setminus \{s\}$ with $ut \in R$. By this and by induction on the cardinality of A_s, it easily follows that $\widehat{B} b_s = e_s$ for all $s \in S$. The result follows. $\square$

By Corollary 4.5.5 and Lemma 4.8.3, we restate Theorem 4.7.7 as follows.

Theorem 4.8.4. *Assume that $\Gamma = (S, R)$ is a tree with a perfect matching. Let $x, y \in S$ such that $xy \in R$. Let Γ' denote the tree obtained from Γ by inserting a new vertex on the edge xy of Γ. Assume that A_x, A_y, defined as Definition 4.8.2, have distinct parities. Then Γ' is 1-lit.*

We now illustrate Theorems 4.8.1 and 4.8.4 with two examples.

Example 4.8.5. Assume that $\Gamma = (S, R)$ is the tree shown in Figure 5.

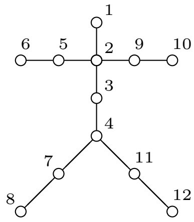

Figure 5: a tree of order 12.

Since Γ contains the perfect matching $\{\{1,2\},\{3,4\},\{5,6\},\{7,8\},\{9,10\},\{11,12\}\}$ and by Theorem 4.8.1, Γ is 1-lit. We next show that any tree obtained from Γ by inserting a new vertex on an edge of Γ is 1-lit. To see this, it suffices to show the four trees shown in Figure 6 are 1-lit.

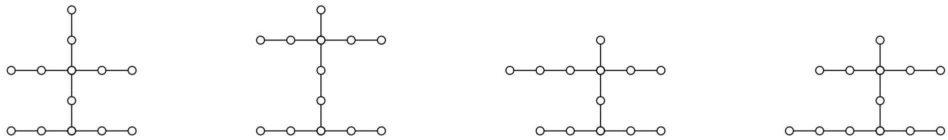

Figure 6: four 1-lit trees of order 13.

Let $D = \{\{1,2\},\{3,4\},\{5,6\},\{4,7\}\}$. Observe that

$$\begin{aligned} &A_1 = \{2,4,6,8,10,12\}, \quad A_2 = \{1\}, \quad A_3 = \{4,8,12\}, \\ &A_4 = \{1,3\}, \quad A_5 = \{6\}, \quad A_6 = \{1,5\}, \quad A_7 = \{8\}. \end{aligned} \tag{4.40}$$

Pick any $xy \in D$. By (4.40) and by Theorem 4.8.4 the tree obtained from Γ by inserting a new vertex on the edge xy of Γ is 1-lit. Therefore the four trees in Figure 6 are 1-lit. $\square$

Example 4.8.6. The aim of this example is to show that the trees shown in class IV of Figure 1 are 1-lit by using Theorem 4.8.1 and Theorem 4.8.4. Let $k \geq 3$ be an integer. Suppose that $\Gamma = (S, R)$ is the tree of order $2k$ shown in Figure 7. Let P denote the path in Γ between the two vertices 2 and $2k$. It suffices to show that Γ and the tree obtained from Γ by inserting a new vertex on some edge of P are 1-lit.

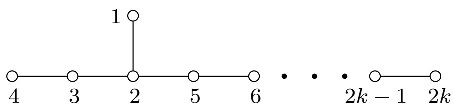

Figure 7: a 1-lit tree of order $2k$.

Since Γ contains the perfect matching $\{\{1,2\},\{3,4\},\ldots,\{2k-1,2k\}\}$ and by Theorem 4.8.1 Γ is 1-lit. It is routine to check that $A_2 = \{1\}$ and $A_6 = \{1,5\}$. Therefore there exists $x \in \{2,6\}$ such that A_5 and A_x have distinct parities. By Theorem 4.8.4 the tree obtained from Γ by inserting a new vertex on the edge $\{5,x\}$ of P is 1-lit. The result follows. $\square$

Chapter 5

The edge-version of lit-only sigma-game

In this chapter we consider the edge-version of lit-only σ-game, which is called *e-lit-only σ-game*. We now describe this game on $\Gamma = (S, R)$. A configuration is an assignment of one of two states, *on* or *off*, to all edges of Γ. Given a configuration, a move allows the player to choose one *on* edge ϵ of Γ and change the states of all adjacent edges ϵ' of ϵ; i.e. $|\epsilon \cap \epsilon'| = 1$. Let $L(\Gamma)$ denote the line graph of Γ. We may view this variation as the lit-only σ-game on $L(\Gamma)$. We denote the flipping group of $L(\Gamma)$ by $\mathbf{W}_R$, and call this the edge-flipping group of Γ. Let $\mathbb{Z}$ denote the additive group of integers. Let n and m denote the numbers of vertices and edges of Γ respectively. Assume $n \geq 3$. The goal of this chapter is to show that $\mathbf{W}_R$ is isomorphic to a semidirect product of $(\mathbb{Z}/2\mathbb{Z})^k$ and the symmetric group S_n of degree n, where $k = (n-1)(m-n+1)$ if n is odd; $k = (n-2)(m-n+1)$ if n is even.

5.1 The edge space and the bond space

In this chapter let $|S| = n$ and $|R| = m$. In this section we mention some properties about the edge space and the bond space of Γ that we will need. The reader may refer to [24, p.23–p.28] for details.

Let $\mathcal{R}$ denote the power set of R. For any $F, F' \in \mathcal{R}$ define $F + F' := \{\epsilon \in R \mid \epsilon \in F \cup F', \epsilon \notin F \cap F'\}$; i.e. the symmetric difference of F and F'. Define $1 \cdot F := F$ and $0 \cdot F := \emptyset$, the empty set. The set $\mathcal{R}$ forms a vector space over $\mathbb{F}_2$ and this is called the *edge space of* Γ. Note that the zero element of $\mathcal{R}$ is $\emptyset$ and $-F = F$ for $F \in \mathcal{R}$. Observe $\{\{\epsilon\} \mid \epsilon \in R\}$ is a basis of $\mathcal{R}$. Therefore the dimension $\dim \mathcal{R}$ of $\mathcal{R}$ is m.

For a subset U of S let $R(U)$ denote the subset of R consisting of all edges of Γ that have exactly one element in U. In graph theory $R(U)$ is often called an *edge cut* of Γ if U is a nonempty and proper subset of S. Notice that $R(\epsilon) = R(\{x, y\})$ for $\epsilon = \{x, y\} \in R$. For convenience $R(s) := R(\{s\})$ for $s \in S$.

Proposition 5.1.1. *The following* (i), (ii) *hold.*

(i) *Each $\epsilon = \{x, y\} \in R$ lies in exactly two edge cuts $R(x)$ and $R(y)$ among $R(s)$ for all $s \in S$.*

(ii) *For* $U \subseteq S$ *we have* $R(U) = \sum_{s \in U} R(s)$.

Proof. (i) is immediate from the definition of $R(s)$ for $s \in S$. (ii) is immediate from (i) and the definition of $R(U)$. □

For the rest of this chapter let $\mathcal{B}$ denote the subspace of $\mathcal{R}$ spanned by $R(s)$ for all $s \in S$. This is called the *bond space of* Γ.

Proposition 5.1.2. *The following* (i)–(iv) *hold.*

(i) $\mathcal{B} = \{R(U) \mid U \subseteq S\}$.

(ii) *The dimension* $\dim \mathcal{B}$ *of* $\mathcal{B}$ *is* $n-1$.

(iii) *For each* $t \in S$, $R(t) = \sum_{s \in S \setminus \{t\}} R(s)$.

(iv) *For each* $t \in S$ *the set* $\{R(s) \mid s \in S \setminus \{t\}\}$ *is a basis of* $\mathcal{B}$.

Proof. (i) follows immediately from Proposition 5.1.1(ii). Similar to the edge space of Γ, the power set $\mathcal{S}$ of S forms a vector space over $\mathbb{F}_2$. Clearly the dimension of $\mathcal{S}$ is n. Observe that the map from the vertex space $\mathcal{S}$ onto the bond space $\mathcal{B}$ of Γ, defined by

$$U \mapsto R(U) \qquad \text{for } U \in \mathcal{S},$$

is a linear transformation with kernel $\{\emptyset, S\}$. It follows that $\dim \mathcal{B} = n-1$. This proves (ii). Let $u \in S$. Since $R(S) = \emptyset$ we have $R(t) = R(t) + R(S)$. By this and Proposition 5.1.1, (iii) follows. (iv) is immediate from (ii), (iii). □

For the rest of this chapter let T denote a minimal subset of R such that (S,T) is connected. We call T a *spanning tree of* Γ. Note that $|T| = n-1$.

Proposition 5.1.3. *The subset* $\{F \in \mathcal{R} \mid F \subseteq R \backslash T\}$ *of* $\mathcal{R}$ *is a set of coset representatives of* $\mathcal{B}$ *in* $\mathcal{R}$.

Proof. There are 2^{m-n+1} cosets of $\mathcal{B}$ in $\mathcal{R}$ because of $\dim \mathcal{B} = n-1$ and $\dim \mathcal{R} = m$. It is clear that $|\{F \mid F \subseteq R \setminus T\}| = 2^{m-n+1}$. For any two distinct $F, F' \subseteq R \setminus T$ the graph $(S, R-(F-F'))$ is still connected since $T \subseteq R-(F-F')$, which implies that $F-F'$ is not an edge cut of Γ. By Proposition 5.1.2(i), $F-F' \notin \mathcal{B}$. Therefore $\{F \mid F \subseteq R \setminus T\}$ is a set of coset representatives of $\mathcal{B}$ in $\mathcal{R}$. □

5.2 The edge-flipping group of Γ

In this section we define the edge-flipping group of Γ.

We interpret each configuration G of the e-lit-only σ-game on Γ as the vector

$$\{\epsilon \in R \mid \epsilon \text{ is assigned the } \textit{on} \text{ state by } G\}$$

of $\mathcal{R}$. For each $\epsilon \in R$ define a linear transformation $\boldsymbol{\rho}_\epsilon : \mathcal{R} \to \mathcal{R}$ by

$$\boldsymbol{\rho}_\epsilon G = \begin{cases} G + R(\epsilon) & \text{if } \epsilon \in G, \\ G & \text{else} \end{cases} \tag{5.1}$$

for $G \in \mathcal{R}$. Observe that $R(\epsilon)$ consists of all edges that are adjacent to ϵ. Therefore we may view $\boldsymbol{\rho}_\epsilon$ as the move for which we select the edge ϵ of Γ and change the states of all adjacent edges of ϵ if the state of ϵ is *on*.

Let $\mathrm{GL}(\mathcal{R})$ denote the general linear group of $\mathcal{R}$. Using (5.1) we find $\boldsymbol{\rho}_\epsilon^2 = 1$ and so $\boldsymbol{\rho}_\epsilon \in \mathrm{GL}(\mathcal{R})$. Here 1 denotes the identity in $\mathrm{GL}(\mathcal{R})$.

Definition 5.2.1. Let $\mathbf{W}_R$ denote the subgroup of $\mathrm{GL}(\mathcal{R})$ generated by $\boldsymbol{\rho}_\epsilon$ for all $\epsilon \in R$. We call $\mathbf{W}_R$ the *edge-flipping group of* Γ.

Definition 5.2.2. Let $F \in \mathcal{R}$. By the $\mathbf{W}_R$*-orbit of* F we mean the set $\mathbf{W}_R F = \{\mathbf{g}F \mid \mathbf{g} \in \mathbf{W}_R\}$. By a $\mathbf{W}_R$*-orbit of* $\mathcal{R}$ we mean a $\mathbf{W}_R$-orbit of F for some $F \in \mathcal{R}$.

Let F denote a subset of R. We say that F is *closed under* $\mathbf{W}_R$ whenever $\mathbf{W}_R F \subseteq F$.

Proposition 5.2.3. ([29, Section 5]). *Each coset of* $\mathcal{B}$ *in* $\mathcal{R}$ *is closed under* $\mathbf{W}_R$.

Proof. Fix any $\epsilon \in R$ and $G \in \mathcal{R}$. It suffices to show that $\boldsymbol{\rho}_\epsilon G - G \in \mathcal{B}$. By (5.1), $\boldsymbol{\rho}_\epsilon G - G$ is equal to either $\emptyset$ or $R(\epsilon)$. Since $\emptyset, R(\epsilon) \in \mathcal{B}$ the result follows. $\square$

5.3 The structure of $\mathbf{W}_R$ in the case Γ is a tree

When Γ is a tree with $n \geq 3$, Yaokun Wu showed that $\mathbf{W}_R$ is isomorphic to the symmetric group of degree n. Here we provide another proof.

Lemma 5.3.1. *We have*

$$|\{R(s) \mid s \in S\}| = \begin{cases} n & \text{if } n \geq 3, \\ 1 & \text{else.} \end{cases}$$

Proof. Suppose $n = 1$. Let $S = \{s\}$. Then $R(s) = \emptyset$. Thus $|\{R(s)\}| = 1$. Suppose $n = 2$. Let $S = \{s, t\}$. Then $R(s) = \{s, t\}$ and $R(t) = \{s, t\}$. Thus $|\{R(s), R(t)\}| = 1$. Now suppose $n \geq 3$. Pick two distinct vertices $s, t \in S$. Since $R(\{s, t\})$ is nonempty and by Proposition 5.1.1(ii), $R(s) + R(t) \neq \emptyset$. Therefore $R(s) \neq R(t)$. The result follows. $\square$

For the rest of this chapter we assume $n \geq 3$ until further notice. In view of Lemma 5.3.1 the symmetric group on $\{R(s) \mid s \in S\}$ has degree n. We denote the group by S_n. Let $\epsilon = \{x, y\} \in R$. By Proposition 5.1.1(i) and (5.1) the transformation $\boldsymbol{\rho}_\epsilon$ fixes the $R(s)$ for all $s \in S \setminus \{x, y\}$. Using Proposition 5.1.1(ii) we find that $\boldsymbol{\rho}_\epsilon R(x) = R(y)$ and $\boldsymbol{\rho}_\epsilon R(y) = R(x)$. By the above comments we have a group homomorphism as follows.

Definition 5.3.2. Let $\alpha : \mathbf{W}_R \to S_n$ denote the group homomorphism defined by

$$\alpha(\mathbf{g})(R(s)) = \mathbf{g}R(s) \qquad \text{for } s \in S \text{ and } \mathbf{g} \in \mathbf{W}_R.$$

Observe that for each $\epsilon = \{x, y\} \in R$, $\alpha(\boldsymbol{\rho}_\epsilon)$ is the transposition $(R(x), R(y))$, which switches $R(x)$ and $R(y)$.

Let $F \subseteq R$. For the rest of this chapter let $\mathbf{W}_{R,F}$ denote the subgroup of $\mathbf{W}_R$ generated by the $\boldsymbol{\rho}_\epsilon$ for all $\epsilon \in F$.

Lemma 5.3.3. *The image of* $\mathbf{W}_{R,T}$ *under* α *is* S_n. *Moreover if* Γ *is a tree with* $n \geq 3$, *then* α *is an isomorphism from* $\mathbf{W}_R$ *to* S_n.

Proof. Let A denote the set of the transpositions $\{(R(x), R(y))$ for all $\{x, y\} \in T$. Let s, t denote any two distinct vertices of Γ. There exists a subset $\{\{s_0, s_1\}, \{s_1, s_2\}, \ldots, \{s_{k-1}, s_k\}\}$ of T with $s_0 = s$ and $s_k = t$. Observe that

$$(R(s), R(t)) = (R(s_{k-1}), R(s_k)) \cdots (R(s_2), R(s_3))(R(s_1), R(s_2))(R(s_0), R(s_1))$$
$$(R(s_1), R(s_2))(R(s_2), R(s_3)) \cdots (R(s_{k-1}), R(s_k)).$$

Thus A generates all transpositions in S_n, so A generates S_n. Therefore $\alpha(\mathbf{W}_{R,T}) = S_n$. Now suppose Γ is a tree. In this case $\mathcal{R} = \mathcal{B}$ by Proposition 5.1.2(ii) and comparing the both dimensions. Let $\mathbf{g} \in \operatorname{Ker} \alpha$. Then $\mathbf{g}R(s) = R(s)$ for all $s \in S$. Since the $R(s)$ for all $s \in S$ span $\mathcal{B}$ it follows that $\mathbf{g} = 1$, the identity map in $\mathrm{GL}(\mathcal{R})$, This shows $\operatorname{Ker} \alpha = \{1\}$. Therefore α is an isomorphism. □

Corollary 5.3.4. ([29, Theorem 8]). *Assume that Γ is a tree with $n \geq 3$. Then $\mathbf{W}_R$ is isomorphic to S_n.*

Proof. Immediate from Lemma 5.3.3. □

Example 5.3.5. Assume that $\Gamma = (S, R)$ is the star graph of $n \geq 3$ vertices. By Corollary 5.3.4 the edge-flipping group $\mathbf{W}_R$ of Γ is isomorphic to S_n.

5.4 The $\mathbf{W}_R$-orbits of $\mathcal{R}$

In this section we give a description of $\mathbf{W}_R$-orbits of $\mathcal{R}$. To do this we fix a vertex t of Γ and let

$$\Delta := \{R(s) \mid s \in S \setminus \{t\}\}$$

in this section. By Proposition 5.1.2(iv), Δ is a basis of $\mathcal{B}$. We call Δ the *simple basis of* $\mathcal{B}$. For each $G \in \mathcal{B}$ let $\Delta(G)$ denote the subset of Δ such that the sum of its elements equals G. Define the *simple weight* $||G||_s$ *of* G to be the cardinality of $\Delta(G)$. For example $\Delta(R(t)) = \{R(s) \mid s \in S \setminus \{t\}\}$ and so $||R(t)||_s = n - 1$.

Lemma 5.4.1. *The $\mathbf{W}_{R,T}$-orbits of $\mathcal{B}$ are*

$$\Omega_i := \{G \in \mathcal{B} \mid ||G||_s = i \text{ or } ||G||_s = n - i\} \qquad (0 \leq i \leq \lceil \tfrac{n-1}{2} \rceil).$$

Proof. By Proposition 5.1.1(ii) and Proposition 5.1.2(i), $\mathcal{B}$ consists of $R(U) = \sum_{s \in U} R(s)$ for all $U \subseteq S$. Recall from Lemma 5.3.3 that $\alpha(\mathbf{W}_{R,T}) = S_n$, the symmetric group on $\{R(s) \mid s \in S\}$. Therefore the $\mathbf{W}_{R,T}$-orbits of $\mathcal{B}$ are $\Omega'_i = \{G \in \mathcal{B} \mid G = R(U),\ |U| = i\}$ for $0 \leq i \leq n$. Since $R(U) = R(S \setminus U)$ for $U \subseteq S$ it follows that $\Omega'_i = \Omega'_{n-i}$ and so both are equal to Ω_i. The result follows. □

For the rest of this chapter let Ω_i $(0 \leq i \leq \lceil \frac{n-1}{2} \rceil)$ denote the sets from Lemma 5.4.1.

Corollary 5.4.2. ([29, Theorem 10]). *The $\mathbf{W}_R$-orbits of $\mathcal{B}$ are Ω_i for $0 \leq i \leq \lceil \frac{n-1}{2} \rceil$.*

Proof. By Lemma 5.3.3, $\alpha(\mathbf{W}_R) = S_n$. Therefore the $\mathbf{W}_R$-orbits of $\mathcal{B}$ are as same as the $\mathbf{W}_{R,T}$-orbits of $\mathcal{B}$. The result follows from Lemma 5.4.1. □

Recall that $\{F \mid F \subseteq R \setminus T\}$ is a set of coset representatives of $\mathcal{B}$ in $\mathcal{R}$, from Proposition 5.1.3.

Lemma 5.4.3. *Let F denote a nonempty subset of $R \setminus T$. For any $\epsilon \in F$ the $\mathbf{W}_{R,T\cup\{\epsilon\}}$-orbits of $F+\mathcal{B}$ are*

$$\begin{cases} F+\mathcal{B} & \textit{if } n \textit{ is odd,} \\ F+\mathcal{B}_e \text{ and } F+\mathcal{B}_o & \textit{if } n \textit{ is even,} \end{cases} \tag{5.2}$$

where $\mathcal{B}_e := \{G \in \mathcal{B} \mid ||G||_s \text{ is even}\}$ *and* $\mathcal{B}_o := \{G \in \mathcal{B} \mid ||G||_s \text{ is odd}\}$.

Proof. Since $F \cap T = \emptyset$ we have $\boldsymbol{\rho}_{\epsilon'}F = F$ for any $\epsilon' \in T$. Therefore $\mathbf{W}_{R,T}F = F$. By this and Lemma 5.4.1 the $\mathbf{W}_{R,T}$-orbits of $F+\mathcal{B}$ are

$$F+\Omega_i \qquad (0 \le i \le \lceil \tfrac{n-1}{2} \rceil). \tag{5.3}$$

It remains to consider how $\boldsymbol{\rho}_\epsilon$ acts on $F+\mathcal{B}$. To do this, pick any i among $0,1,\ldots,n-1$ and pick any $G \in \mathcal{B}$ with $||G||_s = i$. Note that $\boldsymbol{\rho}_\epsilon(F+G) = F+R(\epsilon)+\boldsymbol{\rho}_\epsilon G$ and that $R(\epsilon)+\boldsymbol{\rho}_\epsilon G \in \mathcal{B}$. We now discuss $||R(\epsilon)+\boldsymbol{\rho}_\epsilon G||_s$. If $u \notin \epsilon$ then

$$||R(\epsilon)+\boldsymbol{\rho}_\epsilon G||_s = \begin{cases} i+2 & \text{if } |\Delta(G)\cap\Delta(R(\epsilon))| = 0, \\ i & \text{if } |\Delta(G)\cap\Delta(R(\epsilon))| = 1, \\ i-2 & \text{else.} \end{cases} \tag{5.4}$$

If $u \in \epsilon$ then

$$||R(\epsilon)+\boldsymbol{\rho}_\epsilon G||_s = \begin{cases} i & \text{if } |\Delta(G)\cap\Delta(R(\epsilon))| = i-1, \\ n-i-2 & \text{else.} \end{cases} \tag{5.5}$$

Combining (5.3)–(5.5) we find

$$\bigcup_{j \equiv i, n-i \bmod 2} F+\Omega_j \qquad \text{for } i = 0,1 \tag{5.6}$$

are the $\mathbf{W}_{R,T\cup\{\epsilon\}}$-orbits of $F+\mathcal{B}$. If n is odd then (5.6) equals $F+\mathcal{B}$ for each $i=0,1$; if n is even (5.6) equals $F+\mathcal{B}_e$ (resp. $F+\mathcal{B}_o$) for $i=0$ (resp. $i=1$). The result follows. □

For the rest of this chapter let $\mathcal{B}_e$ and $\mathcal{B}_o$ denote as in Lemma 5.4.3.

Corollary 5.4.4. ([29, Theorem 12]). *Let F denote a nonempty subset of $R \setminus T$. Then the $\mathbf{W}_R$-orbits of $F+\mathcal{B}$ are as* (5.2).

Proof. The group $\mathbf{W}_{R,T\cup F}$ is generated by $\mathbf{W}_{R,T\cup\{\epsilon\}}$ for all $\epsilon \in F$. By this and Lemma 5.4.3 the $\mathbf{W}_{R,T\cup F}$-orbits of $F+\mathcal{B}$ are as described in (5.2). Pick any $\epsilon \in R-(T\cup F)$. Observe that $\boldsymbol{\rho}_\epsilon(F+\mathcal{B}) = F+\mathcal{B}$, and that if n is even then $\boldsymbol{\rho}_\epsilon(F+\mathcal{B}_e) = F+\mathcal{B}_e$ and $\boldsymbol{\rho}_\epsilon(F+\mathcal{B}_o) = F+\mathcal{B}_o$. The result follows. □

Corollary 5.4.5. ([29, Theorem 10, Theorem 12]). *The $\mathbf{W}_R$-orbits of $\mathcal{R}$ are $\Omega_0, \Omega_1, \ldots, \Omega_{\lceil \frac{n-1}{2} \rceil}$ and*

$$\begin{cases} \textit{the } F+\mathcal{B} \quad \textit{for all } F \in \mathcal{R}\setminus\mathcal{B} & \textit{if } n \textit{ is odd,} \\ \textit{the } F+\mathcal{B}_e \quad \textit{for all } F \in \mathcal{R}\setminus\mathcal{B}_e & \textit{if } n \textit{ is even.} \end{cases}$$

Proof. Immediate from Corollary 5.4.2 and Corollary 5.4.4. □

5.5 The minimum light number for e-lit-only σ-game on Γ

Similar to lit-only σ-game we consider the numbers defined below.

Definition 5.5.1. For a subset O of R define $|O|$ to be the number

$$\min_{G \in O} |G|.$$

Definition 5.5.2. Let $k \geq 1$ denote an integer. We say that Γ is *k-lit for e-lit-only σ-game* whenever $|O| \leq k$ for any $\mathbf{W}_R$-orbit O of $\mathcal{R}$.

Definition 5.5.3. Let $\mu_e(\Gamma)$ denote the minimum number k such that Γ is *k-lit for e-lit-only σ-game.* We call $\mu_e(\Gamma)$ the *minimum light number for e-lit-only σ-game on* Γ.

Observe that $\mu_e(\Gamma)$ equals $\max |O|$, where the maximum is over all $\mathbf{W}_R$-orbits O of $\mathcal{R}$. By Corollary 5.4.2 we have

$$\mu_e(\Gamma) = \max\{|\Omega_0|, |\Omega_1|, \ldots, |\Omega_{\lceil \frac{n-1}{2} \rceil}|\} \qquad \text{if } \Gamma \text{ is a tree.} \tag{5.7}$$

By Corollary 5.4.5 we have

$$\mu_e(\Gamma) = \begin{cases} \max\{|\Omega_0|, |\Omega_1|, \ldots, |\Omega_{\lceil \frac{n-1}{2} \rceil}|, \max\limits_{F \in \mathcal{R}} |F + \mathcal{B}|\} & \text{if } n \text{ is odd,} \\ \max\{|\Omega_0|, |\Omega_1|, \ldots, |\Omega_{\lceil \frac{n-1}{2} \rceil}|, \max\limits_{F \in \mathcal{R}} |F + \mathcal{B}_e|\} & \text{if } n \text{ is even.} \end{cases} \tag{5.8}$$

There are some results about $\mu_e(\Gamma)$. Here we provide short proofs.

Definition 5.5.4. For each $0 \leq i \leq n$ define $b_i(\Gamma)$ to be the number

$$\min |R(U)|,$$

where the minimum is over all subsets U of S with $|U| = i$. This number is called the *ith edge-isoperimetric number of* Γ.

Definition 5.5.5. Define $b(\Gamma)$ to be the number $\max\{b_0(\Gamma), b_1(\Gamma), \ldots, b_n(\Gamma)\}$. This number is called the *edge-isoperimetric number of* Γ.

Definition 5.5.6. Let O denote a subset of $\mathcal{R}$. Define $\varrho(O)$ to be the number

$$\max_{F \in \mathcal{R}} |F + O|.$$

This number is called the *covering radius of O in $\mathcal{R}$.*

Definition 5.5.7. Let $\mathcal{A}$ denote the subspace of $\mathcal{R}$ spanned by $R(\epsilon)$ for all $\epsilon \in R$.

Lemma 5.5.8. *The number $b(\Gamma)$ equals* $\max\{|\Omega_0|, |\Omega_1|, \ldots, |\Omega_{\lceil \frac{n-1}{2} \rceil}|\}$.

Proof. For $0 \leq i \leq \lceil \frac{n-1}{2} \rceil$, $b_i(\Gamma) = b_{n-i}(\Gamma)$ and both are equals to $|\Omega_i|$. The result follows. □

Theorem 5.5.9. ([29, Corollary 15]). *Assume that Γ is a tree. Then $\mu_e(\Gamma) = b(\Gamma)$.*

Proof. Immediate from (5.7) and Lemma 5.5.8. $\square$

Theorem 5.5.10. ([29, Theorem 16]). $\mu_e(\Gamma) = \max\{b(\Gamma), \varrho(\mathcal{A})\}$.

Proof. Observe that the $R(U)$ for all $U \subseteq S$ with even sizes span $\mathcal{A}$. Therefore $\mathcal{A} = \mathcal{B}$ if n is odd, and $\mathcal{A} = \mathcal{B}_e$ if n is even. By the above comment, $\varrho(\mathcal{A})$ equals

$$\begin{cases} \max\limits_{F \in \mathcal{R}} |F + \mathcal{B}| & \text{if } n \text{ is odd,} \\ \max\limits_{F \in \mathcal{R}} |F + \mathcal{B}_e| & \text{if } n \text{ is even.} \end{cases} \tag{5.9}$$

Now the result follows from (5.8), (5.9), Lemma 5.5.8. $\square$

5.6 The structure of $\mathbf{W}_R$

In this section we investigate the structure of $\mathbf{W}_R$. For $i = 1, 2, \ldots, m-n+1$ Let $\mathcal{B}_i$ denote a copy of the bond space $\mathcal{B}$ of Γ. Let $\mathcal{B}^{m-n+1}$ denote the (external) direct sum of $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_{m-n+1}$,

$$\bigoplus_{i=1}^{m-n+1} \mathcal{B}_i.$$

We view $\mathcal{B}^{m-n+1}$ as an additive group. Let $\mathrm{Aut}(\mathcal{B}^{m-n+1})$ denote the automorphism group of $\mathcal{B}^{m-n+1}$.

Definition 5.6.1. Let $\beta : \mathbf{W}_R \to \mathrm{Aut}(\mathcal{B}^{m-n+1})$ denote the group homomorphism defined by

$$\beta(\mathbf{g})(G_i)_{i=1}^{m-n+1} = (\mathbf{g}G_i)_{i=1}^{m-n+1}$$

for $\mathbf{g} \in \mathbf{W}_R$ and $(G_i)_{i=1}^{m-n+1} \in \mathcal{B}^{m-n+1}$.

By Lemma 5.3.3 the group homomorphism $\alpha : \mathbf{W}_R \to S_n$ is surjective. We now show that there exists a unique group homomorphism $\theta : S_n \to \mathrm{Aut}(\mathcal{B}^{m-n+1})$ such that the following diagram commutes.

$$\begin{tikzcd} \mathbf{W}_E(X) \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & S_n \arrow[d, "\theta"] \\ & \mathrm{Aut}(\mathcal{B}^{m-n+1}) \end{tikzcd}$$

Lemma 5.6.2. *There exists a unique group homomorphism $\theta : S_n \to \mathrm{Aut}\ (\mathcal{B}^{m-n+1})$ such that $\beta = \theta \circ \alpha$. Moreover θ is determined by the following relation*

$$\theta(\sigma)(R(s_i))_{i=1}^{m-n+1} = (\sigma(R(s_i)))_{i=1}^{m-n+1} \tag{5.10}$$

for all $s_1, s_2, \ldots, s_{m-n+1} \in S$ and $\sigma \in S_n$.

Proof. Since α is surjective, if θ exists then θ is unique. To show the existence of θ, it suffices to show the kernel $\operatorname{Ker}\alpha$ of α is contained in the kernel $\operatorname{Ker}\beta$ of β. Let $\mathbf{g} \in \operatorname{Ker}\alpha$. Then $\mathbf{g}R(s) = R(s)$ for all $s \in S$. By this and since $\{R(s) \mid s \in S\}$ spans $\mathcal{B}$, we have $\mathbf{g}G = G$ for all $G \in \mathcal{B}$. Therefore $\mathbf{g} \in \operatorname{Ker}\beta$. We now show (5.10). Pick any $\sigma \in S_n$. Since α is surjective there exists $\mathbf{h} \in \mathbf{W}_R$ such that $\alpha(\mathbf{h}) = \sigma$. Using $\beta = \theta \circ \alpha$, we write (5.10) as

$$\beta(\mathbf{h})(R(v_i))_{i=1}^{m-n+1} = (\alpha(\mathbf{h})(R(v_i)))_{i=1}^{m-n+1} \tag{5.11}$$

Using Definition 5.3.2 we obtain the right-hand side of (5.11) equals

$$(\mathbf{h}R(v_1), \mathbf{h}R(v_2), \ldots, \mathbf{h}R(v_{m-n+1})). \tag{5.12}$$

Using Definition 5.6.1, we obtain the left-hand side of (5.11) also equals (5.12). This shows (5.10). Since $\{R(s) \mid s \in S\}$ spans $\mathcal{B}$, $\theta(\sigma)$ is uniquely determined by (5.10). By this and since σ is an arbitrary element of S_n, θ is uniquely determined by (5.10). $\square$

In view of Lemma 5.6.2 we can define the (external) semidirect product of $\mathcal{B}^{m-n+1}$ and S_n with respect to θ. We denote this by $\mathcal{B}^{m-n+1} \rtimes_\theta S_n$. This group is the set $\mathcal{B}^{m-n+1} \times S_n$ with the group operation defined by

$$((G_i)_{i=1}^{m-n+1}, \sigma_1)\ ((H_i)_{i=1}^{m-n+1}, \sigma_2) = ((G_i)_{i=1}^{m-n+1} + \theta(\sigma_1)(H_i)_{i=1}^{m-n+1}, \sigma_1\sigma_2)$$

for all $(G_i)_{i=1}^{m-n+1}, (H_i)_{i=1}^{m-n+1} \in \mathcal{B}^{m-n+1}$ and $\sigma_1, \sigma_2 \in S_n$.

Recall that T denotes a spanning tree of R. Note that $|R \setminus T| = m - n + 1$. Let $\epsilon_1, \epsilon_2, \ldots, \epsilon_{m-n+1}$ denote the all elements in $R \setminus T$. By Corollary 5.4.4, $\{\epsilon_i\} + \mathbf{W}_R\{\epsilon_i\}$ is contained in $\mathcal{B}$ for $i = 1, 2 \ldots, m - n + 1$. By the above comment we can define a map from $\mathbf{W}_R$ to $\mathcal{B}^{m-n+1} \rtimes_\theta S_n$ as follows.

Definition 5.6.3. Let $\gamma : \mathbf{W}_R \to \mathcal{B}^{m-n+1} \rtimes_\theta S_n$ denote the map defined by

$$\gamma(\mathbf{g}) = ((\{\epsilon_i\} + \mathbf{g}\{\epsilon_i\})_{i=1}^{m-n+1}, \alpha(\mathbf{g})) \qquad \text{for } \mathbf{g} \in \mathbf{W}_R.$$

Lemma 5.6.4. *γ is a group monomorphism from $\mathbf{W}_R$ into $\mathcal{B}^{m-n+1} \rtimes_\theta S_n$.*

Proof. For $\mathbf{g}, \mathbf{h} \in \mathbf{W}_R$,

$$\begin{aligned}
\gamma(\mathbf{g})\gamma(\mathbf{h}) &= ((\{\epsilon_i\} + \mathbf{g}\{\epsilon_i\})_{i=1}^{m-n+1}, \alpha(\mathbf{g}))((\{\epsilon_i\} + \mathbf{h}\{\epsilon_i\})_{i=1}^{m-n+1}, \alpha(\mathbf{h})) \\
&= ((\{\epsilon_i\} + \mathbf{g}\{\epsilon_i\})_{i=1}^{m-n+1} + \theta(\alpha(\mathbf{g}))(\{\epsilon_i\} + \mathbf{h}\{\epsilon_i\})_{i=1}^{m-n+1}, \alpha(\mathbf{g})\alpha(\mathbf{h})) \\
&= ((\{\epsilon_i\} + \mathbf{g}\{\epsilon_i\})_{i=1}^{m-n+1} + \beta(\mathbf{g})(\{\epsilon_i\} + \mathbf{h}\{\epsilon_i\})_{i=1}^{m-n+1}, \alpha(\mathbf{gh})) \\
&= ((\{\epsilon_i\} + \mathbf{g}\{\epsilon_i\})_{i=1}^{m-n+1} + (\mathbf{g}\{\epsilon_i\} + \mathbf{gh}\{\epsilon_i\})_{i=1}^{m-n+1}, \alpha(\mathbf{gh})) \\
&= ((\{\epsilon_i\} + \mathbf{gh}\{\epsilon_i\})_{i=1}^{m-n+1}, \alpha(\mathbf{gh})) \\
&= \gamma(\mathbf{gh}).
\end{aligned}$$

This shows that γ is a group homomorphism. Since each $\mathbf{g} \in \operatorname{Ker}\gamma$ fixes the spanning set $\{\{\epsilon_1\}, \{\epsilon_2\}, \ldots, \{\epsilon_{m-n+1}\}\} \cup \{R(s) \mid s \in S\}$ of the edge space $\mathcal{R}$ of Γ, $\mathbf{g}$ is the identity map on $\mathcal{R}$. Hence $\operatorname{Ker}\gamma$ is trivial. $\square$

By Lemma 5.6.4, $\mathbf{W}_R$ is isomorphic to the image of $\mathbf{W}_R$ under γ. Fortunately the structure of $\gamma(\mathbf{W}_R)$ is knowable. In Lemma 5.4.3 we define $\mathcal{B}_e = \{G \in \mathcal{B} \mid ||G||_s \text{ is even}\}$. Note that dim $\mathcal{B}_e = n-2$. Let $\mathcal{B}_e^{m-n+1}$ denote the subgroup

$$\bigoplus_{i=1}^{m-n+1} \mathcal{B}_{e,i}$$

of the additive group $\mathcal{B}^{m-n+1}$, where $\mathcal{B}_{e,i}$ $(1 \le i \le m-n+1)$ is the subspace of $\mathcal{B}_i$ as $\mathcal{B}_e$.

Theorem 5.6.5. *The edge-flipping group $\mathbf{W}_R$ of Γ is isomorphic to*

$$\begin{cases} \mathcal{B}^{m-n+1} \rtimes_\theta S_n & \text{if } n \text{ is odd,} \\ \mathcal{B}_e^{m-n+1} \rtimes_\theta S_n & \text{if } n \text{ is even,} \end{cases}$$

provided $n \ge 3$.

Proof. It suffices to show that for any $\sigma \in S_n$, there exists $\mathbf{g} \in \mathbf{W}_R$ such that

$$\gamma(\mathbf{g}) = ((\emptyset)_{i=1}^{m-n+1}, \sigma), \tag{5.13}$$

and that for each $1 \le i \le m-n+1$ and for each

$$G \in \begin{cases} B_i & \text{if } n \text{ is odd,} \\ B_{e,i} & \text{if } n \text{ is even} \end{cases}$$

there exists $\mathbf{h} \in \mathbf{W}_R$ such that

$$\gamma(\mathbf{h}) = (\emptyset, \ldots, \emptyset, G, \emptyset, \ldots, \emptyset, \alpha(\mathbf{h})), \tag{5.14}$$

where G is in the ith coordinate. By Lemma 5.3.3 there exists $\mathbf{g} \in \mathbf{W}_{R,T}$ such that $\alpha(\mathbf{g}) = \sigma$. Such $\mathbf{g}$ satisfies (5.13). By Lemma 5.4.3 there exists $\mathbf{h} \in \mathbf{W}_{R,T\cup\{\epsilon_i\}}$ such that $\mathbf{h}\{\epsilon_i\} = \{\epsilon_i\} + G$. Such $\mathbf{h}$ satisfies (5.14). The result follows. □

Let $\mathbb{Z}$ is the additive group of integers. Since dim $\mathcal{B} = n-1$ and dim $\mathcal{B}_e = n-2$ the additive groups $\mathcal{B}$ and $\mathcal{B}_e$ are isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{n-1}$ and $(\mathbb{Z}/2\mathbb{Z})^{n-2}$.

Example 5.6.6. Assume that Γ is a cycle of $n \ge 3$ vertices. Then the edge-flipping group $\mathbf{W}_R$ of Γ is isomorphic to

$$\begin{cases} (\mathbb{Z}/2\mathbb{Z})^{n-1} \rtimes S_n & \text{if } n \text{ is odd,} \\ (\mathbb{Z}/2\mathbb{Z})^{n-2} \rtimes S_n & \text{if } n \text{ is even} \end{cases}$$

by Theorem 5.6.5.

We now show that there is a unique edge-flipping group of all finite simple connected graphs $\Gamma = (S, R)$ with fixed $|S|$ and fixed $|R|$, up to isomorphism.

Theorem 5.6.7. *Let $\Gamma = (S, R)$ and $\Gamma' = (S', R')$ denote two finite simple connected graphs with $|S| = |S'|$ and $|R| = |R'| \ge 1$. Then the edge-flipping group of Γ and the edge-flipping group of Γ' are isomorphic.*

Proof. Let $\mathbf{W}_R$ and $\mathbf{W}_{R'}$ denote the edge-flipping groups of Γ and Γ', respectively. If $|R| = 1$ and $|R'| = 1$, then Γ and Γ' are isomorphic and so $\mathbf{W}_R$ and $\mathbf{W}_{R'}$ are isomorphic. Now suppose $|R| = |R'| \geq 2$. Without loss of generality we assume that $S' = S$. Define $R'(v)$, $\mathcal{B}'$, $\mathcal{B}'_e$, S'_n, and θ' correspondingly. In view of Theorem 5.6.5 it suffices to show that $\mathcal{B}^{m-n+1} \rtimes_\theta S_n$ and $\mathcal{B}_e^{m-n+1} \rtimes_\theta S_n$ are isomorphic to $\mathcal{B}'^{m-n+1} \rtimes_{\theta'} S'_n$ and $\mathcal{B}_e'^{m-n+1} \rtimes_{\theta'} S'_n$ respectively. Fix $t \in S$. Let $\mu : \mathcal{B} \to \mathcal{B}'$ denote the invertible linear transformation defined by

$$\mu(R(s)) = R'(s) \qquad \text{for } s \in S \setminus \{t\}.$$

There exists a unique isomorphism $\mu_* : S_n \to S'_n$ such that

$$\mu_*(\sigma)(R'(s)) = \mu(\sigma(R(s))) \qquad \text{for all } \sigma \in S_n \text{ and } s \in S.$$

By the above two comments we can define a map $\phi : \mathcal{B}^{m-n+1} \rtimes_\theta S_n \to \mathcal{B}'^{m-n+1} \rtimes_{\theta'} S'_n$ by

$$\phi((G_i)_{i=1}^{m-n+1}, \sigma) = ((\mu(G_i))_{i=1}^{m-n+1}, \mu_*(\sigma))$$

for all $(G_i)_{i=1}^{m-n+1} \in \mathcal{B}^{m-n+1}$ and $\sigma \in S_n$. Observe that ϕ is bijective and that ϕ sends $\mathcal{B}_e^{m-n+1} \rtimes_\theta S_n$ to $\mathcal{B}_e'^{m-n+1} \rtimes_{\theta'} S'_n$. One readily verifies that ϕ is an isomorphism. The result follows. $\square$

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