國立交通大學

應用數學系

博士論文

群試設計，距離正則圖，圖譜理論及它們的關連

Pooling Designs, Distance-regular Graphs, Spectral Graph Theory and Their Links

博 士 生：黃喻培

指導教授：翁志文 教授

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摘要

群試設計有著篩檢 DNA 序列的應用。為因應尋找群試設計統一性的建構，群試空間出現在前人的研究中。群試空間是一個有秩偏序集滿足以下條件：每一元素上方元素導出的偏序子集均具原子性。我們發現在文獻中已被深入研究的幾何晶格結構也是群試空間，因此在同一架構下提供了許多的群試設計。根據相同的概念，我們也可以利用一個距離正則圖及其距離正則子圖來建構群試空間，因此我們研究如何在給定距離正則圖中建構出距離正則子圖。利用此結果我們解決一類距離正則圖的存在問題。距離正則圖常以等號情形出現在組合學或線性代數相關不等式中，如同我們在數列的算幾不等式中所見，等號情形發生在數列具某種規律的狀況。考慮給定圖對應的鄰接矩陣的最大特徵值，我們找到一些準確的上界，達到上界的圖也滿足一種特殊的規律性。

關鍵字：群試空間，群試設計，有秩偏序集，原子性，幾何晶格，距離正則圖。

Pooling Designs, Distance-regular Graphs, Spectral Graph Theory and Their Links

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Abstract

This dissertation contains three quite different subjects: posets, distance-regular graphs, and spectral graph theory. Motivated by the constructions of pooling designs, we study these three subjects through interesting links among them. A *pooling space* is a ranked poset P such that the subposet w^+ induced by the elements above w is atomic for each element w of P. Pooling spaces were introduced in [Discrete Mathematics 282:163-169, 2004] for the purpose of giving a uniform way to construct pooling designs, which have applications to the screening of DNA sequences. We find that a *geometric lattice*, a well-studied structure in literature, is also a pooling space. This provides us many classes of pooling designs, some old and some new. Following the same concept, the poset constructed from a distance-regular graph with its distance-regular subgraphs is also a pooling space. For a special class of distance-regular graphs, we show the existence of their distance-regular subgraphs with any given diameter. The nonexistence of a class of distance-regular graphs follows from the line of study. Distance-regular graphs appear often in some extremal class of combinatorial or linear algebraic inequalities. As we can see from the inequality of arithmetic and geometric means of a sequence of positive real numbers, the equality occurs when the sequence has some regular patterns. We consider the maximum eigenvalues of the adjacency matrices of graphs and present sharp upper bounds of them. The graphs which attain the bounds also satisfy a special kind of regularity.

Keywords: pooling spaces, pooling designs, ranked posets, atomic, geometric lattices, distance-regular graphs.

謝誌

一晃眼，在新竹的這十幾個年頭就這麼過去了，相對於整個人生來說，這已經不僅是可被歸之為過客的配額。經歷這麼多年浮浮沈沈的學術漂流，還能有幸在今天劃下一個段落的句點，實在有道不盡的感恩心情。

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Chapter 1

Introduction

Group testing is a topic about strategies of experiment arrangements. The main idea behind it is that when we want to find some relatively few abnormal items out of a large set of items, testing items gathering together should be efficient with some smart arrangements. In 1964, W. H. Kautz and R. C. Singleton [23] introduced the now so-called *disjunct matrices* that are useful for us to deal with the group testing problems. With the error-tolerance ability being considered, the concepts of b^d-*disjunct* matrices, a generalization of the original disjunct matrices, was introduced by A. G. D'yachkov, V. V. Rykov, and A. M. Rachad in 1983 [11]. A binary matrix M is b^d-*disjunct* if for any $b+1$ columns $x, x_1, x_2, \ldots, x_b$ of M with x different to the others, there exist $d+1$ rows such that x has values 1, and x_1, x_2, $\ldots$, x_b all have values 0 at these $d+1$ rows. In particular, a b^0-disjunct matrix is also called a b-disjunct matrix for short. A b^d-disjunct matrix can be used to construct an error-tolerable design for non-adaptive group testing, which has applications to the screening of DNA sequences, and the corresponding decoding algorithm is efficient. See [9, 18] for details. Hence a b^d-disjunct matrix is also called a *pooling design*.

The constructions of b^d-disjunct matrices were given by many authors, e.g. [29, 30, 32, 10]. These constructions use some properties of a ranked poset. In [19], the name *pooling spaces* was given to describe these ranked posets (formal definition in Section 3.1). Fix a pooling space P and positive integers $r < k$. Let M denote the incidence matrix between the rank r elements and the rank k elements in P. It was shown in [19] that M is b^d-disjunct for $b = r$ and $d = 0$. A binary matrix is *fully* b^d-disjunct if it is b^d-disjunct but neither b^{d+1}- nor $(b+1)^d$-disjunct. Some fully b^d-disjunct matrices are given in [10].

So far we know that the incidence relation between two levels in a pooling space can help us to construct pooling designs. Roughly speaking, the supporting structure behind the pooling space must be "good" enough. In particular, the poset of distance-regular subgraphs in a given distance-regular graph (formal definition in Section 2.2), ordered by the containment relation between subgraphs, forms a pooling space [40].

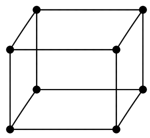

Figure 1. A distance-regular graph with many distance-regular subgraphs.

In Figure 1, the distance-regular subgraphs of this "cube" are the "points", "edges", "faces", and the "cube" itself. In general, the determination of distance-regular subgraphs may not be so obvious. With some restrictions on the intersection numbers of a distance-regular graph, we introduce a systematical way in Chapter 4 that helps us to construct distance-regular subgraphs of it. The results involved also help us to show the nonexistence of a class of distance-regular graphs.

Distance-regular graphs appear often in some extremal class of combinatorial or linear algebraic inequalities. For example it is well-known that the number of edges of a graph of girth 5 and order n is at most $\frac{n\sqrt{n-1}}{2}$, and its maximum number of edges is attained when the graph is distance-regular [27, Theorem 4.2]. Sometimes other graphs with certain regularity appear as extremal class. For example, the average degree of a graph is at most the maximum eigenvalue of its adjacency matrix, and a regular graph attains the maximum [5, Lemma 3.2.1]. As we can see from the inequality of arithmetic and geometric means of a sequence of positive real numbers, extremal conditions for inequalities occur when the sequence has some regular pattern. In the last chapter of this dissertation, we consider the maximum eigenvalues of the adjacency matrices of graphs, and present sharp upper bounds of them. The graphs attain the bound also satisfy a special kind of regularity.

Chapter 2

Preliminaries

In this chapter we review some definitions and basic concepts concerning graphs, distance-regular graphs, posets, nonnegative matrices, and binomial coefficients.

2.1 Graphs

A *graph* Γ is an ordered pair (X, R) consisting of a finite *vertex set* X and an *edge set* R where each element in R is a 2-element subset of X. Two vertices $x, y \in X$ are *adjacent* if $\{x, y\} \in R$ and we use $x \sim y$ to denote that x, y are adjacent. A *path* between vertices x and y in Γ is a sequence $x_0, x_1, \cdots, x_\ell$ of distinct vertices where $x_0 = x$ and $x_\ell = y$, such that $x_i \sim x_{i+1}$ for $i = 0, 1, \cdots, \ell - 1$. The *length* of a path is the number of edges on it. The *distance* between $x, y \in X$ is the length of the shortest path between x and y and is denoted by $\partial(x, y)$. The *diameter* D of Γ is defined as D:=max$\{\, \partial(x, y) \mid x, y \in X\}$. For a vertex $x \in X$ and an integer $0 \leq i \leq D$, set $\Gamma_i(x) := \{\, z \in X \mid \partial(x, z) = i\}$. The *valency* of a vertex $x \in X$ is the cardinality of $\Gamma_1(x)$ and is denoted by d_x. For the *adjacency matrix* $A = (a_{xy})$ of Γ, we mean a binary square matrix of order $|X|$ with rows and columns indexed by the vertices in X, such that for any pair $x, y \in X$, $a_{xy} = 1$ iff $x \sim y$.

A *cycle* of length ℓ, denoted by C_ℓ, is a graph with ℓ vertices and ℓ edges whose vertices can be placed around a circle so that two vertices are adjacent if and only if they appear consequently along the cycle. The *girth* of a graph with a cycle is the length of its shortest cycle.

2.2 Distance-regular Graphs

A graph $\Gamma = (X, R)$ is called *regular* (with valency k) if each vertex in X has valency k. A graph Γ is said to be *distance-regular* whenever for all integers $0 \le h, i, j \le D$, and all vertices $x, y \in X$ with $\partial(x, y) = h$, the number

$$p^h_{ij} = |\Gamma_i(x) \cap \Gamma_j(y)|$$

is independent of x, y. The constants p^h_{ij} are known as the *intersection numbers* of Γ.

Suppose that $\Gamma = (X, R)$ is a distance-regular graph with diameter $D \ge 3$. For two vertices $x, y \in X$, with $\partial(x, y) = i$, set

$$\begin{aligned} B(x,y) &:= \Gamma_1(x) \cap \Gamma_{i+1}(y), \\ C(x,y) &:= \Gamma_1(x) \cap \Gamma_{i-1}(y), \\ A(x,y) &:= \Gamma_1(x) \cap \Gamma_i(y). \end{aligned}$$

Note that

$$\begin{aligned} |B(x,y)| &= p^i_{1\ i+1}, \\ |C(x,y)| &= p^i_{1\ i-1}, \\ |A(x,y)| &= p^i_{1\ i} \end{aligned}$$

are independent of x, y. For convenience, set $c_i := p^i_{1\ i-1}$ for $1 \le i \le D$, $a_i := p^i_{1\ i}$ for $0 \le i \le D$, $b_i := p^i_{1\ i+1}$ for $0 \le i \le D-1$ and put $b_D := 0$, $c_0 := 0$. Note that $k := b_0$ is the valency of each vertex in Γ. It is immediate from the definition of p^h_{ij} that $b_i \ne 0$ for $0 \le i \le D-1$ and $c_i \ne 0$ for $1 \le i \le D$. Moreover

$$k = a_i + b_i + c_i \quad \text{for} \quad 0 \le i \le D. \tag{2.2.1}$$

2.3 Posets

Let P denote a finite set. By a *partial order* on P, we mean a binary relation $\le$ on P such that

(i) $x \le x \quad \text{for} \quad x \in P$,

(ii) $x \le y$ and $y \le z \implies x \le z \quad \text{for} \quad x, y, z \in P$,

(iii) $x \leq y$ and $y \leq x \implies x = y \qquad$ for $x, y \in P$.

By a *partially ordered set* (or *poset*, for short), we mean a pair $(P, \leq)$, where P is a finite set, and where $\leq$ is a partial order on P. We may suppress reference to $\leq$, and just write P instead of $(P, \leq)$ if no confusion occurs. Let P denote a poset with partial order $\leq$, and let x and y denote any two elements in P. As usual, we write $x < y$ whenever $x \leq y$ and $x \neq y$, and write $x \not< y$ whenever $x < y$ is not true. We say that *y covers x* whenever $x < y$, and there is no $z \in P$ such that $x < z < y$. A sequence $x_0, x_1, \ldots, x_t$ of elements of P is said to be a *direct chain of length t* whenever x_i covers x_{i-1} for $1 \leq i \leq t$. A poset can be described by a diagram in the plane in which the elements are corresponding to dots, and y covers x whenever dot y is placed above dot x with an edge connecting them. See Figure 2 for the diagram of the poset with five elements $\{0, x, y, z, w\}$, and x, y cover 0; z, w cover both x and y. Note that $0, x, z$ is a direct chain of length 2.

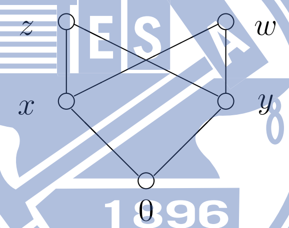

Figure 2. A poset.

Let P denote any finite poset, and let S denote any subset of P. Then there is a unique partial order on S such that for all $x, y \in S$, $x \leq y$ in S if and only if $x \leq y$ in P. This partial order is said to be *induced* from P. By a *subposet* of P, we mean a subset of P, together with the partial order induced from P. An element $x \in S$ is said to be *minimal* (resp. *maximal*) in S whenever there is no $y \in S$ such that $y < x$ (resp. $x < y$). Let $\min(S)$ (resp. $\max(S)$) denote the set of all minimal (resp. maximal) elements in P. Whenever $\min(P)$ (resp. $\max(P)$) consists of a single element, we denote it by 0 (resp. 1), and we say that P has *the least element* 0 (resp. *the greatest element* 1).

Throughout the remaining of the dissertation we assume that P is a poset with the least element 0. By an *atom* in P, we mean an element in P that covers 0. We let A_P

denote the set of atoms in P. By the *interval* $[x, y]$, where $x, y \in P$ with $x \leq y$, we mean the subposet

$$[x, y] := \{z | z \in P, x \leq z \leq y\}$$

of P.

By a *rank function* on P, we mean a function "rank" from P to the set of nonnegative integers such that $\text{rank}(0) = 0$, and for all $x, y \in P$, y covers x implies $\text{rank}(y) - \text{rank}(x) = 1$. Observe that the rank function is unique if it exists. P is said to be *ranked* whenever P has a rank function. In this case, we set

$$\text{rank}(P) := \max\{\text{rank}(x) \mid x \in P\},$$

$$P_i := \{x \mid x \in P, \text{rank}(x) = i\},$$

and observe that $P_0 = \{0\}$, $P_1 = A_P$. Also observe that P is ranked if and only if every direct chain from 0 to x has the same length for any $x \in P$. Let P be a ranked poset of rank n and fix two integers $1 \leq r < k \leq n$. The *incidence matrix* M between P_r and P_k is a $|P_r| \times |P_k|$ binary matrix with rows indexed by P_r and columns indexed by P_k such that

$$M_{xy} := \begin{cases} 1, & x \leq y; \\ 0, & \text{else} \end{cases} \quad \text{for } x \in P_r, y \in P_k.$$

Let S be a subset of P. Fix $z \in P$. Then z is said to be an *upper bound* (resp. *lower bound*) of S, if $z \geq x$ (resp. $z \leq x$) for all $x \in S$. Suppose the subposet of upper bounds (resp. lower bounds) of S has a unique minimal (resp. maximal) element. In this case we call this element *the least upper bound* or *join* (resp. *the greatest lower bound* or *meet*) of S. If $S = \{x_1, x_2, \ldots, x_t\}$ we write $x_1 \vee x_2 \vee \cdots \vee x_t$ for the join of S and $x_1 \wedge x_2 \wedge \cdots \wedge x_t$ for the meet of S. P is said to be *atomic* whenever for each nonzero element x of P, x is the join of atoms in the interval $[0, x]$. Suppose P is atomic and $x < y$ are two elements in P. Observe that the atoms in the interval $[0, x]$ is a proper subset of the atoms in the interval $[0, y]$. P is said to be a *meet semi-lattice* (resp. *join semi-lattice*) whenever P is nonempty, and $x \wedge y$ (resp. $x \vee y$) exists for all $x, y \in P$. A meet semi-lattice (resp. join semi-lattice) has a 0 (resp. 1). A meet and join semi-lattice is called a *lattice*. Note that if a nonempty set S in a meet semi-lattice has an upper bound then the join of S exists.

Suppose P is a lattice. Then P is said to be *upper semi-modular* (resp. *lower semi-modular*) whenever for all $x, y \in P$,

$$
\begin{array}{rccc}
 & y \text{ covers } x \wedge y & \longrightarrow & x \vee y \text{ covers } x \\
\text{(resp.} & x \vee y \text{ covers } x & \longrightarrow & y \text{ covers } x \wedge y).
\end{array}
$$

P is said to be *modular* whenever P is both upper semi-modular and lower semi-modular.

Figure 3 is a diagram of an upper semi-modular lattice with 7 elements. This lattice is not lower semi-modular since $1 = x \vee y$ covers x but y does not cover $0 = x \wedge y$.

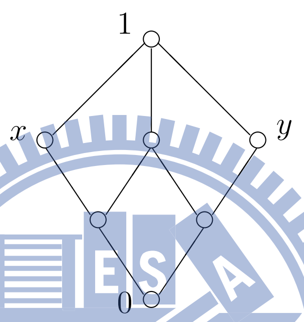

Figure 3. An upper semi-modular lattice that is not lower semi-modular.

2.4 Nonnegative Matrices

Let $A = (a_{ij})$ be a square $n \times n$ matrix. We say that A is *positive* (resp. *nonnegative*) if $a_{ij} > 0$ (resp. $a_{ij} \geq 0$) for all i, j. We say that A is *reducible* if the indices $1, 2, \cdots, n$ can be divided into two disjoint nonempty sets $i_1, i_2, \cdots, i_\mu$ and $j_1, j_2, \cdots, j_\nu$ where $\mu + \nu = n$ such that $a_{i_\alpha j_\beta} = 0$ for $\alpha = 1, 2, \cdots, \mu$ and $\beta = 1, 2, \cdots \nu$. A square matrix is called *irreducible* if it is not reducible. Simply consider the adjacent relation of a graph and the definition of irreducible matrices, we have the following proposition.

Proposition 2.4.1. *The adjacency matrix of a simple graph Γ is irreducible if and only if Γ is connected.* □

The following theorem is a fundamental result of the study on matrix theory. It is referred to as Perron-Frobenius Theorem [31, Chapter 2].

Theorem 2.4.2. *If B is a nonnegative irreducible $n \times n$ matrix with largest eigenvalue $\rho(B)$ and row-sums $r_1, r_2, \ldots, r_n$, then*

$$\rho(B) \leq \max_{1 \leq i \leq n} r_i,$$

with equality if and only if the row-sums of B are all equal. □

2.5 Binomial Coefficients and Their q-analogue

For all nonnegative integers k and n, we define the binomial coefficients $\binom{n}{k}$ as follows.

Definition 2.5.1.

$$\binom{n}{k} := \begin{cases} 0 & \text{if} \quad k > n, \\ 1 & \text{if} \quad k = 0, \\ \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots 1} & \text{otherwise.} \end{cases}$$

A *q-analogue* of a known expression is a generalization of it involving a new parameter q such that as $q \to 1$, the generalization returns to the original expression. The equality

$$\lim_{q \to 1} \frac{1-q^n}{1-q} = n$$

suggests the q-analogue of n, known as the *q-bracket* or *q-number of n*, to be that defined in the following definition.

Definition 2.5.2.

$$[n]_q := \frac{1-q^n}{1-q} = 1 + q + q^2 + \cdots + q^{n-1}.$$

Having the q-analogue of n, we naturally define the *q-factorial* as follows.

Definition 2.5.3.

$$\begin{aligned} [n]_q! &:= [1]_q \cdot [2]_q \cdots [n-1]_q \cdot [n]_q \\ &= \frac{1-q}{1-q} \cdot \frac{1-q^2}{1-q} \cdots \frac{1-q^{n-1}}{1-q} \cdot \frac{1-q^n}{1-q} \\ &= 1 \cdot (1+q) \cdots (1+q+\cdots+q^{n-2}) \cdot (1+q+\cdots+q^{n-1}). \end{aligned}$$

From q-factorial, we also define the following *q-binomial coefficients*.

Definition 2.5.4.

$$\begin{aligned}\begin{bmatrix} n \\ k \end{bmatrix}_q &:= \frac{[n]_q!}{[k]_q![n-k]_q!} \\ &= \frac{(q^n-1)(q^{n-1}-1)\cdots(q^{n-k+1}-1)}{(q^k-1)(q^{k-1}-1)\cdots(q-1).}\end{aligned}$$

In particular, $\begin{bmatrix} n \\ 0 \end{bmatrix}_q := 1$.

The q-binomial coefficients are also called *Gaussian numbers* or *Gaussian coefficients*. It is well known that $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is just the number of k-dimensional subspaces of an n-dimensional vector space over a finite field F_q [27, p. 291].

Chapter 3

Construct Pooling Spaces from Geometric Lattices

The name pooling space was given in [19] to describe a special class of ranked posets which are employed to construct pooling designs. In this chapter, we clarify a few things about the definition of pooling spaces. Then we find that a geometric lattice, a well-studied structure in literature, is also a pooling space. This provides us many classes of pooling designs. In particular we study the pooling designs constructed from affine geometries and then find some of them meet the optimal bounds related to a conjecture of Erdös, Frankl, and Füredi.

3.1 Pooling Spaces

Definition 3.1.1. Let P be a ranked poset. For any $w \in P$, define

$$w^+ = \{y \geq w \mid y \in P\}.$$

P is said to be a *pooling space* whenever w^+ is atomic for each $w \in P$.

In particular a pooling space is atomic. It is immediate from the definition that if P is a pooling space, then so is w^+ for any $w \in P$.

The following theorem evolves the study of pooling spaces.

Theorem 3.1.2. *[19, Corollary 3.2] Let P be a pooling space with rank D. Fix an integer ℓ $(1 \leq \ell \leq D)$. Let $M = M(D, \ell)$ be the matrix over $\{0, 1\}$ whose rows (resp.*

columns) are indexed by P_ℓ (resp. P_D) such that $M_{uv} = 1$ iff $u \le v$. Then for each integer b $(1 \le b \le \ell)$, M is b^d-disjunct, where

$$d := min|\bigcup [y, x] \cap P_\ell| - 1$$

with the minimum taken over all pairs (x, T) such that $x \in P_D$, $T \subseteq P_D$, $x \notin T$, $|T| \le b$, and with the union taken over all y such that $y \in P_b$, $y \le x$, $y \not\le z$ for all $z \in T$. □

Lemma 3.1.3. *Let P be a pooling space. Then each interval in P is atomic.*

Proof. Let $[x, y]$ be an interval in P and $z \in [x, y]$ with $z \ne x$. Suppose $x \in P_i$. Note that the set of atoms contained in $[x, z]$, no matter considered in x^+ or in $[x, y]$, is the same set $[x, z] \cap P_{i+1}$. Since z is the join of $[x, z] \cap P_{i+1}$ in x^+ by assumption, z is also the join of $[x, z] \cap P_{i+1}$ in $[x, y]$. □

Remark 3.1.4. The definition of pooling space was first given in [19]. However in the abstract of that paper, it was stated in an alternative way that a pooling space is a ranked poset with atomic intervals. The following example shows that this is not correct.

Example 3.1.5. Let $P = \{0, x, y, z, w\}$ and the partial order is defined as in Fig. 2 of Section 2.3. Then each interval in P is atomic. Since neither z nor w is the least upper bound of x and y, P is not atomic. Observe that P is not a meet semi-lattice.

We now give a revised version.

Proposition 3.1.6. *Let P be a ranked meet semi-lattice. Then P is a pooling space if and only if each interval in P is atomic.*

Proof. We have just proved the necessary condition in the previous lemma. To prove the sufficient condition we fix an element $w \in P$ and suppose $w \in P_s$ for some integer $0 \le s \le \text{rank}(P)$. We shall prove that w^+ is atomic. To do this fix $x \in w^+ \setminus \{w\}$ and we need to prove that x is the join of $[w, x] \cap P_{s+1}$ in w^+. By the assumption $[w, x]$ being atomic, x is the join of $[w, x] \cap P_{s+1}$ in $[w, x]$. In particular, x is an upper bound of $[w, x] \cap P_{s+1}$. Since P is a meet semi-lattice, the upper bounds of $[w, x] \cap P_{s+1}$ have a least element and denote it by y. Hence $y \leq x$ and clearly $w \leq y$, so equivalently $y \in [w, x]$. This forces $x \leq y$, since x is the least upper bound and y is also an upper bound of $[w, x] \cap P_{s+1}$. Then we obtain $x = y$. □

We give a poling space which is not a meet semi-lattice.

Example 3.1.7. Let $P = \{0, u, x, y, v, z, w\}$ and let the partial order be defined as in the Figure 4 below. Observe $z = u \vee x \vee y$ and $w = x \vee y \vee v$. The remaining properties of a pooling space hold trivially. Hence P is a pooling space. P is not a meet semi-lattice since $z \wedge w$ does not exist.

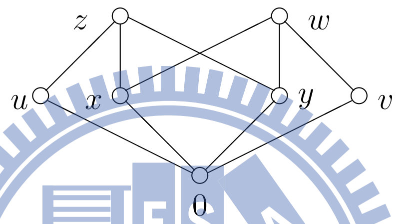

Figure 4. A pooling space which is not a meet semi-lattice.

3.2 The Contractions of Graphs

Many examples of pooling spaces were given in [19]. They are related to the Hamming matroids, the attenuated spaces, and six classical polar spaces. Among these examples there is a common property: each interval is modular. In this section we will construct pooling spaces without modular intervals. The construction in this section also can be obtained as a consequence of our main theorem in the next section. We do it earlier and repeatedly here for the purpose to give the readers a concrete impression of a pooling space, and hope that one can find one's own class of examples in the sequel.

Throughout the section let Γ denote a simple connected graph on n vertices.

Definition 3.2.1. Let $P = P(\Gamma)$ denote the set of partitions S of the vertex set $V(\Gamma)$ such that the subgraph induced by each block of S is connected. For $S, Q \in P$, define

$$S \leq Q \Longleftrightarrow S \text{ is a refinement of } Q.$$

The poset $(P(\Gamma), \leq)$ is called the poset of *contractions* of Γ.

Example 3.2.2. Let Γ denote a graph with the vertex set $\{x, y, z, w\}$ and edge set $\{\overline{xy}, \overline{yz}, \overline{zw}, \overline{wx}\}$, i.e. Γ is the 4-cycle C_4. Then the poset $P(\Gamma)$ is as in Figure 5. We delete the single element blocks in the notation of a partition, e.g. the notation 0 is used to denote the partition with four blocks $\{x\}, \{y\}, \{z\}$, $\{w\}$ respectively, and $\overline{xy}$ is used to denote the partition with three blocks $\{x, y\}$, $\{z\}$, $\{w\}$ respectively. The poset is a lattice, but not a modular lattice. This is because the join of $\overline{xy}\ \overline{zw}$ and $\overline{yz}\ \overline{wx}$ is $\overline{xyzw}$, which covers $\overline{xy}\ \overline{zw}$, but $\overline{yz}\ \overline{wx}$ does not cover their meet 0. Observe the subposet induced on $\overline{xy}^+$ is $P(C_3)$, the poset of contractions of a triangle.

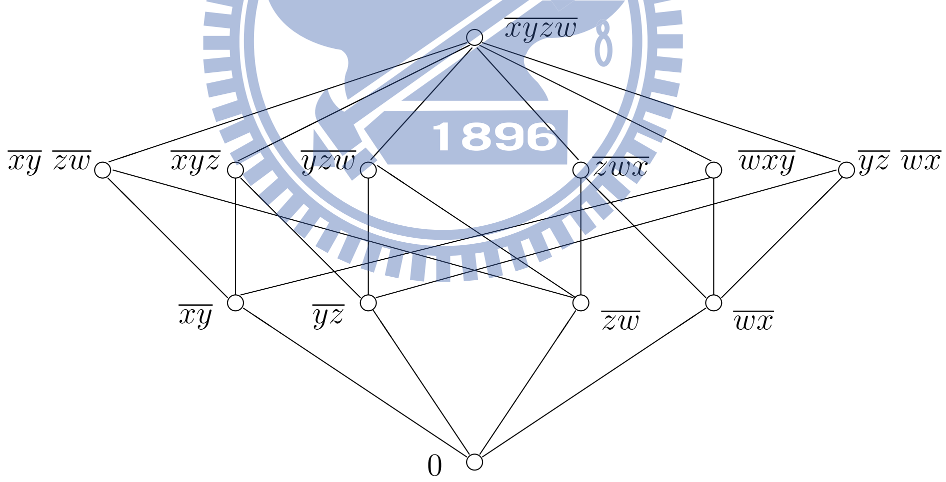

Figure 5. The poset $P(C_4)$ of contractions of C_4.

Lemma 3.2.3. *$P(\Gamma)$ is a ranked poset of rank $n-1$. The rank i elements are those elements in $P(\Gamma)$ with $n-i$ blocks for $0 \leq i \leq n-1$.*

Proof. For $N \in P(\Gamma)$ with $n-i$ blocks define the rank of N to be i, where $0 \leq i \leq n-1$. We claim that this is a rank function. Suppose that Q covers S and $\text{rank(S)} = i$. Since

S is a proper refinement of Q, $\operatorname{rank}(Q) \geq i+1$ and there are two blocks in S that are contained in the same block of Q. Let T be an element in $P(\Gamma)$ that glues these two blocks of S. Then $S < T \leq Q$ and $\operatorname{rank}(T) = \operatorname{rank}(S) + 1$. This shows $T = Q$ and $\operatorname{rank}(Q) = i + 1$. □

Proposition 3.2.4. *$P(\Gamma)$ is a pooling space of rank $n-1$.*

Proof. $P(\Gamma)$ is ranked by previous lemma. From previous lemma and the definition, each atom in $P(\Gamma)$ contains $n-1$ blocks, one block containing two adjacent vertices and each of the remaining $n-2$ blocks containing a single vertex. By identifying the atoms with the edges of Γ we find that each element $S \in P(\Gamma)$ is the join of those edges contained in the induced subgraph of Γ corresponding to each block of S. This shows that $P(\Gamma)$ is atomic. More generally, for $Q \in P(\Gamma)$, the subposet Q^+ is also atomic. This is because the subposet Q^+ is isomorphic to the poset $P(Q_\Gamma)$ of contractions of Q_Γ, where Q_Γ is the graph with the vertex set Q, and for two distinct blocks $x, y \in Q$ x is adjacent to y whenever some vertex in x is adjacent to some vertex in y. □

Remark 3.2.5. Let $\Gamma = K_n$ denote the complete graph on n vertices. Then the elements in $P = P(K_n)$ are all the partitions of the vertex set of K_n. $S(n,k) := |P_{n-k}|$ is called the *Stirling number of the second kind* where $k \geq 1$. It is well known that $S(n,k)$ can be solved by the recurrence relation

$$S(n,k) = S(n-1,k-1) + kS(n-1,k) \qquad \text{for } 1 \leq k \leq n-1$$

with initial condition $S(n,0) := 0$ for $n \geq 1$, $S(0,0) := 1$, and $S(n,n) = 1$ for $n \geq 1$. See [4, Section 8.2] for details.

By applying Proposition 3.2.4 and Remark 3.2.5 with the result in [19, Corollary 3.2] we immediately have the following corollary.

Corollary 3.2.6. *Let Γ denote a simple connected graph on n vertices and $P = P(\Gamma)$. Let $C(\Gamma,k,r)$ denote the incidence matrix between P_r and P_k where $1 \leq r < k \leq n-1$. Then $C(\Gamma,k,r)$ is r-disjunct. In particular if $\Gamma = K_n$, then the matrix $C(\Gamma,k,r)$ has size $S(n,n-r) \times S(n,n-k)$.* □

3.3 Geometric Lattices

The concept of geometric lattices can be described in very different ways. See [27, Chapter 23] for details. For the purpose to derive our main result easily, we adopt the definition that a *geometric lattice* is an upper semi-modular atomic lattice [27, Page 271]. We will show that a geometric lattice is a pooling space in this section. The following lemma is immediate from the definition.

Lemma 3.3.1. *Let P be an upper semi-modular lattice. Then the poset induced on every interval of P is an upper semi-modular lattice.* $\square$

Lemma 3.3.2. *Let P be a geometric lattice. Then the poset induced on every interval of P is a geometric lattice.*

Proof. Let $[x, y]$ denote an interval in P where $x \in P_i$. By previous lemma, it remains to show $[x, y]$ is atomic. Fix $z \in [x, y]$ with $z \neq x$. Suppose that w is the join of $P_{i+1} \cap [x, z]$, the atoms in $[x, z]$. Then $w \leq z$. We are done if $w = z$, so assume $w < z$. Then there exists an atom in $a \in [0, z] \setminus [0, w]$. Note that $a \not\leq x$. By the upper semi-modularity, $a \vee x \in P_{i+1} \cap [x, z]$ is an atom in $[x, z]$, a contradiction to $a \vee x \not\leq w$. $\square$

Lemma 3.3.3. *An upper semi-modular lattice is ranked. In particular, a geometric lattice is ranked.*

Proof. Let P be an upper semi-modular lattice and suppose that P is not ranked. Then there exists $x \in P$ such that $[x, 1]$ is not ranked, but for all atoms a of $[x, 1]$, $[a, 1]$ is ranked. Pick an atom $a \in [x, 1]$. Let f be a rank function on $[a, 1]$. We extend the function f to a function f' in $[x, 1]$ by defining

$$f'(y) := \begin{cases} f(y) + 1, & \text{if } y \in [a, 1]; \\ f(a \vee y), & \text{else.} \end{cases}$$

We shall prove f' is a rank function in $[x, 1]$. Suppose $u, v \in [x, 1]$ and u covers v. We need to show $f'(u) = f'(v) + 1$. This is clear if $v \in [a, 1]$. Assume $v \notin [a, 1]$. Suppose $u \in [a, 1]$. Then $u = a \vee v$ and $f'(u) = f(a \vee v) + 1 = f'(v) + 1$. Suppose

$u \notin [a, 1]$. Since u covers $v = (a \vee v) \wedge u$, we have $a \vee u = (a \vee v) \vee u$ covers $a \vee v$. Then $f'(u) = f(a \vee u) = f(a \vee v) + 1 = f'(v) + 1$. This concludes that $[x, 1]$ is ranked, a contradiction. $\square$

Theorem 3.3.4. *Let P be a geometric lattice. Then P is a pooling space.*

Proof. P is ranked by Lemma 3.3.3. Since each interval of P is a geometry lattice by Lemma 3.3.2, each interval is atomic. The theorem now follows from Proposition 3.1.6. $\square$

By applying Theorem 3.3.4 and Theorem 3.1.2 we immediately have the following corollary.

Corollary 3.3.5. *Let P be a geometric lattice with rank n. Let $G(P, k, r)$ denote the incidence matrix between P_r and P_k where $1 \leq r < k \leq n$. Then $G(P, k, r)$ is r-disjunct.* $\square$

Many examples of geometry lattices are listed in Chapter 23 of [27]. These are related to *linear spaces*, *Steiner systems*, *affine geometries*, *projective geometries*, and contractions of graphs. More examples are given in [17]. In some cases the corresponding results in Corollary 3.3.5 are not fully disjunct. The fully disjunct properties on projective geometries were studied in [10].

3.4 Affine Geometries

In this section we study the fully disjunct properties of the binary matrices constructed from affine geometries. The idea is exactly the same as the study of projective geometries in [10]. In fact this idea works for any geometric lattices with each interval isomorphic to a projective geometry. For completeness of the dissertation, we still provide the proof. Also there are some small computation mistakes in [10]. We will point out these mistakes after Corollary 3.4.6. In the beginning, we give the definition of affine geometries.

Definition 3.4.1. Let V denote an n-dimensional vector space over a finite field F_q, where q is the number of elements in the field. Let $P = P(V)$ denote the poset with element set

$$P = \{u + W \mid u \in V \text{ and } W \subseteq V \text{ is a subspace}\} \cup \{\emptyset\},$$

where $\emptyset$ denote the empty set. The order is defined by inclusion. Note that P is a geometric lattice of rank $n+1$. P is called the *affine geometry* and is denoted by $AG_n(F_q)$. The rank i elements in P_i are referred to as the *affine* $(i-1)$*-subspaces* of V for $1 \leq i \leq n+1$. We say that the affine subspaces $u + W$ and $v + W$ are *parallel* for vectors $u, v \in V$ and subspace $W \subseteq V$.

We immediately have the following lemma.

Lemma 3.4.2. *Let V denote an n-dimensional vector space over a finite field F_q. Let $u_1, u_2 \in V$ be elements and let $W_1, W_2 \subseteq V$ be subspaces. Then $u_1 + W_1 = u_2 + W_2$ if and only if $W_1 = W_2$ and $u_1 - u_2 \in W_1$.* □

Lemma 3.4.3. *Let V denote an n-dimensional vector space over a finite field F_q, and A denote an affine k-subspace of V. Then the number of affine r-subspaces contained in A is*

$$q^{k-r}\begin{bmatrix} k \\ r \end{bmatrix}_q,$$

where $r < k$. These affine r-subspaces in A are partitioned into

$$\begin{bmatrix} k \\ r \end{bmatrix}_q \tag{3.4.1}$$

classes, each class consisting of q^{k-r} parallel affine subspaces.

Proof. The parallel property defines an equivalent relation on the set of affine r-subspaces in A. The number of equivalent classes is as in (3.4.1) and each equivalent class consists of q^{k-r} elements by Lemma 3.4.2. □

Theorem 3.4.4. *Let V denote an n-dimensional vector space over a finite field F_q. Fix integers $1 \leq r < k \leq n$ and a positive integer b. Let $A, A_1, A_2, \ldots, A_b$ denote affine k-subspaces of V with $A \neq A_i$ for $1 \leq i \leq b$. Then there are at least*

$$q^{k-r}\begin{bmatrix} k \\ r \end{bmatrix}_q - bq^{k-r-1}\begin{bmatrix} k-1 \\ r \end{bmatrix}_q \tag{3.4.2}$$

affine r-subspaces contained in A and not contained in any of A_i for $1 \leq i \leq b$.

Proof. There are

$$q^{k-r}\begin{bmatrix}k\\r\end{bmatrix}_q$$

affine r-subspaces contained in A, some of them in some affine subspace $A\cap A_i$ for each $1\le i\le b$ to be deducted. $A\cap A_i$ takes maximal coverage of these affine r-subspaces when $A\cap A_i$ is an affine $(k-1)$-subspace, and in this situation the number of these affine r-subspaces is

$$q^{(k-1)-r}\begin{bmatrix}k-1\\r\end{bmatrix}_q.$$

□

Remark 3.4.5. For positive integers $b\le q$ and $k<n$ the number in (3.4.2) is optimal. We choose A_i to an affine k-subspace with the meet with A corresponding to each of the q parallel affine $(k-1)$-subspaces in A. Then (3.4.2) is exactly the number of affine r-subspaces contained in A and not contained in any of A_i for $1\le i\le b$.

From Lemma 3.4.3, Theorem 3.4.4, and Remark 3.4.5 we have the following corollary.

Corollary 3.4.6. *Let $P=AG_n(F_q)$ and let $E_q(n+1,k+1,r+1)$ denote the incidence matrix between P_{r+1} and P_{k+1} where $r<k$. Let $d=q^{k-r}\left[\begin{smallmatrix}k\\r\end{smallmatrix}\right]_q-bq^{k-r-1}\left[\begin{smallmatrix}k-1\\r\end{smallmatrix}\right]_q-1$. Then $E_q(n+1,k+1,r+1)$ is b^d-disjunct with size*

$$q^{n-r}\begin{bmatrix}n\\r\end{bmatrix}_q\times q^{n-k}\begin{bmatrix}n\\k\end{bmatrix}_q,$$

where b is any positive integer less than $\dfrac{q(q^k-1)}{q^{k-r}-1}$ to ensure $d\ge 0$ by (3.4.2). Moreover, if $k<n$ and b is a positive integer such that

$$\begin{cases} b\le q, & \text{if } r>0;\\ b\le q-1, & \text{if } r=0,\end{cases}\tag{3.4.3}$$

then $E_q(n+1,k+1,r+1)$ is not b^{d+1}-disjunct. □

The result in [10, Corollary 4.6] is similar to Corollary 3.4.6, but the former makes a mistake for not separating the case $r=0$ in (3.4.3) from $r>0$. This mistake inherits an earlier mistake in [10, Theorem 4.4], referring to the last line of its proof. The case $r = 0$ of (3.4.3) will be important in our following discussing. In view of (3.4.2) b increases if and only if d decreases. We set $r = 0$ and $b = q - 1$ to be the largest possible integer in Corollary 3.4.6 to obtain the following result.

Corollary 3.4.7. *$E_q(n+1, k+1, 1)$ is $(q-1)^{q^{k-1}-1}$-disjunct, but not $(q-1)^{q^{k-1}}$-disjunct, with size $q^n \times q^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_q$.* □

We promised in the beginning of this chapter to give some matrices that meet some optimal bound. These matrices are $E_q(3, 2, 1)$, where q is a power of a prime. We describe an optimal bound of an assumption below, and show the relation of this assumption and the conjecture of Erdös, Frankl and Füredi [12] later.

Assumption: Any b-disjunct matrix of size $s \times t$ with $s < t$ must have $s \geq (b+1)^2$.

We don't know if the above assumption is true, but $E_q(3, 2, 1)$ attains the equality $s = (b+1)^2$, since $E_q(3, 2, 1)$ is a $(q-1)$-disjunct matrix of size $q^2 \times (q^2 + q)$ by Corollary 3.4.7. In fact the above assumption is a consequence of the following conjecture of Erdös, Frankl, and Füredi in [12]:

EFF Conjecture: Any b-disjunct matrix of size $s \times (b+1)^2$ must have $s \geq (b+1)^2$.

Also see [9, page 29] for the above conjecture. Suppose that EFF Conjecture is true and suppose that the above assumption fails. Let M be a b-disjunct matrix of size $s \times t$ with $s < t$, but $s < (b+1)^2$. If $t \geq (b+1)^2$ then we obtain a b-disjunct matrix of size $s \times (b+1)^2$ by deleting any $t - (b+1)^2$ columns of M. This contradicts the EFF Conjecture. Suppose $t < (b+1)^2$. Then we make a larger b-disjunct matrix by taking the direct sum of M and the $((b+1)^2 - t) \times ((b+1)^2 - t)$ identity matrix to become a matrix of size $((b+1)^2 - t + s) \times (b+1)^2$. We also have a contradiction to EFF Conjecture since $(b+1)^2 - t + s < (b+1)^2$.

Note that $E_q(3, 2, 1)$ has more columns than rows. In the similar construction of disjunct matrices from a projective geometry of rank 3 [10], only square matrices can be obtained.

The results in this chapter have been included in the following paper.

"H. Huang, Y. Huang, and C. Weng, More on pooling spaces, *Discrete Mathematics*, 308 (2008), 6330-6338."

Chapter 4

Distance-regular Subgraphs in a Distance-regular Graph

Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter D. In this chapter, for given $0 \leq d \leq D$ we present a systematical way to construct a distance-regular subgraph of diameter d containing two given vertices of distance d in X. With some previous work, this construction also helps us to build a criterion that rules out the existence of some distance-regular graphs.

4.1 Strongly Closed Subgraphs

A sequence x, z, y of vertices of Γ is *geodetic* whenever

$$\partial(x, z) + \partial(z, y) = \partial(x, y),$$

where ∂ is the distance function of Γ. A sequence x, z, y of vertices of Γ is *weak-geodetic* whenever

$$\partial(x, z) + \partial(z, y) \leq \partial(x, y) + 1.$$

For a subset $\Delta \subseteq X$, Δ is *strongly closed* if for any weak-geodetic sequence x, z, y of Γ,

$$x,\ y \in \Delta \Longrightarrow z \in \Delta.$$

A subset Δ of X is *strongly closed with respect to a vertex* $x \in \Delta$ if

$$C(y, x) \subseteq \Delta \quad \text{and} \quad A(y, x) \subseteq \Delta \qquad \text{for all} \quad y \in \Delta. \tag{4.1.1}$$

Note that Δ is strongly closed if and only if for any vertex $x \in \Delta$, Δ is strongly closed with respect to x [44, Lemma 2.3]. Strongly closed subgraphs are called *weak-geodetically closed subgraphs* in [44]. If a strongly closed subgraph Δ of diameter d is regular then it has valency $a_d + c_d = b_0 - b_d$, where a_d, c_d, b_0, b_d are intersection numbers of Γ. Furthermore Δ is distance-regular with intersection numbers $a_i(\Delta) = a_i(\Gamma)$ and $c_i(\Delta) = c_i(\Gamma)$ for $1 \leq i \leq d$ [44, Theorem 4.5].

4.2 D-bounded Property and Known Results

Definition 4.2.1. Γ is said to be *d-bounded* whenever for all $x, y \in X$ with $\partial(x, y) \leq d$, there is a regular strongly closed subgraph of diameter $\partial(x, y)$ which contains x and y.

Note that a $(D-1)$-bounded distance-regular graph is clear to be D-bounded. The properties of D-bounded distance-regular graphs were studied in [43], and these properties were used in the classification of classical distance-regular graphs of negative type [45].

We list a few results which will be used later in this chapter.

Theorem 4.2.2. *([44, Theorem 4.6]) Let Γ be a distance-regular graph with diameter $D \geq 3$. Let Ω be a regular subgraph of Γ with valency γ and set $d := min\{i \mid \gamma \leq c_i + a_i\}$. Then the following (i),(ii) are equivalent.*

(i) Ω is strongly closed with respect to at least one vertex $x \in \Omega$.

(ii) Ω is strongly closed with diameter d.

In this case $\gamma = c_d + a_d$. □

The following Theorem is a combination of three previous results.

Theorem 4.2.3. *Let Γ denote a distance-regular graph with diameter $D \geq 3$. Suppose that the intersection numbers a_1, a_2, c_2 satisfy one of the following.*

(i) [14, Theorem 2] $a_2 > a_1 = 0$, $c_2 > 1$;

(ii) [44, Theorem 2] $a_1 \neq 0$, $c_2 > 1$; or

(iii) [38, Theorem 1.1] $a_2 > a_1 \geq c_2 = 1$.

Fix an integer $1 \leq d \leq D-1$ *and suppose that* Γ *contains no parallelograms of any lengths up to* $d+1$. *Then* Γ *is* d*-bounded.* $\square$

We will deal with the complemental case "$a_1 = 0, a_2 \neq 0$, and $c_2 = 1$" in Theorem 4.4.6.

4.3 The Shapes of Pentagons

Throughout this section, let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $D \geq 3$, and intersection numbers $a_1 = 0$, $a_2 \neq 0$. Such graphs are also studied in [14, 26, 33, 34, 35]. By a *pentagon* in Γ, we mean a 5-tuple $u_1u_2u_3u_4u_5$ consisting of distinct vertices in Γ such that $\partial(u_i, u_{i+1}) = 1$ for $1 \leq i \leq 4$ and $\partial(u_5, u_1) = 1$. Fix a vertex $x \in X$, a pentagon $u_1u_2u_3u_4u_5$ has *shape* i_1, i_2, i_3, i_4, i_5 *with respect to* x if $i_j = \partial(x, u_j)$ for $1 \leq j \leq 5$. By a *parallelogram of length* d, we mean a 4-tuple $xyzw$ consisting of vertices of Γ such that $\partial(x, y) = \partial(z, w) = 1$, $\partial(x, w) = d$, and $\partial(x, z) = \partial(y, w) = \partial(y, z) = d-1$. Note that any two vertices at distance 2 are always contained in a pentagon since $a_2 \neq 0$, and two nonconsecutive vertices in a pentagon of Γ have distance 2 since $a_1 = 0$. In this section we give a few lemmas which will be used in the next section.

Lemma 4.3.1. *Let* Γ *be a distance-regular graph with diameter* $D \geq 3$. *Suppose* $a_1 = 0$, $a_2 \neq 0$, *and* Γ *contains no parallelograms of lengths up to* $d+1$ *for some integer* $d \geq 2$. *Let* x *be a vertex of* Γ, *and let* $u_1u_2u_3u_4u_5$ *be a pentagon of* Γ *such that* $\partial(x, u_1) = i-1$ *and* $\partial(x, u_3) = i+1$ *for* $1 \leq i \leq d$. *Then the pentagon* $u_1u_2u_3u_4u_5$ *has shape* $i-1, i, i+1, i+1, i$ *with respect to* x.

Proof. It suffices to prove $\partial(x, u_4) = i+1$. We prove this by induction on i. The case $i = 1$ holds otherwise $\partial(x, u_4) = 1$ and $\partial(x, u_5) = 1$ which contradicts the assumption $a_1 = 0$. Suppose $i \geq 2$. Suppose to the contrary that $\partial(x, u_4) = i$. We can choose $y \in C(x, u_1)$. Thus $\partial(y, u_1) = i-2$ and $\partial(y, u_3) = i$. By the induction hypothesis, the pentagon $u_1u_2u_3u_4u_5$ has shape $i-2, i-1, i, i, i-1$ with respect to y. In particular,

$\partial(y,u_3) = \partial(y,u_4) = i$. Then xyu_4u_3 is a parallelogram of length $i+1$, a contradiction. $\square$

Other versions of Lemma 4.3.1 can be seen in [44, Lemma 6.9] and [38, Lemma 4.1] under various assumptions on intersection numbers.

The following three lemmas were formulated by A. Hiraki in [14] under an additional assumption $c_2 > 1$, but this assumption is essentially not used in his proofs. For the sake of completeness, we still provide the proofs.

Lemma 4.3.2. *Fix an integer $1 \leq d \leq D-1$, and suppose Γ does not contain parallelograms of lengths up to $d+1$. Then for any two vertices $z, z' \in X$ such that $\partial(x,z) \leq d$ and $z' \in A(z,x)$, we have $B(x,z) = B(x,z')$.*

Proof. By symmetry, it suffices to show $B(x,z) \subseteq B(x,z')$. Suppose there exists $w \in B(x,z) \setminus B(x,z')$. Then $\partial(w,z') \neq \partial(x,z)+1$. Note that $\partial(w,z') \leq \partial(w,x)+\partial(x,z') = 1+\partial(x,z)$ and $\partial(w,z') \geq \partial(w,z) - \partial(z,z') = \partial(x,z)$. This implies $\partial(w,z') = \partial(x,z)$ and $wxz'z$ forms a parallelogram of length $\partial(x,z)+1$, a contradiction. $\square$

Lemma 4.3.3. *Fix integers $1 \leq i \leq d \leq D-1$, and suppose Γ does not contain parallelograms of any lengths up to $d+1$. Let x be a vertex of Γ. Then there is no pentagon of shape $i,i,i,i,i+1$ with respect to x.*

Proof. Let $u_1u_2u_3u_4u_5$ be a pentagon of shape $i,i,i,i,i+1$ with respect to x. We derive a contradiction by induction on i. The case $i=1$ is impossible since $a_1 = 0$. Suppose $i \geq 2$. Note that $B(x,u_1) = B(x,u_2) = B(x,u_3) = B(x,u_4)$ by Lemma 4.3.2. We shall prove $C(x,u_1) = C(x,u_2) = C(x,u_3) = C(x,u_4)$.

First we prove $C(x,u_1) = C(x,u_2)$. It suffices to show $C(x,u_2) \subseteq C(x,u_1)$ since both sets have the same size c_i. To the contrary suppose there exists $v \in C(x,u_2) - C(x,u_1)$. Note that $v \in A(x,u_1)$ as $B(x,u_1) = B(x,u_2)$. Then $B(u_1,x) = B(u_1,v)$ by Lemma 4.3.2 and hence $\partial(v,u_5) = i+1$ since $u_5 \in B(u_1,x)$. Now $u_2u_1u_5u_4u_3$ has shape $i-1,i,i+1,i+1,i$ with respect to v by Lemma 4.3.1, a contradiction since $v \notin B(x,u_4) = B(x,u_2)$. This proves $C(x,u_2) \subseteq C(x,u_1)$ as desired. By symmetry, $C(x,u_3) = C(x,u_4)$.

It remains to show $C(x,u_2) \subseteq C(x,u_4)$. To the contrary suppose there exists $u \in C(x,u_2) - C(x,u_4)$. Note that $u \in A(x,u_4)$ as $B(x,u_2) = B(x,u_4)$. Then $B(u_4,x) = B(u_4,u)$ by Lemma 4.3.2 and hence $\partial(u,u_5) = i+1$ since $u_5 \in B(u_4,x)$. Hence $u_2u_1u_5u_4u_3$ has shape $i-1, i, i+1, i+1, i$ with respect to u by Lemma 4.3.1, a contradiction since $u \notin B(x,u_4)$.

Pick a vertex $v \in C(x,u_1) = C(x,u_2) = C(x,u_3) = C(x,u_4)$. Then $u_1u_2u_3u_4u_5$ is a pentagon of shape $i-1, i-1, i-1, i-1, i$ with respect to v, a contradiction to the inductive hypothesis. □

Lemma 4.3.4. *Fix integers $1 \le i \le d \le D-1$, and suppose Γ does not contain parallelograms of any lengths up to $d+1$. Let x be a vertex and $u_1u_2u_3u_4u_5$ be a pentagon of shape $i, i-1, i, i-1, i$ or of shape $i, i-1, i, i-1, i-1$ with respect to x. Then $B(x,u_1) = B(x,u_3)$.*

Proof. It suffices to show $B(x,u_3) \subseteq B(x,u_1)$ since both sets have the same size b_i. Pick $u \in B(x,u_3)$. Then $\partial(u,u_3) = i+1$. Since $\partial(u_3,u_2) = 1$ and $\partial(x,u_2) = i-1$, then $\partial(u,u_2) = i$ and similarly $\partial(u,u_4) = i$. Note that $\partial(u,u_1) \neq i-1$, otherwise by Lemma 4.3.1, the pentagon $u_1u_2u_3u_4u_5$ has shape $i-1, i, i+1, i+1, i$ with respect to u, a contradiction.

Suppose $\partial(u,u_1) = i$ for this moment. Then to avoid obtaining a pentagon $u_5u_4u_3u_2u_1$ of type $i-1, i, i+1, i, i$ or a pentagon $u_4u_5u_1u_2u_3$ of type $i, i, i, i, i+1$ with respect to u we must have $\partial(u,u_5) = i+1$ by Lemma 4.3.1 and Lemma 4.3.3. Then $\partial(x,u_5) = i$ by construction. Now u_5u_1xu is a parallelogram of length $i+1$, a contradiction.

Hence $\partial(u,u_1) = i+1$ or equivalently $u \in B(x,u_1)$. This proves $B(x,u_3) \subseteq B(x,u_1)$ as desired. □

The following lemma rules out a class of pentagons of certain shapes with respect to a given vertex.

Lemma 4.3.5. *Fix integers $1 \le i \le d \le D-1$, and suppose Γ does not contain parallelograms of any lengths up to $d+1$. Let x be a vertex. Then there is no pentagon of shape $i, i, i, i+1, i+1$ with respect to x.*

Proof. Suppose that $u_2u_3u_4u_5u_1$ is a pentagon of shape $i, i, i, i+1, i+1$ with respect to x. We derive a contradiction by induction on i. The case $i = 1$ is impossible since $a_1 = 0$. Suppose $i \geq 2$. Pick $v \in C(x, u_2)$ and note that $\partial(v, u_1) = i$ by construction. In particular $v \notin B(x, u_2)$ and $B(x, u_2) = B(x, u_3) = B(x, u_4)$ by Lemma 4.3.2, so $v \in C(x, u_4) \cup A(x, u_4)$. In fact $v \in C(x, u_4)$; otherwise $\partial(v, u_4) = i$. By considering the shape of the pentagon $u_2u_1u_5u_4u_3$ with respect to v and applying Lemma 4.3.1, we have that $\partial(v, u_5) = i$. Hence xvu_4u_5 is a parallelogram of length $i+1$, a contradiction. Thus $\partial(v, u_4) = i - 1$, and by construction we now also have $\partial(v, u_5) = i$. Note that $\partial(v, u_3) = i$; otherwise $\partial(v, u_3) = i - 1$ and $u_2u_3u_4u_5u_1$ is a pentagon of shape $i-1, i-1, i-1, i, i$ with respect to v, a contradiction to the inductive hypothesis. Now setting $x = v$ in Lemma 4.3.4, we have $B(v, u_1) = B(v, u_3)$, a contradiction since $x \in B(v, u_1) - B(v, u_3)$. $\square$

4.4 D-bounded Property and Nonexistence of Parallelograms

Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $D \geq 3$. Fix an integer $1 \leq d \leq D - 1$. Throughout this section, we assume that Γ satisfies the following conditions.

Assumption:

(i) The intersection numbers satisfy $a_1 = 0$, $a_2 \neq 0, c_2 = 1$, and

(ii) Γ contains no parallelograms of lengths up to $d + 1$.

We shall prove the d-bounded property of Γ in this section. By the definition of strongly closed subgraphs, the following proposition is easily seen.

Proposition 4.4.1. *Suppose $\Delta \subseteq X$ is a strongly closed subgraph of Γ and $ux_1vx_2x_3$ or $ux_1x_2vx_3$ is a pentagon in Γ. If $u, v \in \Delta$, then x_1, x_2, x_3 are all in Δ.*

Proof. Since $a_1 = 0$, it's easily seen that $\partial(u, v) = 2$ and u, x_i, v is weak-geodetic for $i = 1, 2, 3$. $\square$

We then give a definition.

Definition 4.4.2. For any vertex $x \in X$ and any subset $\Pi \subseteq X$, define $[x, \Pi]$ to be the subgraph induced by the set

$$\{v \in X \mid \text{there exists } y' \in \Pi, \text{ such that the sequence } x, v, y' \text{ is geodetic }\}.$$

For any $x, y \in X$ with $\partial(x, y) = d$, set

$$\Pi_{xy} := \{y' \in \Gamma_d(x) \mid B(x, y) = B(x, y')\} \tag{4.4.1}$$

and

$$\Delta(x, y) = [x, \Pi_{xy}]. \tag{4.4.2}$$

Note that $\Delta(x, y)$ contains x, y and $\Gamma_d(x) \cap \Delta(x, y) = \Pi_{xy}$. We can also easily see the following proposition.

Proposition 4.4.3. *For $x, y, z, w \in X$ and $w \in \Delta(x, y)$, if x, z, w is geodetic, then $z \in \Delta(x, y)$.*

Proof. Suppose $\partial(x, y) = d$, $\partial(x, w) = i$, and $\partial(x, z) = j$. Then $\partial(z, w) = i - j$. By the construction of Definition 4.4.2, there exists $y' \in \Pi(x, y)$ such that x, w, y' is geodetic. Hence $\partial(w, y') = d - i$. Note that $\partial(z, y') \leq \partial(z, w) + \partial(w, y') = d - j$, and $\partial(z, y') \geq \partial(x, y') - \partial(x, z) = d - j$. So $\partial(z, y') = d - j$ and thus x, z, y' are geodetic. Hence $z \in \Delta(x, y)$. □

For any $1 \leq j \leq d$, we define the following three kinds of conditions:

(B_j) For any vertices $x, y \in X$ with $\partial(x, y) = j$, $\Delta(x, y)$ is regular strongly closed with valency $a_j + c_j$.

(W_j) For any vertices $x, y \in X$ with $\partial(x, y) = j$, $\Delta(x, y)$ is strongly closed with respect to x.

(R_j) For any vertices $x, y \in X$ with $\partial(x, y) = j$, the subgraph induced on $\Delta(x, y)$ is regular with valency $a_j + c_j$

By referring to Theorem 4.2.2, the statement (B_j) holds for all $1 \leq j \leq d$ is equivalent to the combination of conditions that (W_j) and (R_j) hold for all $1 \leq j \leq d$. Our objective is to prove that (B_j) holds for $1 \leq j \leq d$ under the assumptions in the beginning of this section. We use induction on j to achieve our objective. To adequately proceed the induction process, the following two lemmas are required.

Lemma 4.4.4. *Suppose (W_j), (R_j), and thus (B_j) hold in X for $1 \leq j \leq d-1$. For any vertices $x, y \in X$ with $\partial(x, y) = d$ and for any vertex $z \in \Delta(x, y) \cap \Gamma_i(x)$, where $1 \leq i \leq d$, we have the following (i), (ii).*

(i) $A(z, x) \subseteq \Delta(x, y)$.

(ii) *For any vertex $w \in \Gamma_i(x) \cap \Gamma_2(z)$ with $B(x, w) = B(x, z)$, we have $w \in \Delta(x, y)$.*

In particular (W_d) holds.

Proof. We prove (i), (ii) by induction on $d - i$. In the case $i = d$, $z \in \Pi(x, y)$ and (i) follows by Lemma 4.3.2, and (ii) follows from the construction of $\Delta(x, y)$ in Definition 4.4.2. Suppose $i < d$.

To prove (i) we note that if $i = 1$ then $A(z, x)$ is an empty set as $a_1 = 0$, clearly contained in $\Delta(x, y)$. Hence we suppose $2 \leq i < d$ in this case. We pick a vertex $v \in A(z, x)$ and show $v \in \Delta(x, y)$. Pick $u \in \Delta(x, y) \cap \Gamma_{i+1}(x) \cap \Gamma_1(z)$. Note that (i), (ii) hold if we use u to replace z by the inductive hypothesis. Let uu_2u_3vz be a pentagon of Γ for some $u_2, u_3 \in X$. Note that uu_2u_3vz cannot have shape $i+1, i, i-1, i, i$, shape $i+1, i+2, i+1, i, i$ by Lemma 4.3.1, cannot have shape $i+1, i, i, i, i$ by Lemma 4.3.3, and cannot have shape $i+1, i+1, i, i, i$ by Lemma 4.3.5 with respect to x. Hence uu_2u_3vz has shape $i+1, i+1, i+1, i, i$ or $i+1, i, i+1, i, i$ with respect to x. In the first case we have $u_2 \in A(u, x)$, $u_3 \in A(u_2, x)$, and this implies $u_2, u_3 \in \Delta(x, y)$ by the inductive hypothesis of (i). Then $v \in \Delta(x, y)$ by Proposition 4.4.3 since x, v, u_3 is geodetic. In the latter case we have $B(x, u) = B(x, u_3)$ by Lemma 4.3.4, and consequently $u_3 \in \Delta(x, y)$ by inductive hypothesis of (ii). Then $v \in \Delta(x, y)$ by Proposition 4.4.3 since x, v, u_3 is geodetic.

To prove (ii) we first note that $\Delta(x,z)$ is a regular strongly closed subgraph of diameter i by Theorem 4.2.2 (ii) and since (B_i) holds. Suppose to the contrary that there exists $w \in \Gamma_i(x) \cap \Gamma_2(z)$ with $B(x,w) = B(x,z)$ such that $w \notin \Delta(x,y)$. Note that hence $\Delta(x,z) = \Delta(x,w)$ by construction in Definition 4.4.2 since $B(x,w) = B(x,z)$.

Let v_2 be the unique vertex in $C(w,z)$.

Claim 1. $\partial(x,v_2) = i-1$.

Proof of Claim 1. Let v_2 be the vertex between w and z. Let $zv_2wv_4v_5$ be a pentagon for some $v_2, v_4, v_5 \in X$. Since $w \in \Delta(x,z) = \Delta(x,w)$, $v_2, v_4, v_5 \in \Delta(x,z)$ by Proposition 4.4.1 and thus $v_2, v_4, v_5 \notin \Gamma_{i+1}(x)$. If $v_2 \in A(z,x)$ then $v_2, w \in \Delta(x,y)$ by (i), a contradiction. Hence $\partial(x,v_2) = i-1$.

Let u be a vertex in $\Delta(x,y) \cap \Gamma_{i+1}(x) \cap \Gamma_1(z)$, $y_3 \in A(u,v_2)$, and $y_4 \in C(y_3,v_2)$.

Claim 2. The pentagon $v_2zuy_3y_4$ has shape $i-1, i, i+1, i+1, i$ with respect to x. Moreover the pentagon is contained in $\Delta(x,y)$.

Proof of Claim 2. The shape of the pentagon $v_2zuy_3y_4$ is determined by Lemma 4.3.1. Since $y_3 \in A(u,x)$, $y_3 \in \Delta(x,y)$ by the inductive hypothesis of (i) since $d - \partial(x,u) < d - i$.

Let $w_3 \in A(y_4,w)$ and $w_4 \in C(w_3,w)$.

Claim 3. The pentagon $v_2y_4w_3w_4w$ has shape $i-1, i, i+1, i+1, i$ with respect to x and $\{w_3, w_4\} \cap \{y_3, u\} = \emptyset$.

Proof of Claim 3. Note that $\Delta(x,w) = \Delta(x,z)$ is strongly closed of diameter i since B_i holds. Also note that $v_2 \in \Delta(x,z)$. If $\partial(x,w_4) \leq i$ then $w_4 \in \Delta(x,w)$ and this forces $y_4 \in \Delta(x,z)$ by Proposition 4.4.1. For the same reason, we then have $y_3 \in \Delta(x,z)$ as $z, y_4 \in \Delta(x,z)$. We have a contradiction since $\Delta(x,z)$ has diameter i and $\partial(x,y_3) = i+1 > i = \operatorname{diam} \Delta(x,z)$. Hence $\partial(x,w_4) = i+1$ and $v_2ww_4w_3y_4$ has shape $i-1, i, i+1, i+1, i$ with respect to x by Lemma 4.3.1.

Hence by the inductive hypothesis of (i), if $w_3 \in \Delta(x,y)$ then $w_4 \in \Delta(x,y)$. Thus if $\{w_3, w_4\} \cap \{y_3, u\} \neq \emptyset$, then $w_4 \in \Delta(x,y)$. Hence $w \in \Delta(x,y)$ by Proposition 4.4.3 since x, w, w_4 is geodetic, a contradiction.

The two pentagons $v_2zuy_3y_4$ and $v_2y_4w_3w_4w$ are shown in Figure 6.

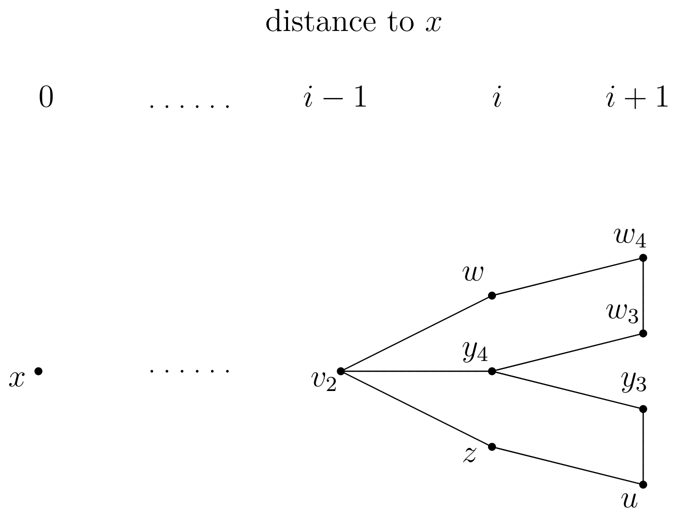

Figure 6. Two pentagons in the proof of Lemma 4.4.4(ii).

Claim 4. $B(x, y_3) \neq B(x, w_3)$.

Proof of Claim 4. Note that $B(x, u) = B(x, y_3)$ and $B(x, w_3) = B(x, w_4)$ by Lemma 4.3.2. If $B(x, y_3) = B(x, w_3)$ then by the inductive hypothesis of (ii) we have $w_3 \in \Delta(x, y)$. We then have $w_4 \in \Delta(x, y)$ by the inductive hypothesis of (i). Thus $w \in \Delta(x, y)$ by Proposition 4.4.3, a contradiction.

Let $p_3 \in A(y_3, w_3)$ and $p_4 \in C(p_3, w_3)$.

Claim 5. The pentagon $y_4y_3p_3p_4w_3$ has shape $i, i+1, i+2, i+2, i+1$ with respect to x.

Proof of Claim 5. Since p_3 is adjacent to y_3, $\partial(x, p_3) = i, i+1$ or $i+2$. Suppose $\partial(x, p_3) = i+1$, then $\partial(x, p_4) \neq i+2$ by Lemma 4.3.1, $\partial(x, p_4) \neq i+1$ by Lemma 4.3.2, and $\partial(x, p_4) \neq i$ by Lemma 4.3.4, a contradiction. Suppose $\partial(x, p_3) = i$, then $\partial(x, p_4) \neq i-1$ by Lemma 4.3.1, $\partial(x, p_4) \neq i$ by Lemma 4.3.4, and $\partial(x, p_4) \neq i+1$ by Lemma 4.3.4, also a contradiction. Thus $\partial(x, p_3) = i+2$ and the pentagon $y_4y_3p_3p_4w_3$ has shape $i, i+1, i+2, i+2, i+1$ with respect to x by Lemma 4.3.1.

Now we have three pentagons and their shapes with respect to x as shown in Figure 7.

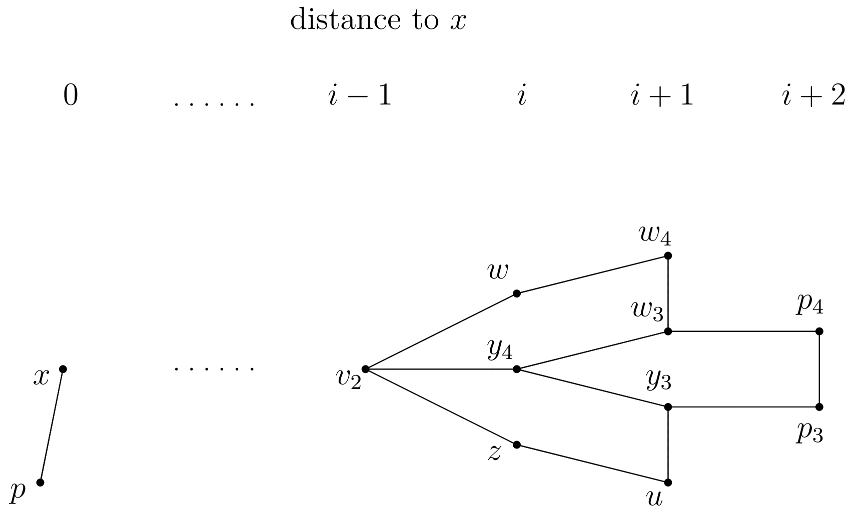

Figure 7. Three pentagons in the proof of Lemma 4.4.4(ii).

Claim 6. $B(x, y_4) \neq B(x, z)$ and thus $B(x, y_4) - B(x, z) \neq \emptyset$.

Proof of Claim 6. If $B(x, y_4) = B(x, z)$, then $\Delta(x, y_4) = \Delta(x, z)$, a strongly closed subgraph of diameter i. Since $y_4, z \in \Delta(x, z)$, we have $y_3 \in \Delta(x, z)$ by Proposition 4.4.1 and $\partial(x, y_3) = i+1$, which is a contradiction as before. The fact $B(x, y_4) - B(x, z) \neq \emptyset$ is easily seen since $|B(x, y_4)| = |B(x, z)| = b_i$.

Pick $p \in B(x, y_4) - B(x, z)$.

Claim 7. $\partial(p, z) = i$.

Proof of Claim 7. Note that $\partial(p, y_4) = i+1$ under this assumption. Also note that for this moment $\partial(p, z) = i-1$ or i.

Suppose $\partial(p, z) = i-1$. Then $zv_2y_4y_3u$ is a pentagon of shape $i-1, i, i+1, i+1, i$ with respect to p by Lemma 4.3.1. Since p_3 is adjacent to both y_3 and $\partial(x, p_3) = i+2$, we have $\partial(p, p_3) = i+2$ or $i+1$.

Next we show that $\partial(p, p_3) = i+2$. If $\partial(p, p_3) = i+1$ then xpy_3p_3 is a parallelogram of length $i+2 \leq d+1$, a contradiction. Thus $\partial(p, p_3) = i+2$.

Next we show that $\partial(p, w_3) = i+2$. We know that $\partial(p, w_3) = i, i+1$ or $i+2$. Consider the shape of the pentagon $y_4y_3p_3p_4w_3$ with respect to p. We have $\partial(p, w_3) \neq i$ by Lemma 4.3.1. If $\partial(p, w_3) = i+1$, then $\partial(p, p_4) \neq i+1$ by Lemma 4.3.3, and

$\partial(p, p_4) \neq i+2$ by Lemma 4.3.5, a contradiction to the fact that p_4 is adjacent to both w_3 and p_3. Thus we have $\partial(p, w_3) = i+2$.

We finally consider the shape of the pentagon $v_2y_4w_3w_4w$ with respect to p and get a contradiction. Consider the relative distance among x, p, v_2, and y_4, we have $\partial(p, v_2) = i$. Hence $v_2y_4w_3w_4w$ is a pentagon of shape $i, i+1, i+2, i+2, i+1$ with respect to p by Lemma 4.3.1. That is $p \in B(x, w)$, a contradiction to our assumptions $B(x, z) = B(x, w)$ and $p \in B(x, y_4) - B(x, z)$.

Claim 8. $\partial(p, w) = i$.

Proof of Claim 8. We know that $\partial(p, w) = i-1, i$ or $i+1$ since p is adjacent to x and $\partial(x, w) = i$. Suppose $\partial(p, w) = i+1$, then $p \in B(x, w)$ but $p \notin B(x, z)$ which is a contradiction to our assumption that $B(x, w) = B(x, z)$. Hence $\partial(p, w) = i-1$ or i. Most of the following arguments are similar as the ones in the previous Step 7.

Suppose $\partial(p, w) = i-1$. First we have that the pentagon $wv_2y_4w_3w_4$ is of shape $i-1, i, i+1, i+1, i$ with respect to p by Lemma 4.3.1.

Next we show that then $\partial(p, p_4) = i+2$. To avoid xpw_3p_4 to be a parallelogram of length $i+2 \leq d+1$, we have $\partial(p, p_4) = i+2$.

Then we show that $\partial(p, y_3) = i+2$. By applying Lemma 4.3.1, Lemma 4.3.3, and Lemma 4.3.5 to the pentagon $y_4w_3p_4p_3y_3$, we have that $\partial(p, y_3) = i+2$.

We finally consider the shape of the pentagon $v_2y_4y_3uz$ with respect to p and get a contradiction. Consequently $v_2y_4y_3uz$ is a pentagon of shape $i, i+1, i+2, i+2, i+1$ with respect to p by Lemma 4.3.1, which is a contradiction to $\partial(p, z) = i$.

Claim 9 $\partial(p, u) = \partial(p, w_4) = i+1$.

Proof of Claim 9. Since $\partial(p, z) = \partial(x, z) = i$, we have $p \in A(x, z)$ and thus $B(z, x) = B(z, p)$ by Lemma 4.3.2, in particular $\partial(p, u) = i+1$. Similarly, $\partial(p, w_4) = i+1$.

Claim 10. $\partial(p, y_3) = i$.

Proof of Claim 10. As $p \notin B(x, u) = B(x, y_3)$, we have $\partial(p, y_3) = i$ or $i+1$. We shall prove $\partial(p, y_3) = i$.

Suppose $\partial(p, y_3) = i+1$. We first show that $\partial(p, p_3) = i+2$. By applying Lemma 4.3.2 we have $B(y_3, x) = B(y_3, p)$. Then as $p_3 \in B(y_3, x) = B(y_3, p)$, $\partial(p, p_3) =$

$i+2$.

Next we show that $\partial(p, w_3) = i+2$. Applying Lemma 4.3.3 and Lemma 4.3.5 to the pentagon $w_3y_4y_3p_3p_4$ and considering its shape with respect to p, we find $\partial(p, w_3) \neq i+1$. Applying Lemma 4.3.1 to the pentagon $w_3p_4p_3y_3y_4$, we find $\partial(p, w_3) \neq i$. Thus $\partial(p, w_3) = i+2$.

We finally get a contradiction that pxw_4w_3 is a parallelogram of length $i+2 \leq d+1$.

Claim 11. $\partial(p, w_3) = i$.

Proof of Claim 11. Similar as the arguments in the previous Step 10, as $p \notin B(x, w_4) = B(x, w_3)$, we have $\partial(p, w_3) = i$ or $i+1$. Suppose $\partial(p, w_3) = i+1$. Applying Lemma 4.3.1 to the pentagon $y_3p_3p_4w_3y_4$, we then find $\partial(p, p_4) \neq i+2$ and thus $\partial(p, p_4) = i+1$. Then xpw_3p_4 us a parallelogram of length $i+2 \leq d+1$, a contradiction.

We finally consider the shape of the pentagon $p_4w_3y_4y_3p_3$ with respect to p to get a final contradiction. Since $\partial(x, p_3) = i+2$ and $\partial(p, y_3) = i$, we have $\partial(p, p_3) = i+1$ and similarly $\partial(p, p_4) = i+1$. To sum up, the pentagon $p_4w_3y_4y_3p_3$ has shape $i+1, i, i+1, i, i+1$ with respect to p. However, Lemma 4.3.4 now yields that $B(p, p_4) = B(p, y_4)$, which is a contradiction since $x \in B(p, p_4)$ and $x \in C(p, y_4)$. Consequently, $w \in \Delta(x, y)$ and this completes the (ii) part of this lemma.

By (i) we have $A(z, x) \subseteq \Delta(x, y)$ and by Proposition 4.4.3 we also have $C(z, x) \subseteq \Delta(x, y)$. Hence (W_d) holds by (4.1.1). $\square$

The following lemma proves (R_d) and hence completes the remaining of our goal.

Lemma 4.4.5. *Suppose (W_j), (R_j), and thus (B_j) hold in X for $1 \leq j \leq d-1$. For any vertices $x, y \in X$ with $\partial(x, y) = d$, $\Delta(x, y)$ is regular with valency $a_d + c_d$.*

Proof. Set $\Delta = \Delta(x, y)$. Clearly for any $v \in \Delta$, the construction ensures us that $\partial(x, v) \leq d$. Hence $B(y', x) \cap \Delta = \emptyset$ for any $y' \in \Pi_{xy}$. Applying Lemma 4.4.4, we have $|\Gamma_1(y') \cap \Delta| = a_d + c_d$ for any $y' \in \Pi_{xy}$. Next we show $|\Gamma_1(x) \cap \Delta| = a_d + c_d$. Note that $y \in \Delta \cap \Gamma_d(x)$ by construction of Δ. For any $z \in C(x, y) \cup A(x, y)$,

$$\partial(x, z) + \partial(z, y) \leq \partial(x, y) + 1.$$

This implies $z \in \Delta$ since Δ is strongly closed with respect to x by Lemma 4.4.4. Hence $C(x,y) \cup A(x,y) \subseteq \Delta$. Suppose $B(x,y) \cap \Delta \neq \emptyset$. Choose $t \in B(x,y) \cap \Delta$. Then there exists $y' \in \Pi_{xy}$ such that $t \in C(x,y')$, a contradiction to $B(x,y) = B(x,y')$. Hence $B(x,y) \cap \Delta = \emptyset$ and $\Gamma_1(x) \cap \Delta = C(x,y) \cup A(x,y)$. This proves $|\Gamma_1(x) \cap \Delta| = a_d + c_d$.

Since each vertex in Δ appears in a sequence of vertices $x = x_0, x_1, \ldots, x_d$ in Δ, where $\partial(x, x_\ell) = \ell$, $\partial(x_{\ell-1}, x_\ell) = 1$ for $1 \leq \ell \leq d$, and $x_d \in \Pi_{xy}$, it suffices to show

$$|\Gamma_1(x_i) \cap \Delta| = a_d + c_d \tag{4.4.3}$$

for $1 \leq i \leq d-1$. For each integer $1 \leq i \leq d$, we show

$$|\Gamma_1(x_{i-1}) \setminus \Delta| \leq |\Gamma_1(x_i) \setminus \Delta| \tag{4.4.4}$$

by the 2-way counting of the number of the pairs (z,s) for $z \in \Gamma_1(x_{i-1}) \setminus \Delta$, $s \in \Gamma_1(x_i) \setminus \Delta$ and $\partial(z,s) = 2$. For a fixed $s \in \Gamma_1(x_i) \setminus \Delta$, we have $\partial(s, x_{i-1}) = 2$ since $a_1 = 0$. Hence such a z must be one of the a_2 vertices in $A(x_{i-1}, s)$. The number of such pairs (z,s) is thus at most $|\Gamma_1(x_i) \setminus \Delta| a_2$.

On the other hand, we show this number is $|\Gamma_1(x_{i-1}) \setminus \Delta| a_2$ exactly. Fix a $z \in \Gamma_1(x_{i-1}) \setminus \Delta$. Note that $\partial(x,z) = i$ by Lemma 4.4.4, and $\partial(x_i, z) = 2$ since $a_1 = 0$. Pick any $s \in A(x_i, z)$. We shall prove $s \notin \Delta$. Suppose to the contrary $s \in \Delta$ in the below arguments and choose any $w \in C(s,z)$. Note that $\partial(x,s) \leq i$, otherwise $\partial(x,s) = i+1$ and the pentagon $x_{i-1}x_i swz$ has shape $i-1, i, i+1, i+1, i$ with respect to x by Lemma 4.3.1. Thus $w \in A(s,x)$ and then $w \in \Delta$ by Lemma 4.4.4(i). This forces $z \in \Delta$ by Proposition 4.4.3, a contradiction. We also have $\partial(x,w) \leq i$ by considering the shape of the pentagon $x_{i-1}zwsx_i$ with respect to x and Lemma 4.3.1. If $s \in A(x_i, x)$, $w \in A(s,x)$, and $z \in A(w,x)$, then $z \in \Delta$ by Lemma 4.4.4(i), a contradiction. Hence $\partial(x,w) \leq i-1$ or $\partial(x,s) \leq i-1$. Applying Lemma 4.3.4 to the pentagon $x_i x_{i-1} zws$ in the remaining cases we have $B(x,z) = B(x,x_i)$ and then $z \in \Delta$ by Lemma 4.4.4(ii), a contradiction.

From the above counting, we have

$$|\Gamma_1(x_{i-1}) \setminus \Delta| a_2 \leq |\Gamma_1(x_i) \setminus \Delta| a_2 \tag{4.4.5}$$

for $1 \leq i \leq d$. Eliminating a_2 from (4.4.5), we find (4.4.4) or equivalently

$$|\Gamma_1(x_{i-1}) \cap \Delta| \geq |\Gamma_1(x_i) \cap \Delta| \tag{4.4.6}$$

for $1 \leq i \leq d$. We have shown previously $|\Gamma_1(x_0) \cap \Delta| = |\Gamma_1(x_d) \cap \Delta| = a_d + c_d$. Hence (4.4.3) follows from (4.4.6).

□

Theorem 4.4.6. *Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $D \geq 3$, and intersection numbers $a_1 = 0$, $a_2 \neq 0$, and $c_2 = 1$. Fix an integer $1 \leq d \leq D - 1$ and suppose that Γ contains no parallelograms of any lengths up to $d + 1$. Then Γ is d-bounded.*

Proof. For $1 \leq j \leq d$, we prove (W_j) and (R_j) by induction on j. Since $a_1 = 0$, there are no edges in $\Gamma_1(x)$ for any vertex $x \in X$. If $d = 1$ in Definition 4.4.2, then $\Pi_{xy} = \{y\}$ since for any other $y' \in \Gamma_1(x)$, $y' \in B(x, y)$ but $y' \notin B(x, y')$. Consequently $\Delta(x, y) = \{x, y\}$ is an edge; in particular $\Delta(x, y)$ is regular with valency $1 = a_1 + c_1$ and is strongly closed with respect to x since $a_1 = 0$. This proves (R_1) and (W_1). For $d \geq 2$, assume (W_j), (R_j) and thus (B_j) hold for $1 \leq j \leq d - 1$. By Lemma 4.4.4 and Lemma 4.4.5, we have that (W_d), (R_d), and thus (B_d) hold. Then the proof is completed. □

Theorem 4.4.6 answers the problem proposed in [44, p. 299] and is a generalization of [5, Lemma 4.3.13], [34]. Recall that Theorem 4.2.3 (i) was proved by A. Hiraki [14]. Indeed for the lemmas stated independently in Section 4.3 we are inspired by some lemmas in [14].

Combining Theorem 4.2.3 and Theorem 4.4.6, the following characterization of d-bounded distance-regular graphs is completed.

Theorem 4.4.7. *Suppose Γ is a distance-regular graph with diameter $D \geq 3$ and the intersection number $a_2 \neq 0$. Fix an integer $2 \leq d \leq D - 1$. Then the following two conditions (i), (ii) are equivalent:*

(i) Γ is d-bounded.

(ii) Γ *contains no parallelograms of any lengths up to* $d+1$ *and* $b_1 > b_2$.

Proof. ((i) $\Rightarrow$ (ii)) Suppose that Γ is d-bounded for $d \geq 2$. Let $\Omega \subseteq \Delta$ be two regular strongly closed subgraphs of diameters $1, 2$ respectively. Since Ω and Δ have different valency $b_0 - b_1$ and $b_0 - b_2$ respectively by Theorem 4.2.2, we have $b_1 > b_2$. It is also easy to see that Γ contains no parallelograms of any lengths up to $d+1$ [44, Lemma 6.5].

((ii) $\Rightarrow$ (i)) Under the assumptions Theorem 4.4.7(ii) (hence $b_1 > b_2$) and $a_2 \neq 0$, consider the following four cases.

(a) $a_1 = 0$ and $c_2 > 1$: This case follows from Theorem 4.2.3 (i).

(b) $a_1 = 0$ and $c_2 = 1$: This case follows from Theorem 4.4.6.

(c) $a_1 \neq 0$ and $c_2 > 1$: This case follows from Theorem 4.2.3 (ii).

(d) $a_1 \neq 0$ and $c_2 = 1$: Note that in this case $a_2 > a_1 \geq c_2 = 1$. Then this case follows from Theorem 4.2.3 (iii).

$\square$

Some applications of Theorem 4.4.7 were previously given in [14, 35]. We will give a new application as Theorem 4.5.7 in the following section.

4.5 Classical Parameters

Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $D \geq 3$. Γ is said to have *classical parameters* (D, b, α, β) whenever the intersection numbers of Γ satisfy

$$c_i = \begin{bmatrix} i \\ 1 \end{bmatrix}_b \left(1 + \alpha \begin{bmatrix} i-1 \\ 1 \end{bmatrix}_b\right) \qquad \text{for } \; 0 \leq i \leq D, \tag{4.5.1}$$

$$b_i = \left(\begin{bmatrix} D \\ 1 \end{bmatrix}_b - \begin{bmatrix} i \\ 1 \end{bmatrix}_b\right)\left(\beta - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix}_b\right) \qquad \text{for } \; 0 \leq i \leq D, . \tag{4.5.2}$$

Applying (2.2.1) with (4.5.1), (4.5.2), we have

$$\begin{aligned} a_i &= \begin{bmatrix} i \\ 1 \end{bmatrix}_b \left(\beta - 1 + \alpha(\begin{bmatrix} D \\ 1 \end{bmatrix}_b - \begin{bmatrix} i \\ 1 \end{bmatrix}_b - \begin{bmatrix} i-1 \\ 1 \end{bmatrix}_b)\right) && (4.5.3) \\ &= \begin{bmatrix} i \\ 1 \end{bmatrix}_b \left(a_1 - \alpha(\begin{bmatrix} i \\ 1 \end{bmatrix}_b + \begin{bmatrix} i-1 \\ 1 \end{bmatrix}_b - 1)\right) && (4.5.4) \end{aligned}$$

for $1 \le i \le D$.

Classical parameters were introduced in [5, Chapter 6]. Graphs with such parameters yield P- and Q-polynomial association schemes. Bannai and Ito proposed the classification of such schemes in [1]. Suppose Γ has classical parameters (D, b, α, β) and $D \ge 3$. Then b is an integer, $b \ne 0$, and $b \ne -1$ [5, p. 195]. Two known classes of distance-regular graphs with classical parameters (D, b, α, β) and $b < -1$ are the dual polar graphs ${}^2A_{2D-1}(-b)$ and the Hermitian forms graphs $Her_{-b}(D)$ as listed in [5, Table 6.1]. Here we use the notation in [5, page 274]. A.A. Ivanov and S.V. Shpectorov show that if Γ has the same intersection numbers as the dual polar graph ${}^2A_{2D-1}(-b)$ then Γ is the dual polar graph ${}^2A_{2D-1}(-b)$ [20]. They also show that if Γ does not contain parallelograms of length 2 and has the same intersection numbers as the Hermitian forms graph $Her_{-b}(D)$ then Γ is the Hermitian forms graph $Her_{-b}(D)$ [21, 22]. P. Terwilliger shows the following theorem.

Theorem 4.5.1. *([39, Theorem 2.12], [44, Lemma 7.3(ii)]) Let Γ denote a distance-regular graph with classical parameters (D, b, α, β), $b < -1$, and $D \ge 3$. Then Γ contains no parallelograms of any lengths.* $\square$

More general versions of Theorem 4.5.1 can be found in [42, 26, 33]. The following is a by-product of Theorem 4.5.1.

Lemma 4.5.2. *([39, Theorem 2.11], [44, Lemma 7.3(ii)]) Let Γ denote a distance-regular graph with classical parameters (D, b, α, β) and $D \ge 3$. Suppose Γ contains no parallelograms of lengths 2. Then Γ contains no parallelograms of any lengths.* $\square$

By applying Theorem 4.5.1, the D-bounded property of Γ is proved by different authors according to different assumptions [38, 44, 34, 14]. Recall that if Γ has intersection numbers $b_1 > b_2$ and $a_2 \ne 0$ then Γ is D-bounded as stated in Theorem 4.4.7.

A poset associated with a D-bounded distance-regular graph was constructed in [43] and further studied in [45]. This produces the following two useful theorems.

Theorem 4.5.3. *([43, Corollary 3.7, Theorem 4.2]) Let Γ denote a distance-regular graph with classical parameters (D, b, α, β) and $b < -1$. Suppose that Γ is D-bounded with $D \geq 4$. Then*

$$\beta = \alpha \frac{1 + b^D}{1 - b}. \tag{4.5.5}$$

□

Theorem 4.5.4. *([45, Lemma 10.2, Theorem 10.3]) Let Γ denote a distance-regular graph with classical parameters (D, b, α, β) and $b < -1$. Suppose that Γ is D-bounded with $D \geq 4$, and Γ is neither the dual polar graph ${}^2A_{2D-1}(-b)$ nor the Hermitian forms graph $Her_{-b}(D)$. Then*

$$\alpha = (b - 1)/2, \quad \beta = -(1 + b^D)/2, \tag{4.5.6}$$

where $-b$ is a power of an odd prime. □

Recently, J. Guo and K. Wang investigated other posets associated with a D-bounded distance-regular graph [13]. F. Vanhove shows that the existence of a $(-b+1)/2$-*ovoid* in the dual polar graph ${}^2A_{2D-1}(-b)$ will imply the existence of Γ with parameters as in (4.5.6) of Theorem 4.5.4 [41].

The following two lemmas have been obtained by applying Theorem 4.5.3.

Lemma 4.5.5. *([43, Corollary 6.4]) There is no distance-regular graph Γ with classical parameters (D, b, α, β), $D \geq 4$, $c_2 = 1$, and $a_2 > a_1 > 1$.* □

Lemma 4.5.6. *([35, Theorem 2.2]) Let Γ denote a distance-regular graph with classical parameters (D, b, α, β) and $D \geq 3$. Assume the intersection numbers $a_1 = 0$, $a_2 \neq 0$, and $c_2 = 1$. Then $(b, \alpha, \beta) = (-2, -2, ((-2)^{D+1} - 1)/3)$.* □

Theorem 4.5.7. *There is no distance-regular graph with classical parameters $(D, b, \alpha, \beta) = (D, -2, -2, ((-2)^{D+1} - 1)/3)$, where $D \geq 4$.*

Proof. Let Γ denote a distance-regular graph with classical parameters $(D, b, \alpha, \beta) = (D, -2, -2, ((-2)^{D+1} - 1)/3)$, where $D \geq 4$. Then Γ contains no parallelograms of any lengths by Theorem 4.5.1. By (4.5.1), (4.5.3), we have $c_2 = 1$ and $a_2 = 2 > 0 = a_1$. Hence Γ is D-bounded by Theorem 4.4.7 and since $b_1 > b_2$. By (4.5.5), $\beta = ((-2)^{D+1} - 2)/3)$, a contradiction. $\square$

Since Witt graph M_{23} [5, Table 6.1] is a distance-regular graph with classical parameters (D, b, α, β) with $D = 3$, $b = -2$, $\alpha = -2$, and $\beta = 5$, the condition $D \geq 4$ in Theorem 4.5.7 can not be loosened to $D \geq 3$. A consequence of Theorem 4.5.7 is the following.

Corollary 4.5.8. *Let Γ denote a distance-regular graph with classical parameters (D, b, α, β), $D \geq 4$, and $c_2 = 1$. Then $a_2 = a_1$ and $a_1 \neq 0$.*

Proof. First note that $a_2 < a_1$ is impossible since $c_2 = c_1 = 1$ and this implies $b_2 > b_1$. Since $c_2 = 1$, Γ contains no parallelograms of length 2 and then contains no parallelograms of any lengths by Lemma 4.5.2. By Lemma 4.5.5, Lemma 4.5.6, and Theorem 4.5.7, only the case $a_2 > a_1 = 1$ and the case $a_2 = a_1$ remain. For the first case, H. Suzuki proves that Γ contains a regular strongly closed subgraph Ω of diameter 2 with $a_1(\Omega) = 1 = c_1(\Omega)$ in [38]. Since the Friendship Theorem [46, Theorem 8.6.39] rules out such Ω, there must be no such distance-regular graph Γ. For the latter case, we have $\alpha = -b/(1 + b)$ since $c_2 = 1$ and by (4.5.1). Applying this to (4.5.4) we find the impossibility of $a_2 = a_1 = 0$ since $b \neq -1$. $\square$

We close this chapter by proposing a few conjectures for further study. The next step to work after Corollary 4.5.8 might be the following conjecture.

Conjecture 4.5.9. *There is no distance-regular graph Γ with classical parameters (D, b, α, β), $D \geq 4$, and $c_2 = 1$.*

There is a mistake in [5, Proposition 6.1.2] which proves the above conjecture. This mistake is corrected in [6].

Remark 4.5.10. (See [5, p. 194]) The Triality graph ${}^3D_{4,2}(q)$ is a distance-regular graph with classical parameters $(3, -q, q/(1 - q), q^2 + q)$, $c_2 = 1$, and $a_1 = a_2 = q - 1$.

Hence the assumption $D \geq 4$ in Conjecture 4.5.9 is necessary. Note that the Triality graph ${}^3D_{4,2}(q)$ is not 3-bounded by Theorem 4.4.7 since $b_1 = b_2$.

In [14] A. Hiraki assumes that $D \geq 3$, $a_1 = 0$, $a_2 \neq 0$, and $c_2 > 1$ and shows that Γ is either the Hermitian forms graph $Her_2(D)$ or α, β satisfy (4.5.6) with $b = -3$. Hence the following conjecture is the first step to study the unknown case of (4.5.6).

Conjecture 4.5.11. *There is no distance-regular graph with classical parameters* $(D, b, \alpha, \beta) = (3, -3, -2, 13)$.

The results in this chapter have been included in the following paper.

"Y. Huang, Y. Pan, and C. Weng, Nonexistence of a class of distance-regular graphs, to appear."

Chapter 5

Spectral Radius and Average 2-Degree Sequence of a Graph

Let Γ=(X, R) be a connected graph and let A be the adjacency matrix of Γ. Since A is a real symmetric matrix, the eigenvalues of A are all real numbers. We represent the distinct eigenvalues of A with their corresponding multiplicities by an array as follows:

$$\begin{pmatrix} \theta_0 & \theta_1 & \cdots & \theta_D \\ \\ m_0 & m_1 & \cdots & m_D \end{pmatrix}$$

where $\theta_0 > \theta_1 > \cdots > \theta_D$. Note $m_0 + m_1 + \cdots + m_D = |X|$. This array is said to be the *spectrum* of Γ. The *spectral radius* $\rho(\Gamma)$ of Γ is the largest eigenvalue of its adjacency matrix. This parameter has been studied by many authors [2, 3, 15, 16, 24, 25, 28, 36, 37, 47] and can be used to induce some other bounds such as the upper bounds of signless Laplacian eigenvalues [7, 8]. We shall give a sharp upper bound of the spectral radius of a graph in terms of average 2-degree sequence of a graph.

5.1 Average 2-degree Sequence of a Graph

For $x \in X$, we define the *average 2-degree* M_x of x to be the average degree of the neighbors of x. In other words, $M_x = \sum_{y \sim x} d_y / d_x$, where d_x is the degree of x. Label the vertices of Γ by $1, 2, \cdots, n$ such that $M_1 \geq M_2 \geq \cdots \geq M_n$. It's trivial that a regular graph of order n with valency k has average 2-degree sequence $M_1 = M_2 = \cdots = M_n = k$. A graph of order n with identical average 2-degree (i.e. $M_1 = M_2 = \cdots = M_n$) is called *pseudo-regular* in [47]. An interesting problem could be characterizing all the nonregular pseudo-regular graphs. We provide some examples of pseudo-regular graphs that are not regular in the following Example 5.1.1.

Example 5.1.1. The following graphs are pseudo-regular but not regular.

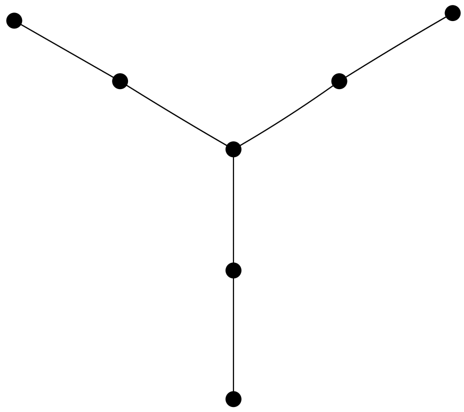

Figure 8. A graph with $M_i = 2$.

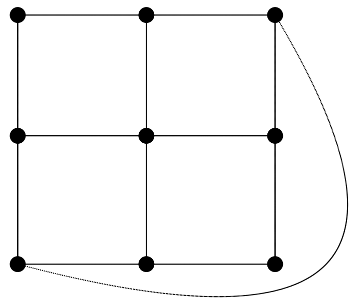

Figure 9. A graph with $M_i = 3$.

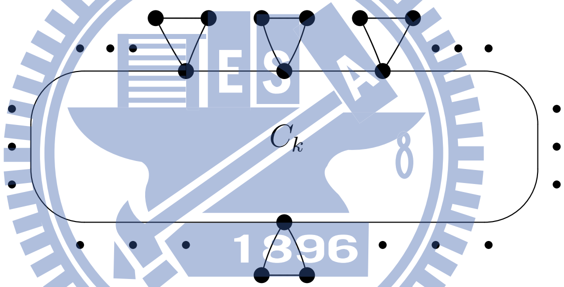

Figure 10. Graphs with $M_i = 3$.

The graph in Figure 10 has a cycle C_k of k vertices, and shares each vertex of C_k with a triangle K_3.

5.2 Upper Bounds of Spectral Radii

By setting $B = U^{-1}AU$, where $U = \text{diag}\,(d_1, d_2, \cdots, d_n)$, the following fact is easily seen from Theorem 2.4.2.

Theorem 5.2.1.

$$\rho(\Gamma) \leq M_1$$

with equality if and only if Γ *is pseudo-regular.* $\square$

In 2011 [7, Theorem 2.1], Chen, Pan, and Zhang gave the following bound.

Theorem 5.2.2. *Let* $a := max\ \{d_i/d_j \mid 1 \leq i, j \leq n\}$*. Then*

$$\rho(\Gamma) \leq \frac{M_2 - a + \sqrt{(M_2 + a)^2 + 4a(M_1 - M_2)}}{2},$$

with equality if and only if Γ *is pseudo-regular.* $\square$

We will show in Corollary 5.4.3 that Theorem 5.2.2 is indeed a generalization of Theorem 5.2.1.

5.3 Main Result

The following Theorem is our main result which is a generalization of Theorem 5.2.2.

Theorem 5.3.1. *For any* $b \geq max\ \{d_i/d_j \mid i \sim j\}$ *and* $1 \leq \ell \leq n$,

$$\rho(\Gamma) \leq \frac{M_\ell - b + \sqrt{(M_\ell + b)^2 + 4b\sum_{i=1}^{\ell-1}(M_i - M_\ell)}}{2},$$

with equality if and only if Γ *is pseudo-regular.*

Proof. For each $1 \leq i \leq \ell - 1$, let $x_i \geq 1$ be a variable to be determined later. Let $U = diag(d_1x_1, \ldots, d_{\ell-1}x_{\ell-1}, d_\ell, \ldots, d_n)$ be a diagonal matrix of size $n \times n$. Consider the matrix $B = U^{-1}AU$. Note that A and B have the same eigenvalues. Let $r_1, r_2, \ldots, r_n$ be the row-sums of B. Then for $1 \leq i \leq \ell - 1$ we have

$$\begin{aligned} r_i &= \sum_{k=1}^{\ell-1} \frac{1}{d_ix_i} a_{ik}d_kx_k + \sum_{k=\ell}^{n} \frac{1}{d_ix_i} a_{ik}d_k \\ &= \frac{1}{x_i}\sum_{k=1}^{\ell-1}(x_k - 1)a_{ik}\frac{d_k}{d_i} + \frac{1}{x_i}\sum_{k=1}^{n} a_{ik}\frac{d_k}{d_i} \\ &\leq \frac{b}{x_i}\left(\sum_{k=1,k\neq i}^{\ell-1} x_k - (\ell-2)\right) + \frac{1}{x_i}M_i, \end{aligned} \tag{5.3.1}$$

since $a_{ik}d_k/d_i \le b$. Similarly for $\ell \le j \le n$ we have

$$\begin{aligned} r_j &= \sum_{k=1}^{\ell-1} x_k a_{jk}\frac{d_k}{d_j} + \sum_{k=\ell}^{n} a_{jk}\frac{d_k}{d_j} \\ &= \sum_{k=1}^{\ell-1}(x_k-1)a_{jk}\frac{d_k}{d_j} + \sum_{k=1}^{n} a_{jk}\frac{d_k}{d_j} \\ &\le b\left(\sum_{k=1}^{\ell-1} x_k - (\ell-1)\right) + M_\ell. \end{aligned} \tag{5.3.2}$$

Let

$$\phi_\ell = \frac{M_\ell - b + \sqrt{(M_\ell+b)^2 + 4b\sum_{i=1}^{\ell-1}(M_i - M_\ell)}}{2}.$$

For $1 \le i \le \ell-1$ let

$$x_i = 1 + \frac{M_i - M_\ell}{\phi_\ell + b} \ge 1. \tag{5.3.3}$$

Then for $1 \le i \le \ell-1$ we have

$$\begin{aligned} r_i &\le \frac{b}{x_i}\left(\sum_{k=1,k\ne i}^{\ell-1} x_k - (\ell-2)\right) + \frac{1}{x_i}M_i \\ &= \frac{b\sum_{k=1}^{\ell-1}(M_k - M_\ell) + \phi_\ell M_i + bM_\ell}{\phi_\ell + b + M_i - M_\ell} \\ &= \frac{\frac{1}{4}[(M_\ell-b)^2 + (M_\ell+b)^2 + 4b\sum_{k=1}^{\ell-1}(M_k - M_\ell) - 2M_\ell^2 - 2b^2 + 4bM_\ell] + \phi_\ell M_i}{\phi_\ell + b + M_i - M_\ell} \\ &= \frac{\phi_\ell^2 + \phi_\ell b - \phi_\ell M_\ell + \phi_\ell M_i}{\phi_\ell + b + M_i - M_\ell} \\ &= \phi_\ell. \end{aligned}$$

For $\ell \le j \le n$ we have

$$\begin{aligned} r_i &\le b\left(\sum_{k=1}^{\ell-1} x_k - (\ell-1)\right) + M_\ell \\ &= \frac{b\sum_{k=1}^{\ell-1}(M_k - M_\ell) + \phi_\ell M_\ell + bM_\ell}{\phi_\ell + b} \\ &= \frac{\frac{1}{4}[4b\sum_{k=1}^{\ell-1}(M_k - M_\ell) + 2M_\ell\sqrt{(M_\ell+b)^2 + 4b\sum_{k=1}^{\ell-1}(M_k - M_\ell)} + 2M_\ell^2 + 2bM_\ell]}{\phi_\ell + b} \\ &= \frac{\phi_\ell^2 + \phi_\ell b}{\phi_\ell + b} \\ &= \phi_\ell. \end{aligned}$$

Hence by Theorem 2.4.2,

$$\rho(\Gamma) = \rho(B) \le \max_{1\le i\le n}\{r_i\} \le \phi_\ell.$$

The first part of Theorem 5.3.1 follows.

Suppose $M_1 = M_2 = \cdots = M_n$. Then $\rho(\Gamma) = M_1 = \phi_\ell$ by Theorem 5.2.1. Hence the equality in Theorem 5.3.1 follows.

To prove the necessary condition, suppose $\rho(\Gamma) = \phi_\ell$. Applying Theorem 2.4.2 and the inequalities in (5.3.1) and (5.3.2), $\phi_\ell = \rho(\Gamma) \le \max_{1\le i\le n} r_i \le \phi_\ell$. Hence $r_1 = r_2 = \cdots = r_n = \phi_\ell$, and the equalities in (5.3.1) and (5.3.2) hold. In particular,

$$b = a_{ik}\frac{d_k}{d_i} \tag{5.3.4}$$

for any $1 \le i \le n$ and $1 \le k \le \ell - 1$ with $x_k - 1 > 0$, and $M_\ell = M_n$. We consider three cases:

(i) Suppose $M_1 = M_\ell$: Clearly $M_1 = M_n$.

(ii) $M_{t-1} > M_t = M_\ell$ for some $3 \le t \le \ell$: Then $x_k > 1$ for $1 \le k \le t-1$ by (5.3.3). Hence by (5.3.4)

$$b = a_{12}\frac{d_2}{d_1} = a_{21}\frac{d_1}{d_2} = 1,$$

and $d_i = n-1$ for all $i = 1, 2, \cdots, n$. This implies Γ is regular, a contradiction.

(iii) $M_1 > M_2 = M_\ell$: Then $x_1 > 1$ by (5.3.3). Hence by (5.3.4), $b = a_{i1}d_1/d_i$ for $2 \le i \le n$. Hence $d_1 = n-1$ and $d_2 = d_3 = \cdots = d_n = (n-1)/b$. Then $(n-1)/b = M_1 > M_2 = M_n = (n-1)/b - 1 + b$. This implies $b < 1$, a contradiction. This completes the proof of the theorem.

$\square$

Note that Theorem 5.2.2 is a special case of Theorem 5.3.1 by taking $b = a$ and $\ell = 2$. The proof of Theorem 1.4 is a subtle application of Perron-Frobenius Theorem. This idea was previously employed in [28, 36]. Indeed, our proof is an edited version of the proof of Theorem 1.7 in [28].

5.4 The Shape of the Sequence $\phi_1, \phi_2, \ldots, \phi_n$

In this section, we investigate the lowest upper bound among the choices of b and ℓ. Given a decreasing sequence $M_1 \geq M_2 \geq \cdots \geq M_n$ of positive integers, consider the functions

$$\phi_\ell(x) = \frac{M_\ell - x + \sqrt{(M_\ell + x)^2 + 4x \sum_{i=1}^{\ell-1}(M_i - M_\ell)}}{2}$$

for $x \in [1, \infty)$. Note that $\phi_\ell(b)$ is the upper bound of $\rho(\Gamma)$ in Theorem 5.3.1.

The following proposition shows that the smaller the b in Theorem 5.3.1 is, the lower the upper bound of $\rho(\Gamma)$ reaches.

Proposition 5.4.1. *For any $1 \leq \ell \leq n$, $\phi_\ell(x)$ is increasing on $[1, \infty)$.*

Proof. For convenience, let

$$S = \sum_{i=1}^{\ell-1}(M_i - M_\ell).$$

To show that $\phi_\ell(x)$ is increasing on $[1, \infty)$, it is sufficient to show that the derivative of $\phi_\ell(x)$ is nonnegative. This follows from the following equivalent steps.

$$\begin{aligned}
& & \phi'_\ell(x) &\geq 0 \\
&\Leftrightarrow & -1 + \frac{M_\ell + x + 2S}{\sqrt{(M_\ell + x)^2 + 4Sx}} &\geq 0 \\
&\Leftrightarrow & \frac{M_\ell + x + 2S}{\sqrt{(M_\ell + x)^2 + 4Sx}} &\geq 1 \\
&\Leftrightarrow & (M_\ell + x + 2S)^2 &\geq (M_\ell + x)^2 + 4Sx \\
&\Leftrightarrow & 4SM_\ell + 4S^2 &\geq 0.
\end{aligned}$$

□

Note that for $1 \leq s \leq n-1$, $M_s = M_{s+1}$ implies $\phi_s(x) = \phi_{s+1}(x)$. We adopt the same viewpoint as [28, Proposition 3.1] to describe when the bound gets improved throughout the sequence $\phi_1(x)$, $\phi_2(x)$, $\ldots$, $\phi_n(x)$ in the following proposition.

Proposition 5.4.2. *Suppose $M_s > M_{s+1}$ for some $1 \leq s \leq n-1$, and let the symbol $\succeq$ denote $>$ or $=$. Then*

$$\phi_s(x) \succeq \phi_{s+1}(x) \quad \textit{iff} \quad \sum_{i=1}^{s} M_i \succeq xs(s-1).$$

Proof. Consider the following equivalent relations step by step.

$$\begin{aligned}
& \phi_s(x) > \phi_{s+1}(x) \\
\Leftrightarrow \quad & M_s - M_{s+1} + \sqrt{(M_s + x)^2 + 4x \sum_{i=1}^{s-1}(M_i - M_s)} \\
& \qquad\qquad > \sqrt{(M_{s+1} + x)^2 + 4x \sum_{i=1}^{s}(M_i - M_{s+1})} \\
\Leftrightarrow \quad & \sqrt{(M_s + x)^2 + 4x \sum_{i=1}^{s-1}(M_i - M_s)} > 2xs - (M_s + x) \\
\Leftrightarrow \quad & (M_s + x)^2 + 4x \sum_{i=1}^{s}(M_i - M_s) > 4x^2 s^2 - 4xs(M_s + x) + (M_s + x)^2 \\
\Leftrightarrow \quad & \sum_{i=1}^{s} M_i > xs(s-1),
\end{aligned}$$

where the third relation is obtained from the second by taking square on both sides, simplifying it, and deleting the common term $M_s - M_{s+1}$. Note that even if $2xs - (M_s + x) < 0$ in the third relation, squaring both sides would be proper since then $\sqrt{(M_s + x)^2 + 4x \sum_{i=1}^{s-1}(M_i - M_s)} \geq |M_s + x| \geq |2xs - (M_s + x)|$. Similarly, note that if $\sum_{i=1}^{s} M_i = xs(s-1)$, then $M_s \leq xs$ and $2xs - (M_s + x) \geq 0$. Hence

$$\begin{aligned}
& \phi_s(x) = \phi_{s+1}(x) \qquad (5.4.1) \\
\Leftrightarrow \quad & \sqrt{(M_s + x)^2 + 4x \sum_{i=1}^{s-1}(M_i - M_s)} = 2xs - (M_s + x) \\
\Leftrightarrow \quad & (M_s + x)^2 + 4x \sum_{i=1}^{s}(M_i - M_s) = 4x^2 s^2 - 4xs(M_s + x) + (M_s + x)^2 \\
\Leftrightarrow \quad & \sum_{i=1}^{s} M_i = xs(s-1).
\end{aligned}$$

□

The following corollary shows that Theorem 5.2.2 is an improvement of Theorem 5.2.1.

Corollary 5.4.3. *For any $x \in [1, \infty)$, $\phi_2(x) \leq M_1$ with equality iff $M_2 = M_1$.*

Proof. If $M_2 = M_1$ then $\phi_2(x) = M_2 \leq M_1$. Suppose $M_2 < M_1$. Choose $s = 1$ and the symbol $\succeq$ to be $>$ in Proposition 5.4.2,

$$M_1 = \phi_1(x) > \phi_2(x).$$

$\square$

Choosing $b = \max \{d_i/d_j \mid i \sim j\}$, by Proposition 5.4.2 with $s = 2$ and $x = b$, if $M_2 > M_3$ and $M_1 + M_2 > 2b$, then $\phi_2(b) > \phi_3(b)$. This is a case when Theorem 5.3.1 is truly an improvement of Theorem 5.2.2.

Example 5.4.4. In the following graph, $M_1 = M_2 = 4$, $M_3 = 7/2$, $b = 4/3$, $\phi_1(b) = \phi_2(b) = 4$, $\phi_3(b) \doteqdot 3.762$, and $\rho(\Gamma) = 1 + \sqrt{7} \doteqdot 3.646$.

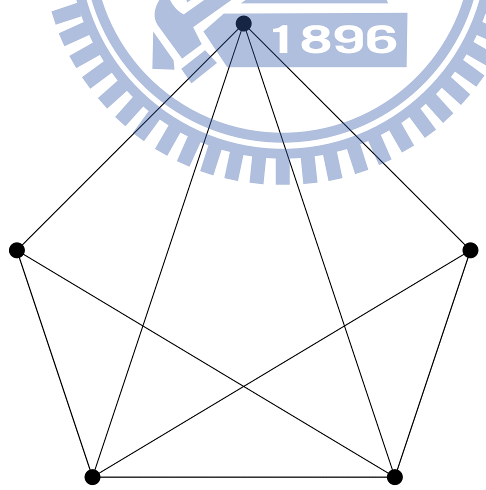

Figure 11. A graph with $\phi_2 > \phi_3$.

Note that $\phi_1(x) = M_1 \geq \phi_2(x)$ by Corollary 5.4.3, and for $2 \leq t \leq n-1$, $\sum_{i=1}^{t} M_i < xt(t-1)$ implies $M_t < x(t-1)$, and hence $\sum_{i=1}^{t+1} M_i < xt(t-1)+x(t-1) < xt(t+1)$. This implies that the sequence $\phi_1(x), \phi_2(x), \ldots, \phi_n(x)$ is composed by two parts. The first part is decreasing and the second part is increasing. In particular, if we choose $x = M_1$, $M_2 > M_3$, $s = 2$, and $\succeq$ to be $>$ in Proposition 5.4.2, then $M_1 + M_2 \not> 2M_1 = xs(s-1)$, so $\phi_2(M_1) \leq \phi_3(M_1)$. Hence $\phi_2(M_1)$ is smallest among $\phi_1(M_1), \phi_2(M_1), \ldots, \phi_n(M_1)$.

The results in this chapter have been included in the following paper.

"Y. Huang and C. Weng, Spectral radius and average 2-degree sequence of a graph, to appear."

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