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應用數學系

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圖譜剩餘定理及其應用

Spectral Excess Theorem and its Applications

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摘　要

圖譜剩餘定理 (Spectral excess theorem) 用於刻劃一個正則圖是否為距離正則圖。存在例子顯示出圖譜剩餘定理無法直接應用於非正則圖，為了使其可應用於非正則圖，在這篇論文中，我們給出一個加權版本的圖譜剩餘定理，並且用此加權版本來證明奇圍長定理 (Odd-girth theorem)，此結果解決了 E.R. van Dam 和 W.H. Haemers 兩位學者在一篇論文中所提出的問題。接著，我們應用圖譜剩餘定理及其證明的精神到二分圖的研究。眾所周知，一個二分距離正則圖的兩個半圖 (halved graphs) 皆為距離正則圖。首先我們提供幾個例子來說明兩個半圖皆為距離正則圖的二分圖不一定會是距離正則圖，然後證明在一些附加條件之下，此二分圖將會是距離正則圖。

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Abstract

The spectral excess theorem gives a quasi-spectral characterization for a regular graph to be distance-regular. An example demonstrates that this theorem cannot be directly applied to nonregular graphs. In order to make it applicable to nonregular graphs, a 'weighted' version of the spectral excess theorem is given. As an application, we show that a connected graph with $d+1$ distinct eigenvalues and odd-girth $2d+1$ is distance-regular, generalizing a result of van Dam and Haemers. We then apply this line of study to the class of bipartite graphs. It is well-known that the halved graphs of a bipartite distance-regular graph are distance-regular. Examples are given to show that the converse does not hold. Thus, a natural question is to find out when the converse is true. We give a quasi-spectral characterization of a connected bipartite weighted 2-punctually distance-regular graph whose halved graphs are distance-regular. In the case the spectral diameter is even we show that the graph characterized above is distance-regular.

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Chapter 1

Introduction

The study of characterizing the graphs whose eigenvalues and/or multiplicities satisfy a prescribed identity has a long history. For example, a well-known and real-world applicable result asserts that a connected graph is bipartite if and only if its largest eigenvalue and smallest eigenvalue have the same absolute value (see e.g. [8, Proposition 3.5.1]). Recently, the eigenvectors, especially the one associated with the largest eigenvalue, are also taking into consideration, for instances, in mathematical theory: [30, 31, 27, 28, 25]; in applications: Google's PageRank [10], Topological structures of complicated protein-protein interaction networks [11]. See [8, p. 65–69] for more applications.

Distance-regularity of graphs is a crucial concept in Algebraic Combinatorics [32]. However, it is in general not determined by the spectrum of the graph. See [21, 23, 7] for some results on spectral characterizations of distance-regular graphs. The spectral excess theorem, proposed by Fiol and Garriga [27], gives a quasi-spectral characterization for a regular graph to be distance-regular: For a connected regular graph, its average excess (the mean of the numbers of vertices at extremal distance from each vertex) is, at most, its spectral excess (a number which can be determined from its spectrum), and equality holds if and only if the graph is distance-regular. For short proofs, see

[16, 29]. Therefore, besides the spectrum, a simple combinatorial property suffices for a regular graph to be distance-regular. An example (given in Section 3.1) demonstrates that this theorem is invalid for nonregular graphs. Motivated by this, a variation of the spectral excess theorem, called the 'weighted' spectral excess theorem (Theorem 3.10), is given in order to make it applicable to nonregular graphs, using a global approach. It is worth mentioning that Fiol, Garriga and Yebra [31] also considered nonregular graphs, using a local approach, however.

Applying the spectral excess theorem, van Dam and Haemers [22] proved the 'odd-girth theorem' for regular graphs: A connected regular graph with $d+1$ distinct eigenvalues and odd-girth (that is, the length of its shortest odd cycle) $2d+1$ is distance-regular, generalizing results of Huang and Liu [37]. In the same paper, they posed the question of determining whether the regularity assumption can be removed. Moreover, they showed that the answer is affirmative for the case $d+1 = 3$, and claimed that they also had proofs for the cases $d+1 \in \{4, 5\}$. As an application of the 'weighted' spectral excess theorem, we demonstrate that the regularity assumption is indeed not necessary, that is, the odd-girth theorem is not restricted to regular graphs (Theorem 3.19). Because the odd-girth is determined by the spectrum, this result is also a generalization of the spectral characterization of the generalized odd graphs [36, 37].

We then apply this line of study to the class of bipartite graphs. It is well-known that the halved graphs of a bipartite distance-regular graph are distance-regular ([15], [6, Proposition 4.2.2]). Examples are given (in Section 4.4) to show that the converse does not hold, that is, a connected bipartite graph whose halved graphs are distance-regular may not be distance-regular. Thus, a natural question is to find out when the converse is true. We will give a quasi-spectral characterization of graphs when an identity involving eigenvalues, multiplicities, the eigenvector corresponding to the largest eigenvalue, and partial graph structure is satisfied (Theorem 4.15).

The contents of the following two papers are included in this dissertation:

1. G.-S. Lee and C.-w. Weng, A spectral excess theorem for nonregular graphs, *J. Combin. Theory Ser. A* 119 (2012), 1427–1431.

2. G.-S. Lee and C.-w. Weng, A characterization of bipartite distance-regular graphs, *Linear Algebra Appl.* 446 (2014), 91–103.

This dissertation is organized as follows. In the next chapter we review some basic notation and results on which our study is based. The spectral excess theorem and its 'weighted' version for nonregular graphs, together with an application (Odd-girth theorem) and related results, are given in Chapter 3. In the last chapter, we focus on bipartite graphs, and give a characterization of bipartite distance-regular graphs.

Chapter 2

Preliminaries

Let us first recall some basic notation and results on which our study is based.

2.1 Basic notation

A *graph* $G = (V, E)$ consists of a vertex set V and an edge set E, where each element in E (called an *edge* of G) is a 2-element subset of V. Two vertices u and v are *adjacent*, or *neighbors*, if $\{u, v\} \in E$. Two edges e and f are *incident* to a common vertex u of G if $e \cap f = u$. A *complete graph* is a simple graph in which any two vertices are adjacent. Let K_n denote a complete graph of n vertices. The *degree* of a vertex u, denoted by $\deg_G(u)$, is the number of vertices adjacent to u. Define $\overline{k} := \sum_{u \in V} \deg_G(u)/n$ to be the *average degree* of G, where n is the number of vertices of G. If all vertices have the same degree then the graph is called *regular*. A *walk* in a graph G is a sequence $x_0, x_1, \ldots, x_t$ of vertices, not necessary distinct, such that any two successive elements of which are adjacent. A walk without repeated (internal) vertices is called a *path*. A *cycle* is a path $x_0, x_1, \ldots, x_t$ with $x_0 = x_t$. The *length* of a walk, path, or cycle is the number of edges on it. Let C_n denote a cycle of length n. A cycle is *odd* or *even* depending on whether its length is odd or even. The *girth* of a graph is the length of its shortest cycle. A *u, v-path* in G is a path whose vertices of degree 1 (called its *endpoints*) are u and v. A graph G is *connected* if a u, v-path exists for every pair of vertices u, v of G. In this dissertation, G denotes a finite, simple, and connected graph with n vertices. The *distance* between two vertices u and v of G, denoted by $\partial(u, v)$, is the length of a shortest u, v-path. The parameter $D := \max\{\partial(u, v) \mid u, v \in V\}$ is called the *diameter* of G. For a vertex $u \in V$ and $0 \leq i \leq D$, let $G_i(u)$ be the set of vertices at distance i from u. The *adjacency matrix* A of G is the binary matrix indexed by the vertex set V, where the entry $(A)_{uv} = 1$ if $\partial(u, v) = 1$, and $(A)_{uv} = 0$ otherwise. The *eigenvalues* of a matrix M are the numbers λ such that $Mx = \lambda x$ has a nonzero solution vector; each such solution is an *eigenvector* associated with λ. The *eigenvalues* of a graph are the eigenvalues of its adjavency matrix. Let the *spectral diameter* d of G be the number of distinct eigenvalues minus one. The *spectrum* of G, denoted by sp $G = \{\lambda_0^{m_0}, \lambda_1^{m_1}, \ldots, \lambda_d^{m_d}\}$, is the list of distinct eigenvalues λ_i's in decreasing order: $\lambda_0 > \lambda_1 > \cdots > \lambda_d$, and the superscripts stand for their multiplicities $m_i = m(\lambda_i)$, $0 \leq i \leq d$. Note that $m_0 = 1$ since G is connected. It is well-known that $Z(x) := \prod_{i=0}^{d}(x - \lambda_i)$ is the *minimal polynomial* of G and $D \leq d$ [3, Chapter 2]. Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are *isomorphic* if there is a bijection $f : V_1 \to V_2$ such that $\{u, v\} \in E_1$ if and only if $\{f(u), f(v)\} \in E_2$. Two nonisomorphic graphs are said to be *cospectral* if they have the same spectrum. A graph is *bipartite* if its vertex set can be partitioned into two subsets X and Y such that all edges meet both X and Y; such a partition (X, Y) is called a *bipartition* of the graph. Then, its adjacency matrix is of the form

$$A = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix},$$

where B is an $|X| \times |Y|$ matrix. It follows that the spectrum of a bipartite graph is symmetric with respect to zero: If $(x, y)^T$ is an eigenvector of A associated with eigenvalue λ, then $(x, -y)^T$ is an eigenvector associated with eigenvalue $-\lambda$. (The converse also holds, see e.g. [8, Proposition 3.5.1].) Note that a graph is bipartite if and only if it has no odd cycle (see e.g. [45, Theorem 1.2.18]). A bipartite graph with bipartition (X, Y) is called a *complete bipartite graph* if every vertex in X is adjacent to every vertex in Y. Let $K_{s,t}$ denote a complete bipartite graph with bipartition (X, Y), where $|X| = s$ and $|Y| = t$. A graph is *multipartite* if its vertex set can be partitioned into non-empty subsets, called *parts*, such that any two vertices in the same part are nonadjacent. Furthermore, a *complete multipartite graph* is a multipartite graph such that any two vertices in different parts are adjacent. Let $K_{m_1,m_2,\dots,m_t}$ denote a complete multipartite graph with t parts, where m_i is the number of vertices in the i^{th} part, $1 \leq i \leq t$. The *line graph* $L(G)$ of G is the graph whose vertices are the edges of G, and two such vertices are adjacent in $L(G)$ if the corresponding edges are incident to a common vertex of G. A *strongly regular graph* with parameter (n, k, λ, μ) (for short, an $srg(n, k, \lambda, \mu)$) is a graph on n vertices which is regular of degree k such that any two adjacent vertices have exactly λ common neighbors, and any two nonadjacent vertices have exactly μ common neighbors. For example, the cycle graph C_5 is an $srg(5, 2, 0, 1)$. Note that a connected regular graph with exactly three distinct eigenvalues is strongly regular [32].

2.2 Distance-regular graphs

Recall that $G_i(u)$ denotes the set of vertices at distance i from a given vertex u. For $0 \leq i \leq D$ and two vertices $u, v \in V$ at distance i, set

$$\begin{aligned} c_i(u, v) &:= |G_1(v) \cap G_{i-1}(u)|, \\ a_i(u, v) &:= |G_1(v) \cap G_i(u)|, \text{ and} \\ b_i(u, v) &:= |G_1(v) \cap G_{i+1}(u)|. \end{aligned}$$

We say that these parameters are *well-defined* if they are independent of the choice of u, v. In this case we use the symbols c_i, a_i and b_i for short. A connected graph

G with diameter D is called *distance-regular* if the above-mentioned parameters are well-defined. In other words, a connected graph G with diameter D is distance-regular if there are constants c_i, a_i, b_i, $0 \le i \le D$, such that for any two vertices u and v at distance i, among the neighbors of v, there are c_i at distance $i-1$ from u, a_i at distance i, and b_i at distance $i+1$. The mentioned constants c_i, a_i, b_i are called the *intersection numbers*. Note that a distance-regular graph is regular with valency $k := b_0$. Moreover, a distance-regular graph with diameter 2 is the same thing as a connected strongly regular graph, where $\lambda = a_1$ and $\mu = c_2$. Here we give some simple examples of distance-regular graphs that will be used later (in Section 4.4): complete graphs, complete bipartite graph $K_{s,t}$ with $s = t$, and complete multipartite graphs with each part having the same number of vertices.

For $0 \le i \le D$, define the *distance-i matrix* A_i of G to be the matrix indexed by the vertex set V such that the entry $(A_i)_{uv} = 1$ if $\partial(u, v) = i$, and $(A_i)_{uv} = 0$ otherwise. In particular, $A_0 = I$ is the identity matrix and $A_1 = A$ is the adjacency matrix. Clearly, $A_0+A_1+\cdots+A_D = J$, the all-ones matrix. Now the above definition of distance-regular graphs is equivalent to the equations

$$AA_i = c_{i+1}A_{i+1} + a_iA_i + b_{i-1}A_{i-1} \quad (0 \le i \le D), \tag{2.1}$$

where $b_{-1} = c_{D+1} := 0$ (see e.g. [6]). Define $f_0 = 1$ and $f_1 = x$. If G is distance-regular, by iteratively applying (2.1), there exist polynomials f_i's, with $\deg f_i = i$, such that $A_i = f_i(A)$ for $0 \le i \le D$. These polynomials $f_0, f_1, \ldots, f_D$ are called the *distance polynomials* of a distance-regular graph. Since $(xf_D - b_{D-1}f_{D-1} - a_Df_D)(A) = 0$, by definition of the minimal polynomial, it follows that $d + 1 \le D + 1$ and thus $D = d$.

2.3 Predistance polynomials

In this section we introduce the concept of orthogonal polynomials [42] related to a graph. The basic idea is to generalize the study of distance-regular graphs (see [6, 43, 3, 24]).

2.3.1 General setting

Let G be a finite, simple, and connected graph of n vertices, with $d+1$ distinct eigenvalues $\lambda_0 > \lambda_1 > \cdots > \lambda_d$. Recall that $Z(x) := \prod_{i=0}^{d}(x-\lambda_i)$ is the minimal polynomial of G. From the spectrum $\operatorname{sp} G = \{\lambda_0^{m_0}, \lambda_1^{m_1}, \ldots, \lambda_d^{m_d}\}$ of G we consider the $(d+1)$-dimensional vector space $\mathbb{R}_d[x] \cong \mathbb{R}[x]/(Z(x))$ of polynomials of degrees at most d over the real number field $\mathbb{R}$ with inner product

$$\langle p, q\rangle_G := \sum_{i=0}^{d} \frac{m_i}{n} p(\lambda_i) q(\lambda_i) = \operatorname{tr}(p(A)q(A))/n, \tag{2.2}$$

and norm

$$\|p\|_G := \sqrt{\langle p, p\rangle_G}$$

for $p, q \in \mathbb{R}_d[x]$, where $\operatorname{tr}(M)$ denotes the trace of the square matrix M (i.e., the sum of the diagonal entries of M). It is well-known that $\operatorname{tr}(A^\ell) = \sum_{i=0}^{d} m_i \lambda_i^\ell$ for $\ell \geq 0$. Note that $\langle p, p\rangle_G \geq 0$ with equality if and only if $p = 0$. Moreover, the defined inner product satisfies the property that $\langle xp, q\rangle_G = \langle p, xq\rangle_G$. For $p, q \in \mathbb{R}_d[x]$, let

$$\operatorname{Proj}_p(q) := \frac{\langle p, q\rangle_G}{\|p\|_G^2} p \tag{2.3}$$

denote the projection of q onto p. Define polynomials $p'_0 = 1, p'_1, \ldots, p'_d$ of $\mathbb{R}_d[x]$ recursively by the Gram–Schmidt procedure:

$$p'_{i+1} = x^{i+1} - \sum_{k=0}^{i} \operatorname{Proj}_{p'_k}(x^{i+1}) \tag{2.4}$$

for $0 \le i \le d-1$. Then $p'_0 = 1, p'_1 = x, \ldots, p'_d$ is an orthogonal basis of $\mathbb{R}_d[x]$ such that p'_i has degree i and leading coefficient 1. We claim that $p'_i(\lambda_0) > 0$ for $0 \le i \le d$ [16]. Let $\theta_1, \theta_2, \ldots, \theta_h$ be zeros of p'_i in (λ_d, λ_0) for which p'_i takes opposite signs in $(\theta_j - \epsilon, \theta_j)$ and in $(\theta_j, \theta_j + \epsilon)$ for all $1 \le j \le h$ and for some $\epsilon > 0$. Set $q = \prod_{j=1}^{h}(x - \theta_j)$. Then $qp'_i \ge 0$ for all $x \in [\lambda_d, \lambda_0]$ or $qp'_i \le 0$ for all $x \in [\lambda_d, \lambda_0]$. Since p'_i has at most i real roots and $h \le i < d+1$, there exists an eigenvalue λ_j such that $q(\lambda_j)p'_i(\lambda_j) \ne 0$. As a result, $\langle q, p'_i \rangle_G \ne 0$ by (2.2). Since q can be written as a linear combination of $p'_0, p'_1, \ldots, p'_h$, we deduce that $h = i$ and all zeros of p'_i are zeros of q. Thus $q = p'_i$ and hence $p'_i(\lambda_0) = q(\lambda_0) > 0$.

Set

$$p_i = \frac{p'_i(\lambda_0)}{\|p'_i\|_G^2} p'_i. \tag{2.5}$$

Then $p_0 = 1, p_1 = \lambda_0 x/\overline{k}, \ldots, p_d$ satisfy $\deg p_i = i$ and $\langle p_i, p_j \rangle_G = \delta_{ij} p_i(\lambda_0)$ for $0 \le i, j \le d$, where $\delta_{ij} = 1$ if $i = j$, and 0 otherwise [27]. A simple example will be given later (Example 2.1) in order to demonstrate the computational procedures. Moreover, $p_0, p_1, \ldots, p_d$ is the unique system of orthogonal polynomials in $\mathbb{R}_d[x]$ with such properties. To prove uniqueness, first note that $p_0, p_1, \ldots, p_d$ is a basis of the vector space $\mathbb{R}_d[x]$, that is, for any polynomial $p \in \mathbb{R}_d[x]$ with $\deg p = i$, $0 \le i \le d$, we have $p = \sum_{j=0}^{i} \alpha_j p_j$ for some $\alpha_j \in \mathbb{R}$ with $\alpha_i \ne 0$. Suppose that there exist $q_0, q_1, \ldots, q_d$ satisfying $\deg q_i = i$ and $\langle q_i, q_j \rangle_G = \delta_{ij} q_i(\lambda_0)$ for $0 \le i, j \le d$. We claim that $q_i = p_i$ for $0 \le i \le d$. As mentioned above, write $q_i = \sum_{j=0}^{i} \alpha_{ij} p_j$ for some $\alpha_{ij} \in \mathbb{R}$ with $\alpha_{ii} \ne 0$. Since $q_0 = \alpha_{00} p_0$ for some nonzero real number α_{00}, we have

$$\alpha_{00} p_0(\lambda_0) = q_0(\lambda_0) = \langle q_0, q_0 \rangle_G = \langle \alpha_{00} p_0, \alpha_{00} p_0 \rangle_G = \alpha_{00}^2 p_0(\lambda_0),$$

which implies that $\alpha_{00} = 1$ and thus $q_0 = p_0$. Likewise, $q_1 = \alpha_{10} p_0 + \alpha_{11} p_1$ for some real numbers α_{10}, α_{11} with $\alpha_{11} \neq 0$. Then

$$0 = \langle q_1, q_0 \rangle_G = \langle \alpha_{10} p_0 + \alpha_{11} p_1, p_0 \rangle_G = \alpha_{10} p_0(\lambda_0),$$

which implies that $\alpha_{10} = 0$, and thus

$$\alpha_{11} p_1(\lambda_0) = q_1(\lambda_0) = \langle q_1, q_1 \rangle_G = \langle \alpha_{11} p_1, \alpha_{11} p_1 \rangle_G = \alpha_{11}^2 p_1(\lambda_0),$$

which implies that $\alpha_{11} = 1$. Hence $q_1 = p_1$. Using the same argument, it follows that $q_i = p_i$ for $0 \leq i \leq d$.

These polynomials p_0, p_1, $\ldots, p_d$ are called the *predistance polynomials* of G, which satisfy a three-term recurrence of the form

$$xp_i = \gamma_{i+1} p_{i+1} + \alpha_i p_i + \beta_{i-1} p_{i-1} \tag{2.6}$$

for $0 \leq i \leq d$, where γ_{i+1}, α_i, β_{i-1} are scalars in $\mathbb{R}$, called the *preintersection numbers* of G, with $\beta_{-1} = \gamma_{d+1} := 0$ [20]. This property can be easily explained in the following [16]. The polynomial xp_i has degree $i+1$ and thus can be expressed as $xp_i = \sum_{j=0}^{i+1} \alpha_{ij} p_j$ for some $\alpha_{ij} \in \mathbb{R}$. For $j < i-1$, $\alpha_{ij} = 0$, since $\alpha_{ij} \langle p_j, p_j \rangle_G = \langle xp_i, p_j \rangle_G = \langle p_i, xp_j \rangle_G = 0$. Hence there are only three terms remained in the expression of xp_i. After renaming the coefficients, the above three-term recurrence follows. Note that, for $0 \leq i \leq d-1$,

$$\gamma_{i+1} = \frac{\langle xp_i, p_{i+1} \rangle_G}{\|p_{i+1}\|_G^2} \neq 0 \quad \text{and} \quad \beta_i = \frac{\langle xp_{i+1}, p_i \rangle_G}{\|p_i\|_G^2} = \frac{\langle p_{i+1}, xp_i \rangle_G}{\|p_i\|_G^2} \neq 0.$$

Moreover, $\alpha_i + \beta_i + \gamma_i = \lambda_0$ for $0 \leq i \leq d$, where $\gamma_0 := 0$ and $\beta_d := 0$ [13].

For a pair of $n \times n$ symmetric matrices M, N over real number field $\mathbb{R}$, define the inner product

$$\langle M, N \rangle := \frac{1}{n} \mathrm{tr}(MN) = \frac{1}{n} \sum_{i,j} M_{ij} N_{ij} = \frac{1}{n} \sum_{i,j} (M \circ N)_{ij}, \tag{2.7}$$

and the norm

$$\|M\| := \sqrt{\langle M, M \rangle},$$

where " $\circ$ " is the entrywise or Hadamard product of matrices. Thus, by (2.2) and (2.7), we obtain that

$$\langle p, q\rangle_G = \langle p(A), q(A)\rangle \tag{2.8}$$

for $p, q \in \mathbb{R}_d[x]$. Note that the equation (2.8) is a useful property that can be used to compute predistance polynomials of a graph. As mentioned before, we demonstrate in the following the computational procedures for a simple example: P_3, a path of three vertices.

Example 2.1. The spectrum of P_3 is $\{\sqrt{2}, 0, -\sqrt{2}\}$. Note that $\mathrm{tr}(I) = 3$, $\mathrm{tr}(A) = 0$, $\mathrm{tr}(A^2) = 4$, $\mathrm{tr}(A^3) = 0$ and $\mathrm{tr}(A^4) = 8$. By (2.3), (2.4), (2.7) and (2.8),

$$\begin{aligned}
p'_0 &= 1,\\
p'_1 &= x - \frac{\langle 1, x\rangle_G}{\langle 1, 1\rangle_G}\cdot 1 = x - \frac{\mathrm{tr}(A)}{\mathrm{tr}(I)} = x,\\
p'_2 &= x^2 - \frac{\langle 1, x^2\rangle_G}{\langle 1, 1\rangle_G}\cdot 1 - \frac{\langle x, x^2\rangle_G}{\langle x, x\rangle_G}\cdot x = x^2 - \frac{\mathrm{tr}(A^2)}{\mathrm{tr}(I)}\cdot 1 - \frac{\mathrm{tr}(A^3)}{\mathrm{tr}(A^2)}\cdot x = x^2 - 4/3.
\end{aligned}$$

Note that, by (2.7),

$$\begin{aligned}
\langle p'_0, p'_0\rangle_G &= \langle 1, 1\rangle_G = \mathrm{tr}(I)/n = 1,\\
\langle p'_1, p'_1\rangle_G &= \langle x, x\rangle_G = \mathrm{tr}(A^2)/n = 4/3\ (= \overline{k}),\\
\langle p'_2, p'_2\rangle_G &= \langle x^2 - 4/3, x^2 - 4/3\rangle_G = \mathrm{tr}(A^4 - 8A^2/3 + 16I/9)/n = 8/9.
\end{aligned}$$

Thus, by (2.5),

$$\begin{aligned}
p_0 &= \frac{1}{\langle 1, 1\rangle_G}\cdot 1 = 1,\\
p_1 &= \frac{\lambda_0}{\langle x, x\rangle_G}\cdot x = 3\sqrt{2}x/4\ (= \lambda_0 x/\overline{k}),\\
p_2 &= \frac{\lambda_0^2 - 4/3}{\langle x^2 - 4/3, x^2 - 4/3\rangle_G}\cdot (x^2 - 4/3) = 3(x^2 - 4/3)/4.
\end{aligned}$$

Note that $\overline{k} \leq \lambda_0$ with equality if and only if G is regular [8, Proposition 3.1.2], and $p_1 = \lambda_0 x/\overline{k}$. As a result, $p_1 = x$ if and only if G is regular. Recall that the distance polynomials $f_0, f_1, \ldots, f_D$ of a distance-regular graph satisfy $A_i = f_i(A)$ for $0 \leq i \leq D$. Note that, in general, the equations $A_i = p_i(A)$ does not hold. For example, if G is nonregular, then $p_1(A) = \lambda_0 A/\overline{k} \neq A_1$. The following result gives a characterization of distance-regular graphs.

Proposition 2.2. *A graph G with $d+1$ distinct eigenvalues is distance-regular if and only if $A_i = p_i(A)$ for every $0 \leq i \leq d$.*

Proof. ($\Rightarrow$) Suppose that G is distance-regular. Then G is regular, the distance polynomials f_i's satisfy $\deg f_i = i$ and $A_i = f_i(A)$ for $0 \leq i \leq D$, and we have $D = d$ as mentioned before. Now it suffices to show that $f_i = p_i$ for every $0 \leq i \leq d$. Since G is regular, $AJ = \lambda_0 J$, where J is the all-ones matrix. Then A_i has constant row sum $f_i(\lambda_0)$ since $A_i J = f_i(A)J = f_i(\lambda_0)J$. By uniqueness of predistance polynomials, the result follows from

$$\langle f_i, f_j \rangle_G = \langle f_i(A), f_j(A) \rangle = \langle A_i, A_j \rangle = \begin{cases} f_i(\lambda_0) & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

($\Leftarrow$) Suppose that $A_i = p_i(A)$ for every $0 \leq i \leq d$. Since $A_d = p_d(A)$, by taking norms, we have $\|A_d\|^2 = p_d(\lambda_0) > 0$, which implies that $D = d$. Then the three-term recurrence (2.6) turns into the equations (2.1) and thus the graph is distance-regular. $\square$

The parameter $p_d(\lambda_0)$, which is called the *spectral excess* of G, can be expressed in terms of the spectrum:

$$p_d(\lambda_0) = \frac{n}{\pi_0^2} \left(\sum_{i=0}^{d} \frac{1}{m_i \pi_i^2} \right)^{-1},$$

where $\pi_i = \prod_{j \neq i} |\lambda_i - \lambda_j|$ for $0 \leq i \leq d$ [27]. The idea of the proof will appear in Lemma 2.4 below. In Section 3.1, we will introduce a characterization of distance-regular graphs in terms of the spectral excess of G.

2.3.2 Bipartite case

Now we consider the case that G is bipartite. Then $\alpha_i = 0$ for $0 \le i \le d$ [18], and thus $xp_i = \gamma_{i+1}p_{i+1} + \beta_{i-1}p_{i-1}$. By this observation, the following lemma gives a three-term recurrence for bipartite graphs.

Lemma 2.3. *If G is bipartite, then the predistance polynomials satisfy a three-term recurrence of the form*

$$x^2 p_i = X_{i+2}p_{i+2} + Y_i p_i + Z_{i-2}p_{i-2} \tag{2.9}$$

for $0 \le i \le d$, where $X_{i+2} := \gamma_{i+1}\gamma_{i+2}$, $Y_i := \beta_i\gamma_{i+1} + \beta_{i-1}\gamma_i$ and $Z_{i-2} := \beta_{i-2}\beta_{i-1}$. Moreover, by directly computing, it follows that $X_i + Y_i + Z_i = \lambda_0^2$ for $0 \le i \le d$. □

A polynomial p is *odd* (resp. *even*) if all its nonzero terms are of odd degrees (resp. even degrees). If G is bipartite, then p_i is odd or even only depending on its degree i being odd or even [18]. The following lemma gives an expression of $p_{d-1}(\lambda_0)$ for bipartite graphs in terms of the spectrum. The proof is essentially identical to [16, p. 8–9], except for the setting of the polynomials h_i.

Lemma 2.4. *Let G be a connected bipartite graph. Then*

$$p_{d-1}(\lambda_0) = n\left(2 + \sum_{i=1}^{d-1} \frac{(h_i(\lambda_0) + (-1)^{d-1}h_i(-\lambda_0))^2}{m_i h_i(\lambda_i)^2}\right)^{-1},$$

where $h_i = \prod_{j\ne 0,i,d}(x - \lambda_j)$ for $1 \le i \le d-1$.

Proof. Note first that $\lambda_d = -\lambda_0$ and $m_d = 1$ since G is bipartite. For $1 \le i \le d-1$, $\deg h_i = d-2 < d-1$ and $h_i(\lambda_j) = 0$ if $j \ne 0, i, d$. Then

$$0 = n\langle h_i, p_{d-1}\rangle_G = h_i(\lambda_0)p_{d-1}(\lambda_0) + m_i h_i(\lambda_i)p_{d-1}(\lambda_i) + h_i(-\lambda_0)p_{d-1}(-\lambda_0),$$

and thus

$$p_{d-1}(\lambda_i) = \frac{p_{d-1}(\lambda_0)h_i(\lambda_0) + p_{d-1}(-\lambda_0)h_i(-\lambda_0)}{m_i h_i(\lambda_i)}.$$

Since G is bipartite, p_{d-1} is odd or even only depending on its degree $d-1$ being odd or even, which implies that $p_{d-1}(-\lambda_0) = (-1)^{d-1} p_{d-1}(\lambda_0)$. By definition of the inner product, we have

$$\begin{aligned} np_{d-1}(\lambda_0) =& n\langle p_{d-1}, p_{d-1}\rangle_G \\ =& 2p_{d-1}(\lambda_0)^2 + \sum_{i=1}^{d-1} m_i p_{d-1}(\lambda_i)^2 \\ =& 2p_{d-1}(\lambda_0)^2 + \sum_{i=1}^{d-1} m_i \left(\frac{p_{d-1}(\lambda_0)h_i(\lambda_0) + (-1)^{d-1} p_{d-1}(\lambda_0)h_i(-\lambda_0)}{m_i h_i(\lambda_i)} \right)^2 . \end{aligned}$$

Therefore,

$$p_{d-1}(\lambda_0) = n \left(2 + \sum_{i=1}^{d-1} \frac{(h_i(\lambda_0) + (-1)^{d-1} h_i(-\lambda_0))^2}{m_i h_i(\lambda_i)^2} \right)^{-1} ,$$

as claimed. □

2.4 Hoffman polynomial

The polynomial

$$H(x) := n \prod_{i=1}^{d} \frac{x - \lambda_i}{\lambda_0 - \lambda_i}$$

is called the *Hoffman polynomial* [35]. The relationship between predistance polynomials and Hoffman polynomial is that the sum of all predistance polynomials gives the Hoffman polynomial:

$$H = p_0 + p_1 + \cdots + p_d, \tag{2.10}$$

no matter whether the graph is regular or not. For completeness, we explian (2.10) in the following, by the same argument as in [16, p. 6–7]. Define $s_i = \sum_{j=0}^{i} p_j$ for $0 \le i \le d$. To prove (2.10), we first show that s_i is the polynomial p of degree i maximizing $p(\lambda_0)$ subject to $\langle p, p\rangle_G = \langle s_i, s_i\rangle_G$. Write $p = \sum_{j=0}^{i} \alpha_j p_j$ for some $\alpha_j \in \mathbb{R}$.

Then $s_i(\lambda_0) = \langle s_i, s_i\rangle_G = \langle p, p\rangle_G = \sum_{j=0}^{i} \alpha_j^2 p_j(\lambda_0)$. By Cauchy–Schwarz inequality,

$$p(\lambda_0)^2 = \left(\sum_{j=0}^{i} \alpha_j p_j(\lambda_0)\right)^2 \leq \left(\sum_{j=0}^{i} \alpha_j^2 p_j(\lambda_0)\right)\left(\sum_{j=0}^{i} p_j(\lambda_0)\right) = s_i(\lambda_0)^2,$$

with equality if and only if all α_j are equal; indeed $\alpha_j^2 = 1$. Since we want to maximize $p(\lambda_0)$ and $p_j(\lambda_0) > 0$, it follow that $\alpha_j = 1$, and thus s_i is the optimal p. On the other hand, since

$$s_i(\lambda_0) = \langle p, p\rangle_G = \frac{1}{n} p(\lambda_0)^2 + \frac{1}{n}\sum_{j=1}^{d} m_j p(\lambda_j)^2, \tag{2.11}$$

the maximality of $p(\lambda_0)$ is equivalent to the minimality of $\sum_{j=1}^{d} m_j p(\lambda_j)^2$. For the case $i = d$, there exists a nonzero polynomial that is zero on λ_j for $1 \leq j \leq d$. Then we can conclude that

$$s_d(\lambda_j) = 0 \quad \text{for} \quad 1 \leq j \leq d, \tag{2.12}$$

and thus, by taking $i = d$ and $p = s_d$ into (2.11), it follows that

$$s_d(\lambda_0) = n. \tag{2.13}$$

Since $\deg s_d = d$, by (2.12) and (2.13), we deduce that

$$s_d = n \prod_{i=1}^{d} \frac{x - \lambda_i}{\lambda_0 - \lambda_i},$$

proving (2.10).

Hoffman [35] proved that a connected graph G is regular if and only if $H(A) = J$, the all-ones matrix. Let α be the eigenvector of A associated with λ_0 such that $\alpha^t\alpha = n$ and all entries of α are positive. Note that such an eigenvector α exists by Perron-Frobenius theorem (see e.g. [8, Theorem 2.2.1]), and is usually called the *Perron vector*. Moreover, $\alpha = (1, 1, \ldots, 1)^t$ if and only if G is regular. The following result, given first in [26, p. 117], gives a generalization of Hoffman's result to nonregular graphs.

Lemma 2.5. *Let G be a connected graph with adjacency matrix A and Perron vector α. Then, $H(A) = \alpha\alpha^t$. Moreover, G is regular if and only if $H(A) = J$, the all-ones matrix.*

Proof. This follows since the matrix $\alpha\alpha^t$ acts the same as

$$H(A) = n \prod_{i=1}^{d} \frac{A - \lambda_i I}{\lambda_0 - \lambda_i}$$

on the eigenvectors of A. □

Chapter 3

The spectral excess theorem

The spectral excess theorem gives a quasi-spectral characterization for a regular graph to be distance-regular. For a graph G with $d+1$ distinct eigenvalues, the number

$$\overline{k}_d := \frac{1}{n}\sum_{u\in V}|G_d(u)| = ||A_d||^2$$

is called the *average excess* of G. Note that $\overline{k}_d > 0$ if and only if $D = d$. Recall that the number $p_d(\lambda_0)$ is the *spectral excess* of G. The spectral excess theorem (Theorem 3.3), proposed by Fiol and Garriga [27], states that

$$\overline{k}_d \leq p_d(\lambda_0)$$

for a regular graph G, and equality is attained if and only if G is distance-regular. See [16, 29] for short proofs, and [18, 17] for some generalizations.

3.1 A simple proof

For completeness, a simple proof [29] is given in this section. Let

$$\mathrm{Proj}_N(M) := \frac{\langle N, M\rangle}{||N||^2}N$$

denote the projection of M onto Span$\{N\}$, where M and N are symmetric matrices over real number field $\mathbb{R}$. By (2.10) and Lemma 2.5, any connected regular graph has the property that

$$A_0 + A_1 + \cdots + A_D = H(A) = p_0(A) + p_1(A) + \cdots + p_d(A). \tag{3.1}$$

It is well-known that $(A^i)_{uv}$ counts the number of walks of length i in G from u to v. By (2.7), we have

$$\langle A_i, p_j(A)\rangle = 0 \ \text{ for } \ j < i. \tag{3.2}$$

Lemma 3.1. ([29, Lemma 1]) *Let G be a regular graph with $d+1$ distinct eigenvalues. Then, $\overline{k}_d \le p_d(\lambda_0)$, and equality is attained if and only if $A_d = p_d(A)$.*

Proof. Note that $p_d(\lambda_0) > 0$. If $D < d$, then $\overline{k}_d = 0$ and clearly $\overline{k}_d \le p_d(\lambda_0)$. Suppose that $D = d$. Now we have $\overline{k}_d > 0$. By (3.1) and (3.2),

$$\mathrm{Proj}_{A_d}(p_d(A)) = \frac{\langle A_d, p_d(A)\rangle}{\|A_d\|^2} A_d = \frac{\langle A_d, H(A)\rangle}{\overline{k}_d} A_d = A_d.$$

Then

$$0 \le \|p_d(A)\|^2 - \|\mathrm{Proj}_{A_d}(p_d(A))\|^2 = p_d(\lambda_0) - \overline{k}_d.$$

Since equality can be attained only when $D = d$, the above argument tells us that $\overline{k}_d = p_d(\lambda_0)$ if and only if $A_d = \mathrm{Proj}_{A_d}(p_d(A)) = p_d(A)$. □

Recall that Proposition 2.2 states that a graph G is distance-regular if and only if $A_i = p_i(A)$ for every $0 \le i \le d$. As it was shown in [31, Theorem 6.4], the following proposition indicates that, for regular graphs, the condition on the highest degree predistance polynomial suffices.

Proposition 3.2. ([31, Theorem 6.4], [16], [29, Proposition 2], [19]) *A regular graph G with $d+1$ distinct eigenvalues is distance-regular if and only if $A_d = p_d(A)$.*

Proof. The necessity has already been proved in Proposition 2.2. To prove sufficiency, by Proposition 2.2, we only need to show that $A_i = p_i(A)$ for $0 \leq i \leq d$, which follows by (backward) induction on $0 \leq i \leq d$. The base case is the assumption that $A_d = p_d(A)$. Suppose now that $A_k = p_k(A)$ for $d \geq k \geq i$. Then deleting these common terms from both sides of (3.1), we have

$$A_0 + A_1 + \cdots + A_{i-1} = p_0(A) + p_1(A) + \cdots + p_{i-1}(A), \tag{3.3}$$

and by induction hypothesis to the three-term recurrence in (2.6),

$$\begin{aligned} AA_i = Ap_i(A) &= \gamma_{i+1}p_{i+1}(A) + \alpha_i p_i(A) + \beta_{i-1}p_{i-1}(A) \\ &= \gamma_{i+1}A_{i+1} + \alpha_i A_i + \beta_{i-1}p_{i-1}(A). \end{aligned} \tag{3.4}$$

It remains to show that $A_{i-1} = p_{i-1}(A)$. To this end, consider the following two cases:

(i) For $\partial(u,v) \geq i-1$, $(A_{i-1})_{uv} = (p_{i-1}(A))_{uv}$ by (3.3).

(ii) For $\partial(u,v) < i-1$, $(AA_i)_{uv} = \sum_{w \in G_1(u)} (A_i)_{wv} = 0$, where the last equality follows since $\partial(w,v) \leq 1+ \partial(u,v) < i$. Then $(p_{i-1}(A))_{uv} = 0$ by (3.4) and since $\beta_{i-1} \neq 0$.

This proves the sufficiency. □

The spectral excess theorem, which we restate below, is proved by Lemma 3.1 and Proposition 3.2.

Theorem 3.3. ([29, Theorem 3]) *Let G be a regular graph with $d+1$ distinct eigenvalues. Then, $\overline{k}_d \leq p_d(\lambda_0)$, and equality is attained if and only if G is distance-regular.*

The following example shows that the regularity assumption of G in the spectral excess theorem is necessary.

Example 3.4. Let G be a path on three vertices. Then $\mathrm{sp}(G) = \{\sqrt{2}, 0, -\sqrt{2}\}$. By Example 2.1, $p_0 = 1$, $p_1 = 3\sqrt{2}x/4$ and $p_2 = 3(x^2 - 4/3)/4$. Note that $\overline{k}_2 = 2/3$ and $p_2(\lambda_0) = 1/2$. This shows that the inequality $\overline{k}_d \leq p_d(\lambda_0)$ does not hold.

3.2 A weighted spectral excess theorem

In this section, we give a 'weighted' version of the spectral excess theorem.

3.2.1 For nonregular graphs

Recall that α is the Perron vector. For $u \in V$, let α_u be the entry corresponding to the vertex u in the vector α. For $0 \le i \le D$, define the *weighted distance-i matrix* $\widetilde{A}_i$ of G to be the matrix indexed by the vertex set V such that the entry $(\widetilde{A}_i)_{uv} = \alpha_u \alpha_v$ if $\partial(u, v) = i$, and $(\widetilde{A}_i)_{uv} = 0$ otherwise. In particular, for the case that G is regular, $\widetilde{A}_i$ is a $(0, 1)$-matrix and thus turns out to be the distance-i matrix A_i of G. Note by (2.7) that $\widetilde{A}_0, \widetilde{A}_1, \ldots, \widetilde{A}_D$ are orthogonal. Define

$$\widetilde{\delta}_i = \langle \widetilde{A}_i, \widetilde{A}_i \rangle = \sum_{u,v \in V} (\widetilde{A}_i \circ \widetilde{A}_i)_{uv}/n, \quad 0 \le i \le D.$$

For $0 \le i \le d$, define $\widetilde{A}_{\ge i} = \sum_{j \ge i} \widetilde{A}_j$, $p_{\ge i} = \sum_{j \ge i} p_j$ and $\widetilde{\delta}_{\ge i} = \sum_{j \ge i} \widetilde{\delta}_j$. Similarly for $\widetilde{A}_{\le i}$, $p_{\le i}$ and $\widetilde{\delta}_{\le i}$. The parameter $\widetilde{\delta}_D$ is referred to as the *average weighted excess* and $p_{\ge D}(\lambda_0)$ as the *generalized spectral excess* of G. Note that if $D = d$ then $p_{\ge D}(x) = p_D(x)$. By the construction of $\widetilde{A}_i$, Lemma 2.5 and (2.10), any connected graph G has the property that

$$\widetilde{A}_0 + \widetilde{A}_1 + \cdots + \widetilde{A}_D = H(A) = p_0(A) + p_1(A) + \cdots + p_d(A). \tag{3.5}$$

Recall that $(A^i)_{uv}$ counts the number of walks of length i in G from u to v. Although $\widetilde{A}_i$ might be different to A_i, they are similar: by (2.7), we have

$$\langle A_i, p_j(A) \rangle = 0 = \langle \widetilde{A}_i, p_j(A) \rangle \quad \text{for} \quad j < i. \tag{3.6}$$

Now we are ready to give a 'weighted' spectral excess theorem (in Theorem 3.10). Lemma 3.5 is a 'weighted' version of Lemma 3.1. In fact, the approach of giving weights, the entries of the Perron vector, to the vertices of a nonregular graph has been recently used many times in the literature (see, for instance, [30, 31, 27, 28, 25]).

Lemma 3.5. *Let G be a graph with diameter D. Then $\widetilde{\delta}_D \leq p_{\geq D}(\lambda_0)$ with equality if and only if $\widetilde{A}_D = p_{\geq D}(A)$.*

Proof. By (3.5) and (3.6),

$$\mathrm{Proj}_{\widetilde{A}_D}(p_{\geq D}(A)) = \frac{\langle \widetilde{A}_D, p_{\geq D}(A)\rangle}{\|\widetilde{A}_D\|^2}\widetilde{A}_D = \frac{\langle \widetilde{A}_D, H(A)\rangle}{\widetilde{\delta}_D}\widetilde{A}_D = \widetilde{A}_D.$$

Then

$$0 \leq \|p_{\geq D}(A)\|^2 - \|\mathrm{Proj}_{\widetilde{A}_D}(p_{\geq D}(A))\|^2 = p_{\geq D}(\lambda_0) - \widetilde{\delta}_D.$$

Moreover, the equality is attained if and only if $\widetilde{A}_D = \mathrm{Proj}_{\widetilde{A}_D}(p_{\geq D}(A)) = p_{\geq D}(A)$. □

Example 3.6. Revisiting the case when G is a path on three vertices described in Example 2.1 and Example 3.4, note that $D = d = 2$, $\alpha = (\sqrt{3}/2, \sqrt{6}/2, \sqrt{3}/2)^t$ and

$$\widetilde{A}_D = \begin{pmatrix} 0 & 0 & 3/4 \\ 0 & 0 & 0 \\ 3/4 & 0 & 0 \end{pmatrix}.$$

Then $\widetilde{\delta}_D = 3/8 \leq 1/2 = p_{\geq D}(\lambda_0)$ satisfies inequality in Lemma 3.5.

Recall that $p_0 = 1$, and $p_1 = x$ if and only if G is regular.

Remark 3.7. If G is regular with diameter $D = 2$, then the equality in Lemma 3.5 holds. Indeed, $\widetilde{A}_2 = A_2 = J - I - A = H(A) - I - A = p_{\geq 2}(A)$.

The graph described in Remark 3.7 is a special case of *distance-polynomial graphs* [44]. It would be interesting to characterize graphs which satisfy equality in Lemma 3.5. Here we give two characterizations: one is under the assumption $D = d$ (Theorem 3.10), and the other is for bipartite graphs (Theorem 3.14). Since $p_0(A) = I$, Lemma 3.8 is obvious, but plays a crucial role in proving the regularity of a graph. Proposition 3.9 is a 'weighted' version of Proposition 3.2.

Lemma 3.8. *$\widetilde{A}_0 = p_0(A)$ if and only if G is regular.* □

Proposition 3.9. *A graph G with $d+1$ distinct eigenvalues is distance-regular if and only if $\widetilde{A}_d = p_d(A)$.*

Proof. (⇒) Suppose that G is distance-regular. Then G is regular, and thus the result follows from Proposition 3.2. (⇐) Suppose that $\widetilde{A}_d = p_d(A)$. We claim that $\widetilde{A}_i = p_i(A)$ for $0 \le i \le d$, which follows by (backward) induction on $0 \le i \le d$. If the proof is finished, then the condition $\widetilde{A}_0 = p_0(A)$ implies that G is regular by Lemma 3.8, and the remaining follows from Proposition 3.2. The base case is the assumption that $\widetilde{A}_d = p_d(A)$. Suppose now that $\widetilde{A}_k = p_k(A)$ for $d \ge k \ge i$. Then deleting these common terms from both sides of (3.5), we have

$$\widetilde{A}_0 + \widetilde{A}_1 + \cdots + \widetilde{A}_{i-1} = p_0(A) + p_1(A) + \cdots + p_{i-1}(A), \tag{3.7}$$

and by induction hypothesis to the three-term recurrence in (2.6),

$$\begin{aligned} A\widetilde{A}_i = Ap_i(A) &= \gamma_{i+1}p_{i+1}(A) + \alpha_i p_i(A) + \beta_{i-1}p_{i-1}(A) \\ &= \gamma_{i+1}\widetilde{A}_{i+1} + \alpha_i \widetilde{A}_i + \beta_{i-1}p_{i-1}(A). \end{aligned} \tag{3.8}$$

It remains to show that $\widetilde{A}_{i-1} = p_{i-1}(A)$. To this end, consider the following two cases:

(*i*) For $\partial(u,v) \ge i-1$, $(p_{i-1}(A))_{uv} = (\widetilde{A}_{i-1})_{uv}$ by (3.7).

(*ii*) For $\partial(u,v) < i-1$, $(A\widetilde{A}_i)_{uv} = \sum_{w \in G_1(u)} (\widetilde{A}_i)_{wv} = 0$, where the last equality follows since $\partial(w,v) \le 1 + \partial(u,v) < i$. Then $(p_{i-1}(A))_{uv} = 0$ by (3.8) and since $\beta_{i-1} \ne 0$.

Thus the proof is completed. □

Combining Lemma 3.5 with Proposition 3.9, a variation of the spectral excess theorem is given below.

Theorem 3.10. *Let G be a connected graph with diameter D. Then $\widetilde{\delta}_D \leq p_{\geq D}(\lambda_0)$ with equality if and only if $\widetilde{A}_D = p_{\geq D}(A)$. Moreover, suppose further that $D = d$. Then equality holds if and only if G is distance-regular.* □

3.2.2 Bipartite case

If G is bipartite, then we can rewrite (3.5) (in Lemma 3.11) more precisely by only taking the 'odd' or 'even' part, which was also considered in [18]. Define $\widetilde{A}^{\text{odd}} = \sum_{\text{odd } i} \widetilde{A}_i$, $p^{\text{odd}} = \sum_{\text{odd } i} p_i$ and $\widetilde{\delta}^{\text{odd}} = \sum_{\text{odd } i} \widetilde{\delta}_i$. Similarly for $\widetilde{A}^{\text{even}}$, p^{even} and $\widetilde{\delta}^{\text{even}}$. Then $\langle \widetilde{A}^*, \widetilde{A}^* \rangle = \widetilde{\delta}^*$ and $\langle p^*(A), p^*(A) \rangle = p^*(\lambda_0)$ for $* \in \{\text{odd, even}\}$. Recall that, if G is bipartite, then p_i is odd or even only depending on its degree i being odd or even. The following lemma is proved by (3.5) and the fact that $(p_i(A))_{uv} = 0$ if $\partial(u,v)$ and i have distinct parity (since bipartite graphs contain no odd cycle).

Lemma 3.11. *If G is bipartite, then $\widetilde{A}^{\text{odd}} = p^{\text{odd}}(A)$ and $\widetilde{A}^{\text{even}} = p^{\text{even}}(A)$. Moreover, by taking norms, $\widetilde{\delta}^{\text{odd}} = p^{\text{odd}}(\lambda_0)$ and $\widetilde{\delta}^{\text{even}} = p^{\text{even}}(\lambda_0)$.* □

Remark 3.12. Observe that $p^{\text{odd}} = (H(x) - H(-x))/2$, $H(\lambda_0) = n$ and $H(\lambda_d) = 0$. Thus for bipartite graphs, we deduce that $\widetilde{\delta}^{\text{odd}} = \widetilde{\delta}^{\text{even}} = p^{\text{odd}}(\lambda_0) = p^{\text{even}}(\lambda_0) = n/2$.

Summing the recurrence relation (2.6) from the terms with index $i+1$ to d, and using the fact that $\alpha_i + \beta_i + \gamma_i = \lambda_0$ for $0 \leq i \leq d$, it follows that

$$xp_{\geq i+1} = \beta_i p_i + \lambda_0 p_{\geq i+1} - \gamma_{i+1} p_{i+1} \tag{3.9}$$

[17, Proposition 2.5]. Note that, if $\widetilde{A}_{\geq i+1} = p_{\geq i+1}(A)$ and $\partial(u,v) < i$ for $u, v \in V$, then $(Ap_{\geq i+1}(A))_{uv} = 0 = (\lambda_0 p_{\geq i+1}(A))_{uv}$, and thus $\beta_i(p_i(A))_{uv} = \gamma_{i+1}(p_{i+1}(A))_{uv}$ by (3.9). Using this fact, we have the following result.

Proposition 3.13. *Let G be a connected bipartite graph and $i \leq d-1$. Then $\widetilde{A}_{\leq i} = p_{\leq i}(A)$ if and only if $\widetilde{A}_j = p_j(A)$ for $0 \leq j \leq i$.*

Proof. The sufficiency is clear. To prove necessity, we only need to show that $\widetilde{A}_i = p_i(A)$ (the remaining follows by similar argument). If $\partial(u,v) \geq i$, then $(\widetilde{A}_i)_{uv} = (p_i(A))_{uv}$ by assumption. Suppose $\partial(u,v) < i$. Note that the assumption $\widetilde{A}_{\leq i} = p_{\leq i}(A)$ is equivalent to the condition $\widetilde{A}_{\geq i+1} = p_{\geq i+1}(A)$. If $\partial(u,v)$ and i have different parity, then clearly $(\widetilde{A}_i)_{uv} = 0 = (p_i(A))_{uv}$. Assume that $\partial(u,v)$ and i have the same parity. Then, by the above argument, $\beta_i(p_i(A))_{uv} = \gamma_{i+1}(p_{i+1}(A))_{uv} = 0$. Since $\beta_i \neq 0$ for $i \leq d-1$, $(\widetilde{A}_i)_{uv} = 0 = (p_i(A))_{uv}$. □

Define $\widetilde{A}^{odd}_{\geq i} = \sum_{\text{odd } j \geq i} \widetilde{A}_j$, $\widetilde{\delta}^{odd}_{\geq i} = \sum_{\text{odd } j \geq i} \widetilde{\delta}_j$ and $p^{odd}_{\geq i} = \sum_{\text{odd } j \geq i} p_j$. Similarly for $\widetilde{A}^*_\Omega$, $\widetilde{\delta}^*_\Omega$ and p^*_Ω, where $(\Omega, *) \in \{(\geq i, \text{even}), (\leq i, \text{odd}), (\leq i, \text{even})\}$. The following result gives a characterization for bipartite graphs satisfying equality in Lemma 3.5 (or equivalently, $\widetilde{A}_D = p_{\geq D}(A)$). Unlike Theorem 3.10, there is no need for the assumption $D = d$ in Theorem 3.14.

Theorem 3.14. *A connected bipartite graph with $\widetilde{A}_D = p_{\geq D}(A)$ is distance-regular.*

Proof. Note first that the assumption $\widetilde{A}_D = p_{\geq D}(A)$ is equivalent to the condition $\widetilde{A}_{\leq D-1} = p_{\leq D-1}(A)$. By Proposition 3.13, $\widetilde{A}_i = p_i(A)$ for $0 \leq i \leq D-1$. By Lemma 3.11, it follows that $p^*_{\geq D+1}(A)$ is the zero matrix, where $* \in \{\text{odd, even}\}$ has the same parity as $D+1$. This happens only for the case $D = d$, since otherwise $p^*_{\geq D+1}(\lambda_0) = 0$, contradicting the fact that $p_i(\lambda_0) > 0$ for $0 \leq i \leq d$. The remaining follows from Theorem 3.10. □

Lemmas 3.15–3.16 present some inequalities related to the spectral excess theorem. The proofs are essentially the same as in Lemma 3.5.

Lemma 3.15. *Let G be a connected graph. For $0 \le i \le d$,*

(i) $\widetilde{\delta}_{\geq i} \leq p_{\geq i}(\lambda_0)$ *with equality if and only if* $\widetilde{A}_{\geq i} = p_{\geq i}(A)$, *and*

(ii) $\widetilde{\delta}_{\leq i} \geq p_{\leq i}(\lambda_0)$ *with equality if and only if* $\widetilde{A}_{\leq i} = p_{\leq i}(A)$. □

Lemma 3.16. *Let G be a connected bipartite graph. For $0 \le i \le d$ and $* \in \{\text{odd, even}\}$,*

(i) $\widetilde{\delta}^*_{\geq i} \leq p^*_{\geq i}(\lambda_0)$ *with equality if and only if* $\widetilde{A}^*_{\geq i} = p^*_{\geq i}(A)$, *and*

(ii) $\widetilde{\delta}^*_{\leq i} \geq p^*_{\leq i}(\lambda_0)$ *with equality if and only if* $\widetilde{A}^*_{\leq i} = p^*_{\leq i}(A)$. □

3.3 An application: Odd-girth theorem

Recall that a cycle is odd or even as its length is odd or even. The *odd-girth* of a graph is the length of its shortest odd cycle. Applying the (standard) spectral excess theorem, van Dam and Haemers [22] proved the 'odd-girth theorem' for regular graphs: A connected regular graph with $d+1$ distinct eigenvalues and odd-girth $2d+1$ is distance-regular. In the same paper, the authors posed the problem to determine whether the regularity assumption can be removed. As an application of the 'weighted' spectral excess theorem (Theorem 3.10), we demonstrate that the regularity assumption is not necessary, that is, the odd-girth theorem is not restricted to regular graphs (Theorem 3.19).

Let G be a connected graph, not necessarily regular, with $d+1$ distinct eigenvalues $\lambda_0 > \lambda_1 > \cdots > \lambda_d$. Let $c = n/\prod_{i=1}^{d}(\lambda_0 - \lambda_i)$, which is the leading coefficient of the Hoffman polynomial H, and hence also of p_d, in view of (2.10). Recall that $Z(x) = \prod_{i=0}^{d}(x - \lambda_i)$ is the minimal polynomial of A. For vertices $u, v \in V$ with $\partial(u, v) = d$, we have

$$(A^d)_{uv} = H(A)_{uv}/c \tag{3.10}$$

and thus

$$(A^{d+1})_{uv} = Z(A)_{uv} + (\sum_{i=0}^{d} \lambda_i)(A^d)_{uv} = (\sum_{i=0}^{d} \lambda_i) H(A)_{uv}/c. \tag{3.11}$$

In order to apply the 'weighted' spectral excess theorem, Lemma 3.17 and Lemma 3.18 are given to determine the average weighted excess $\widetilde{\delta}_D$ and the generalized spectral excess $p_d(\lambda_0)$, respectively, for graphs with odd-girth $2d+1$.

Lemma 3.17. *Let G be a connected graph with $d+1$ distinct eigenvalues and odd-girth $2d+1$. Then the average weighted excess $\widetilde{\delta}_D$ of G equals $c^2\mathrm{tr}(A^{2d+1})/(n\sum_{i=0}^{d}\lambda_i)$, where $c = n/\prod_{i=1}^{d}(\lambda_0 - \lambda_i)$. In particular, $D = d$.*

Proof. Note first that the trace $\mathrm{tr}(A^{2d+1})$ of A^{2d+1} is nonzero since G has odd-girth $2d+1$. For vertices $u, v \in V$ with $\partial(u,v) < d$, we have $(A^d)_{uv} = 0$ or $(A^{d+1})_{vu} = 0$ as G has no odd cycle with length less than $2d+1$. Then by (2.7), (3.5), (3.10) and (3.11), we have

$$\begin{aligned} n(\sum_{i=0}^{d}\lambda_i)\widetilde{\delta}_d &= (\sum_{i=0}^{d}\lambda_i)\sum_{u,v\in V}[(\widetilde{A}_d)_{uv}]^2 \\ &= (\sum_{i=0}^{d}\lambda_i)\sum_{u\in V}\sum_{v\in G_d(u)}[H(A)_{uv}]^2 \\ &= c^2\sum_{u\in V}\sum_{v\in V}(A^d)_{uv}(A^{d+1})_{uv} \\ &= c^2\mathrm{tr}(A^{2d+1}). \end{aligned}$$

Since $c \neq 0$ and $\mathrm{tr}(A^{2d+1}) \neq 0$, we have $\sum_{i=0}^{d}\lambda_i \neq 0$ and $\widetilde{\delta}_d \neq 0$. Thus $\widetilde{\delta}_d = c^2\mathrm{tr}(A^{2d+1})/(n\sum_{i=0}^{d}\lambda_i)$. Moreover, $\widetilde{\delta}_d > 0$ since we always have $\widetilde{\delta}_d \geq 0$ and now $\widetilde{\delta}_d \neq 0$. This implies $D = d$ and the result follows. □

Recall that a polynomial p is odd (resp. even) if all its nonzero terms are of odd degrees (resp. even degrees).

Lemma 3.18. *Let G be a connected graph with $d+1$ distinct eigenvalues and odd-girth $2d+1$. Referring to the notations of three-term recurrence in* (2.6)*, the following* (i)–(ii) *hold:*

(i) *the preintersection number $\alpha_{j-1} = 0$ for $1 \le j \le d$;*

(ii) *the predistance polynomial p_j is even or odd depending on whether j is even or odd for $0 \le j \le d$.*

Moreover, the generalized spectral excess $p_d(\lambda_0)$ of G is $c^2 \mathrm{tr}(A^{2d+1})/(n \sum_{i=0}^{d} \lambda_i)$, where $c = n/\prod_{i=1}^{d}(\lambda_0 - \lambda_i)$.

Proof. Clearly, the polynomial $p_0 = 1$ is even. We prove (i)–(ii) by induction on $j \ge 1$. Note that $p_1 = \lambda_0 x/\overline{k}$ is odd. Setting $i = 0$ in (2.6), we have $\alpha_0 = 0$. Hence (i)–(ii) hold in the base case $j = 1$. In view of (2.6) with $i = k$,

$$\alpha_k p_k(\lambda_0) = \langle \alpha_k p_k, p_k \rangle_G = \langle x p_k, p_k \rangle_G = \mathrm{tr}(A p_k^2(A))/n \tag{3.12}$$

for $0 \le k \le d$. Now suppose that (i)–(ii) hold for $j = k < d$. Since the polynomial xp_k^2 is an odd polynomial of degree $2k+1 < 2d+1$ and G has odd-girth $2d+1$, the last term in (3.12) is zero. Hence $\alpha_k = 0$ and (i) holds for $j = k+1$. From (i) and setting $i = k$ in (2.6), the polynomial p_{k+1} satisfies (ii). This proves (i)–(ii) for any j. For the remaining, by the fact that xp_d^2 is an odd polynomial of degree $2d+1$ and leading coefficient c^2, the last term in (3.12) with $k = d$ equals $c^2 \mathrm{tr}(A^{2d+1})/n$. Thus it suffices to show that $\alpha_d = \sum_{i=0}^{d} \lambda_i$. Choose two vertices u and v at distance d. Then by (2.6), (2.10) and (3.11),

$$\alpha_d H(A)_{uv} = \alpha_d p_d(A)_{uv} = (A p_d(A))_{uv} = c(A^{d+1})_{uv} = (\sum_{i=0}^{d} \lambda_i) H(A)_{uv},$$

where the third equality follows because xp_d has no term of degree d. Dividing both sides by $H(A)_{uv}$, we have $\alpha_d = \sum_{i=0}^{d} \lambda_i$. □

From Lemmas 3.17–3.18, and Theorem 3.10, the odd-girth theorem immediately follows.

Theorem 3.19. (Odd-girth theorem) *A connected graph with $d+1$ distinct eigenvalues and odd-girth $2d+1$ must be distance-regular.* □

Note that, after our result, van Dam and Fiol [19] give a short and more direct proof of Theorem 3.19 which does not rely on the spectral excess theorem, but only a known characterization of distance-regularity in terms of the predistance polynomial p_d of highest degree (Proposition 3.2).

3.4 Some related results

A natural question motivated by Lemmas 3.15–3.16 is to study the relation between the parameters $\widetilde{\delta}_i$ and $p_i(\lambda_0)$ for $0 \leq i \leq d-1$ (the case $i = d$ is given in Theorem 3.10). We give some results in this section. Recall that $p_0 = 1$. Proposition 3.20 is straightforward, but plays a crucial role in proving the regularity of a graph, which follows from Lemma 3.8 and the inequality $\widetilde{\delta}_{\leq 0} \geq p_{\leq 0}(\lambda_0)$ mentioned in Lemma 3.15. In fact, it can also be derived by the Cauchy–Schwarz inequality: $\sum_{u \in V} \alpha_u^4 \geq (\sum_{u \in V} \alpha_u^2)^2/n = n$.

Proposition 3.20. *Let G be a connected graph. Then $\widetilde{\delta}_0 \geq 1$ $(= p_0(\lambda_0))$ (which is equivalent to $\sum_{u \in V} \alpha_u^4 \geq n$), with equality if and only if any of the following conditions holds:*

(*i*) $\widetilde{A}_0 = I$ $(= p_0(A))$,

(*ii*) *G is regular.* □

Recall that, for $u \in V$, the number α_u denotes the entry corresponding to u in the

Perron vector α. Note that

$$\langle A, \widetilde{A}_1 \rangle = \frac{1}{n} \sum_{u,v} (\widetilde{A}_1)_{uv} = \frac{1}{n}\, \mathbf{1}^t \widetilde{A}_1 \mathbf{1} = \frac{1}{n}\, \mathbf{1}^t DAD\mathbf{1} = \lambda_0, \tag{3.13}$$

where $\mathbf{1}$ is the all-ones vector, and D is the diagonal matrix with entries $D_{uu} = \alpha_u$ for $u \in V$. A bipartite graph with bipartition (X, Y) is called (k_1, k_2)-*biregular* if all n_1 vertices in X have degree k_1 and all n_2 vertices in Y have degree k_2. Thus, by counting the number of edges of a (k_1, k_2)-biregular graph in two different ways, we have $n_1 k_1 = n_2 k_2$. Moreover, it is well-known that, for such a graph, $\lambda_0 = \sqrt{k_1 k_2}$ (see e.g. [34, p. 172–173]). Proposition 3.21 characterizes the graphs satisfying $\widetilde{\delta}_1 = p_1(\lambda_0)$, which is useful for checking the regularity or biregularity of a graph. Recall that $p_1 = \lambda_0 x/\overline{k}$ (by the Gram–Schmidt procedure), where $\overline{k}$ is the average degree of G.

Proposition 3.21. *Let G be a connected graph. Then $\widetilde{\delta}_1 \geq \lambda_0^2/\overline{k}$ $(= p_1(\lambda_0))$, with equality if and only if any of the following conditions holds:*

(i) $\widetilde{A}_1 = p_1(A)$,

(ii) *G is regular or biregular.*

Proof. By (3.13),

$$\mathrm{Proj}_{p_1(A)}(\widetilde{A}_1) = \frac{\langle p_1(A), \widetilde{A}_1 \rangle}{\|p_1(A)\|^2} p_1(A) = \frac{\langle \lambda_0 A/\overline{k}, \widetilde{A}_1 \rangle}{p_1(\lambda_0)} p_1(A) = \frac{\lambda_0^2/\overline{k}}{p_1(\lambda_0)} p_1(A) = p_1(A).$$

Then

$$0 \leq \|\widetilde{A}_1\|^2 - \|\mathrm{Proj}_{p_1(A)}(\widetilde{A}_1)\|^2 = \widetilde{\delta}_1 - p_1(\lambda_0).$$

Moreover, the equality is attained if and only if $\widetilde{A}_1 = \mathrm{Proj}_{p_1(A)}(\widetilde{A}_1) = p_1(A)$. Now it remains to show that $(i) \Leftrightarrow (ii)$. To prove necessity, we give the weight α_u to the vertex $u \in V$, and the weight $\alpha_u \alpha_v$ to the edge connecting u and v. Since $\widetilde{A}_1 = p_1(A) = \lambda_0 A/\overline{k}$, all edges receive the same weight, $\lambda_0/\overline{k}$. If G is not bipartite, then it contains an odd cycle, and this implies that all vertices on this cycle must have the same weight. As a result, the connectedness assumption on G implies that all vertices are of the same weight. Thus G is regular. For the case G is bipartite, the condition 'all edges receive the same weight' implies that vertices in the same partite set have the same weight. Thus G is biregular. Now we prove sufficiency. If G is regular, then clearly $p_1(A) = \lambda_0 A/\overline{k} = A = \widetilde{A}_1$. Suppose that G is (k_1, k_2)-biregular with bipartition (X, Y), where $|X| = n_1, |Y| = n_2$. Then $\lambda_0 = \sqrt{k_1 k_2}$, $n_1 k_1 = n_2 k_2$ and the Perron vector

$$\alpha = (\underbrace{\alpha', \cdots, \alpha'}_{n_1}\underbrace{\alpha'', \cdots, \alpha''}_{n_2})^t,$$

where $\alpha' = \sqrt{\dfrac{n_1+n_2}{2n_1}}$ and $\alpha'' = \sqrt{\dfrac{n_1+n_2}{2n_2}}$. Thus

$$p_1(A) = \frac{\lambda_0}{\overline{k}} A = \frac{\sqrt{k_1 k_2}(n_1+n_2)}{n_1 k_1 + n_2 k_2} A = \frac{n_1+n_2}{2\sqrt{n_1 n_2}} A = \alpha' \alpha'' A = \widetilde{A}_1.$$

□

The next question is to discuss the relation between $\widetilde{\delta}_2$ and $p_2(\lambda_0)$. We give the answer under the assumption that G is regular, and provide an example to show that the regularity condition is necessary. Therefore, there is no hope to determine the order of $\widetilde{\delta}_2$ and $p_2(\lambda_0)$ uniformly.

Lemma 3.22. *Let G be a connected regular graph. Then $\widetilde{\delta}_2 \geq p_2(\lambda_0)$, with equality if and only if $\widetilde{A}_2 = p_2(A)$.*

Proof. This follows by the inequality $\widetilde{\delta}_{\leq 2} \geq p_{\leq 2}(\lambda_0)$ mentioned in Lemma 3.15 and Propositions 3.20–3.21. □

Example 3.23. Revisiting again the path on three vertices described in Example 2.1, Example 3.4 and Example 3.6. Note that $\widetilde{\delta}_2 = 3/8 < 1/2 = p_2(\lambda_0)$.

Chapter 4

A characterization of bipartite distance-regular graphs

For an integer $h \leq d$, we say that G is *weighted h-punctually distance-regular* if $\widetilde{A}_h = p_h(A)$. The *distance-i graph* of G is the graph whose adjacency matrix is the distance-i matrix of G. For a connected bipartite graph G with bipartition (X, Y), the *halved graphs* G^X and G^Y are the two connected components of the distance-2 graph of G. It is well-known that the halved graphs of a bipartite distance-regular graph are distance-regular ([15], [6, Proposition 4.2.2]). Examples 4.10–4.12 are given (in Section 4.4) to show that the converse does not hold, that is, a connected bipartite graph whose halved graphs are distance-regular may not be distance-regular. Thus, a natural question is to find out when the converse is true. We give an answer in the following.

Theorem 4.1. *Let G be a connected bipartite graph with bipartition (X, Y). Suppose that G is weighted 2-punctually distance-regular with even spectral diameter, and both halved graphs G^X and G^Y are distance-regular. Then G is distance-regular.*

In order to prove Theorem 4.1, we study the concepts of weighted punctual distance-regularity and weighted partial distance-regularity in Section 4.1, which can be regarded as generalizations of the concepts of punctual distance-regularity and partial distance-regularity [18, 17]. The proof of Theorem 4.1 is given in Section 4.5.

4.1 Weighted punctual distance-regularity

A connected graph is called *h-punctually distance-regular* if $A_h = p_h(A)$; and is called *m-partially distance-regular* if $A_i = p_i(A)$ for $i \leq m$. These two concepts have been recently studied [18, 17]. In this section, we study two concepts, which are basically the same as that in [18, 17], except that the use of weighted distance matrices is taking into account. A connected graph is called *weighted h-punctually distance-regular* if $\widetilde{A}_h = p_h(A)$; and is called *weighted m-partially distance-regular* if $\widetilde{A}_i = p_i(A)$ for $i \leq m$. Clearly, the concepts of weighted 0-punctual distance-regularity and weighted 0-partial distance-regularity are identical. However, the weighted 1-punctual distance-regularity and the weighted 1-partial distance-regularity are not equivalent. For example, by Propositions 3.20–3.21, the path graph of three vertices P_3 is weighted 1-punctually distance-regular, but not weighted 1-partially distance-regular. Proposition 4.2 indicates that the concepts of weighted 2-punctual distance-regularity and (weighted) 2-partial distance-regularity coincide. Recall that $p_1 = x$ if and only if G is regular.

Proposition 4.2. *Let G be a connected graph. Then $\widetilde{A}_2 = p_2(A)$ if and only if G is weighted 2-partially distance-regular.*

Proof. The sufficiency is clear. We only need to prove necessity. Since $\widetilde{A}_2 = p_2(A) = aA^2 + bA + cI$ for some real numbers a, b, c with $a \neq 0$, we conclude that A^2 has a constant diagonal, which implies that G is regular. The remaining follows from Propositions 3.20–3.21. □

Proposition 4.3 states an equivalent condition of the weighted 2-punctual distance-regularity for bipartite graphs with spectral diameter $d \geq 3$. Note that the assumption

$d \geq 3$ is necessary, since otherwise the path graph P_3 of three vertices gives a counterexample.

Proposition 4.3. *Let G be a connected bipartite graph with spectral diameter $d \geq 3$. Then $\widetilde{\delta}_{\leq 2} = p_{\leq 2}(\lambda_0)$ if and only if G is weighted 2-punctually distance-regular.*

Proof. ($\Rightarrow$) Suppose that $\widetilde{\delta}_{\leq 2} = p_{\leq 2}(\lambda_0)$. By Lemma 3.15, we have $\widetilde{A}_{\leq 2} = p_{\leq 2}(A)$, and the result follows by Proposition 3.13. ($\Leftarrow$) Suppose that G is weighted 2-punctually distance-regular. By Proposition 4.2, we have $\widetilde{A}_i = p_i(A)$ for $0 \leq i \leq 2$, and the result follows by taking norms. □

4.2 Halved graphs with the same spectrum

Lemma 4.5 demonstrates that for a connected bipartite weighted 2-punctually distance-regular graph, its two halved graphs have the same spectrum (with appropriate spectral diameter), and, under further assumption, it gives a lower bound or exact value of the diameter, depending on the parity of its spectral diameter. To prove Lemma 4.5, we need some knowledge about Matrix Theory (Theorem 4.4).

Theorem 4.4. ([46, Theorem 2.8]) *Let P and Q be $m \times n$ and $n \times m$ complex matrices, respectively. Then PQ and QP have the same nonzero eigenvalues, counting multiplicity. If $m = n$, then PQ and QP have the same eigenvalues.* □

Lemma 4.5. *Let G be a connected bipartite graph with bipartition (X, Y), diameter D, spectral diameter d and $\widetilde{A}_2 = p_2(A)$. Then the halved graphs G^X and G^Y have the same spectrum, and are of spectral diameter $\lfloor d/2 \rfloor$. Suppose further that at least one of G^X and G^Y has spectral diameter which is equal to its diameter. Then $D \geq d - 1$ for odd d, and $D = d$ otherwise.*

Proof. Note first that G is regular by Proposition 4.2 and Proposition 3.20. Since G is bipartite, p_2 is even, that is, $p_2 = ax^2 + b$ for some real numbers a, b with $a \neq 0$. Note that

$$A = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}$$

for some square matrix B (since G is regular). Let X_1 and Y_1 be adjacency matrices of G^X and G^Y, respectively. Then

$$\begin{pmatrix} X_1 & 0 \\ 0 & Y_1 \end{pmatrix} = A_2 = \widetilde{A}_2 = p_2(A) = aA^2 + bI = \begin{pmatrix} aBB^T + bI & 0 \\ 0 & aB^TB + bI \end{pmatrix}.$$

By Theorem 4.4, BB^T and B^TB have the same eigenvalues, and thus G^X and G^Y have the same spectrum. Note that if λ is an eigenvalue of A with eigenvector u then $a\lambda^2+b$ is an eigenvalue of $\widetilde{A}_2$ with the same eigenvector. Thus $\widetilde{A}_2$ has $\lceil (d+1)/2 \rceil = \lfloor d/2 \rfloor + 1$ distinct eigenvalues, and so do G^X and G^Y. Hence G^X and G^Y are of spectral diameter $\lfloor d/2 \rfloor$. If at least one of G^X and G^Y has spectral diameter which is equal to its diameter, we derived that $d \geq D \geq 2\lfloor d/2 \rfloor$, as claimed. □

The following example shows that, though the two halved graphs have the same spectrum, they may not be isomorphic.

Example 4.6. For integers $D > 1$ and $q > 1$, the Hamming graph $H(D, q)$ is the graph with the vertex set X^D the set of ordered D-tuples of elements of X (or sequences of length D from X), where $|X| = q$. Two vertices are adjacent if they differ in exactly one coordinate. Note that the Hamming graph $H(D, q)$ is a distance-regular graph with diameter D ([6, 7]). A *clique* of a graph is a set of mutually adjacent vertices. A *line* of the Hamming graph $H(D, q)$ is a clique of size q. Consider the Gray graph [4, 5] on 54 vertices obtained by taking the point-line incidence graph of the Hamming graph $H(3, 3)$, which is not distance-regular (c_4 is not well-defined: $\partial(u, v) = \partial(v, w) = 4$ but $c_4(u, v) = 1 \neq 3 = c_4(v, w)$, see Figure 2), with spectrum $\{3^1, \sqrt{6}^6, \sqrt{3}^{12}, 0^{16}, (-\sqrt{3})^{12}, (-\sqrt{6})^6, (-3)^1\}$. Note that $D = 6 = d$, $p_0 = 1$, $p_1 = x$,

$p_2 = x^2 - 3$ (we omit the computational procedures and the results for p_i, $3 \leq i \leq 6$). Since the graph is regular of degree 3, bipartite, and of girth 8, the parameters c_i, a_{i-1}, b_{i-2} are well-defined for $1 \leq i \leq 2$. More precisely, $c_0 := 0$, $a_0 := 0$, $a_1 = 0$ (bipartite), $b_0 = 3$ (3-regular), $c_1 := 1$ and $c_2 = 1$ (girth 8). Then, by (2.1), we have $AA_1 = c_2A_2 + a_1A_1 + b_0A_0$ and thus $A^2 = A_2 + 3I$. Hence, $\widetilde{A}_2 = A_2 = A^2 - 3I = p_2(A)$, that is, the graph is weighted 2-punctually distance-regular. By construction, the two halved graphs are the Hamming graph $H(3,3)$ and the dual graph of $H(3,3)$ (i.e., the graph whose vertices are the lines of $H(3,3)$, and two lines are adjacent if they intersect). They have the same spectrum, but are not isomorphic [38]: $H(3,3)$ is distance-regular, but the dual graph of $H(3,3)$ is not.

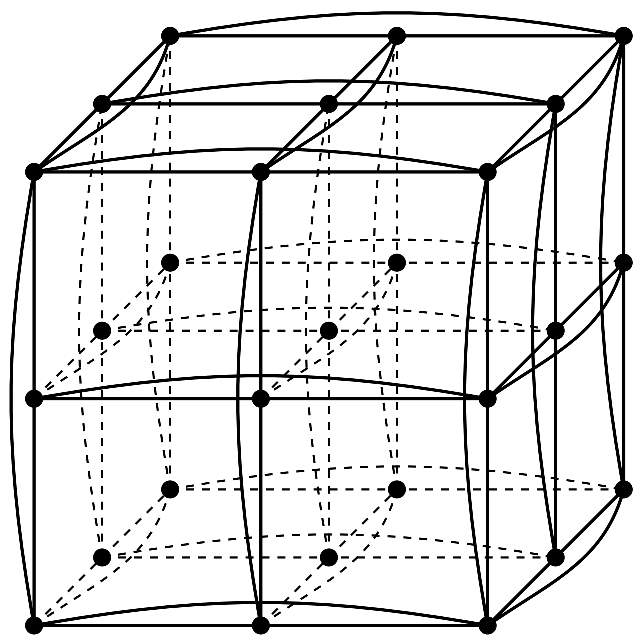

Figure 1. The Hamming graph $H(3,3)$

Note that, even if the bipartite graph is not weighted 2-punctually distance-regular, its two halved graphs are still possible to have the same spectrum (see Example 4.12 in Section 4.4).

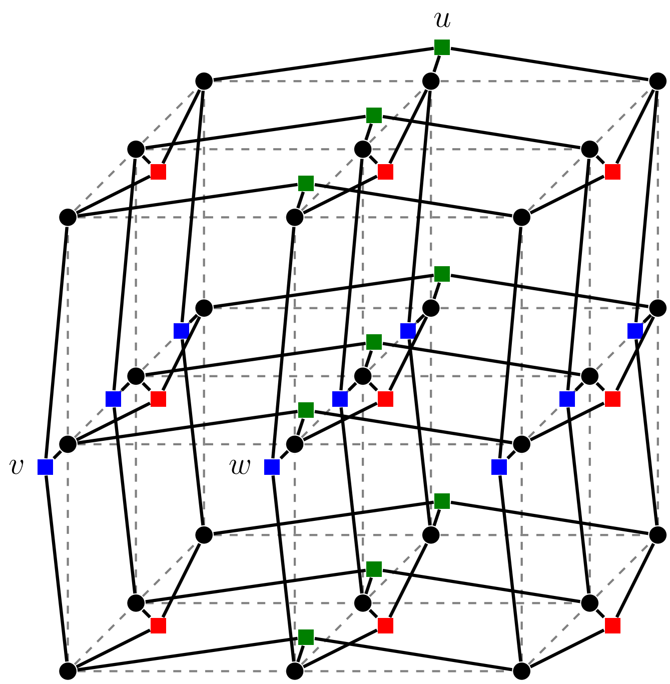

Figure 2. The Gray graph: point-line incidence graph of $H(3, 3)$

4.3 Weighted $(d-1)$-punctual distance-regularity

We have known that a weighted d-punctually distance-regular graph is distance-regular (Proposition 3.9). The bipartite weighted $(d-1)$-punctually distance-regular graphs are studied in this section.

Lemma 4.7. *Let G be a connected bipartite graph. Then $\widetilde{\delta}_{d-1} \le p_{d-1}(\lambda_0)$, with equality if and only if $\widetilde{A}_{d-1} = p_{d-1}(A)$.*

Proof. This follows from Lemma 3.16. □

Proposition 4.8. *Let G be a connected bipartite graph with $\widetilde{\delta}_{d-1} = p_{d-1}(\lambda_0)$. Then $\widetilde{A}_i = p_i(A)$ for all i with the opposite parity of d. In particular, G is regular if d is odd, and biregular otherwise.*

Proof. Since $\widetilde{\delta}_{d-1} = p_{d-1}(\lambda_0)$, we have $\widetilde{A}_{d-1} = p_{d-1}(A)$ by Lemma 4.7. Note that, by

Lemma 3.11,

$$\widetilde{A}_{d-1} + \widetilde{A}_{d-3} + \cdots = p_{d-1}(A) + p_{d-3}(A) + \cdots . \tag{4.1}$$

The proof follows by (backward) induction on i with the opposite parity of d. The base case is $\widetilde{A}_{d-1} = p_{d-1}(A)$. Suppose now that $\widetilde{A}_k = p_k(A)$ for $k \in \{d-1, d-3, \cdots, d-(2i-1)\}$. Then deleting these common terms from both sides of (4.1), we have

$$\widetilde{A}_{d-(2i+1)} + \widetilde{A}_{d-(2i+3)} + \cdots = p_{d-(2i+1)}(A) + p_{d-(2i+3)}(A) + \cdots . \tag{4.2}$$

By induction hypothesis to the three-term recurrence (2.9),

$$\begin{aligned} A^2\widetilde{A}_{d-(2i-1)} &= X_{d-(2i-3)}p_{d-(2i-3)}(A) + Y_{d-(2i-1)}p_{d-(2i-1)}(A) + Z_{d-(2i+1)}p_{d-(2i+1)}(A) \\ &= X_{d-(2i-3)}\widetilde{A}_{d-(2i-3)} + Y_{d-(2i-1)}\widetilde{A}_{d-(2i-1)} + Z_{d-(2i+1)}p_{d-(2i+1)}(A). \end{aligned} \tag{4.3}$$

It remains to show that $\widetilde{A}_{d-(2i+1)} = p_{d-(2i+1)}(A)$. To this end, consider the following two cases:

(*i*) For $\partial(u,v) \geq d-(2i+1)$, $(\widetilde{A}_{d-(2i+1)})_{uv} = (p_{d-(2i+1)}(A))_{uv}$ by (4.2).

(*ii*) For $\partial(u,v) < d-(2i+1)$, $(A^2\widetilde{A}_{d-(2i-1)})_{uv} = \sum_{w\in G_0(u)\bigcup G_2(u)}(\widetilde{A}_{d-(2i-1)})_{wv} = 0$, where the last equality follows since $\partial(w,v) \leq 2 + \ \partial(u,v) < d-(2i-1)$. Then $(p_{d-(2i+1)}(A))_{uv} = 0$ by (4.3) and since $Z_{d-(2i+1)} = \beta_{d-(2i+1)}\beta_{d-2i} \neq 0$.

In particular, $\widetilde{A}_0 = p_0(A)$ if d is odd, and $\widetilde{A}_1 = p_1(A)$ otherwise. Thus the remaining follows from Propositions 3.20–3.21. □

The following example provides a nonregular bipartite graph with even spectral diameter satisfying $p_{d-1}(\lambda_0) = \delta_{d-1}$. A regular example is given in Section 4.4 (Example 4.11). Some (regular) bipartite weighted $(d-1)$-punctually distance-regular graphs with odd spectral diameter are given in Section 4.4 (Example 4.10) and Section 4.6 (Example 4.17).

Example 4.9. Consider the bipartite graph obtained from the Petersen graph by subdividing each edge once (i.e., by replacing each edge with a path of three vertices), with spectrum $\{\sqrt{6}^1, 2^5, 1^4, 0^5, (-1)^4, (-2)^5, (-\sqrt{6})^1\}$. Clearly, this graph is $(2,3)$-biregular on 25 vertices. Note that $D = d = 6$, the Perron vector

$$\alpha = (\underbrace{\sqrt{5/4}, \cdots, \sqrt{5/4}}_{10}, \underbrace{\sqrt{5/6}, \cdots, \sqrt{5/6}}_{15})^t,$$

$p_0 = 1$, $p_1 = 5\sqrt{6}x/12$, $p_2 = 15(x^2 - 12/5)/16$, $p_3 = 5\sqrt{6}(x^3 - 4x)/12$, $p_4 = 25(x^4 - 21x^2/4 + 3)/28$, $p_5 = 5\sqrt{6}(x^5 - 7x^3 + 10x)/24$, $p_6 = 5(x^6 - 65x^4/7 + 22x^2 - 48/7)/24$ (here we omit the computational procedures). Moreover, $\widetilde{A}_i = p_i(A)$ for $i \in \{1, 3, 5\}$ ($\widetilde{\delta}_1 = p_1(\lambda_0) = 5/2$, $\widetilde{\delta}_3 = p_3(\lambda_0) = 5$, $\widetilde{\delta}_5 = p_5(\lambda_0) = 5$, $\widetilde{\delta}_0 = 25/24$, $\widetilde{\delta}_2 = 85/24$, $\widetilde{\delta}_4 = 85/12$, $\widetilde{\delta}_6 = 5/6$, $p_0(\lambda_0) = 1$, $p_2(\lambda_0) = 27/8$, $p_4(\lambda_0) = 375/56$, $p_6(\lambda_0) = 10/7$), and its two halved graphs are the Petersen graph (with spectrum $\{3^1, 1^5, (-2)^4\}$) and the line graph of the Petersen graph (with spectrum $\{4^1, 2^5, (-1)^4, (-2)^5\}$), which are both distance-regular [9].

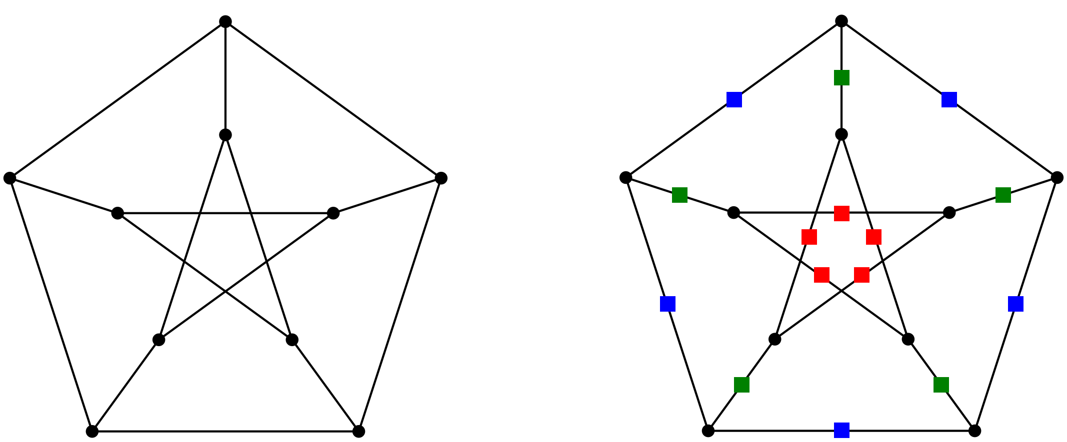

Figure 3. The Petersen graph and its subdivision

4.4 A few examples

In this section, we provide three examples to show that, a connected bipartite graph whose halved graphs are distance-regular, may not be distance-regular. Here we omit the computational details which are straightforward by definitions. Recall that a connected regular graph with exactly three distinct eigenvalues is distance-regular (strongly regular in fact).

Example 4.10. (weighted 2-punctually distance-regular and odd spectral diameter) Consider the Möbius–Kantor graph, i.e., the generalized Petersen graph $G(8,3)$ [40], with spectrum $\{3^1, \sqrt{3}^4, 1^3, (-1)^3, (-\sqrt{3})^4, (-3)^1\}$. Note that $D = 4 < 5 = d$, $p_0 = 1$, $p_1 = x$, $p_2 = x^2 - 3$, $p_3 = 2(x^3 - 5x)/5$, $p_4 = (x^4 - 10x^2 + 15)/6$, $p_5 = (x^5 - 56x^3/5 + 21x)/18$, $\widetilde{A}_i = p_i(A)$ for $i \in \{0,1,2,4\}$ ($\widetilde{\delta}_0 = p_0(\lambda_0) = 1$, $\widetilde{\delta}_1 = p_1(\lambda_0) = 3$, $\widetilde{\delta}_2 = p_2(\lambda_0) = 6$, $\widetilde{\delta}_4 = p_4(\lambda_0) = 1$, $\widetilde{\delta}_3 = 5$, $p_3(\lambda_0) = 24/5$, $p_5(\lambda_0) = 1/5$), and both halved graphs the complete multipartite graphs $K_{2,2,2,2}$ (with spectrum $\{6^1, 0^4, (-2)^3\}$), which are distance-regular.

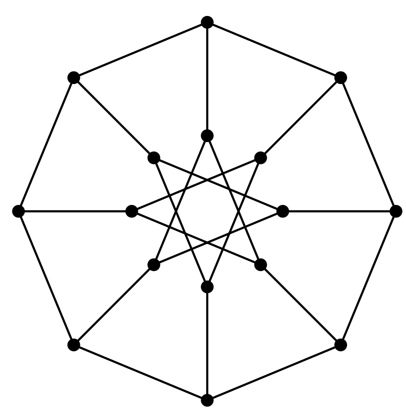

Figure 4. The Möbius–Kantor graph

Example 4.11. (not weighted 2-punctually distance-regular and even spectral diameter) Consider the Hoffman graph with spectrum $\{4^1, 2^4, 0^6, (-2)^4, (-4)^1\}$, which is cospectral to the Hamming 4-cube (or the Hamming graph $H(4,2)$) but not distance-regular [35, 6, 16]. Note that $D = d = 4$, $p_0 = 1$, $p_1 = x$, $p_2 = (x^2 - 4)/2$, $p_3 = (x^3 - 10x)/6$, $p_4 = (x^4 - 16x^2 + 24)/24$, $\widetilde{A}_i = p_i(A)$ for $i \in \{0,1,3\}$ ($\widetilde{\delta}_0 = p_0(\lambda_0) = 1$, $\widetilde{\delta}_1 = p_1(\lambda_0) = 4$, $\widetilde{\delta}_3 = p_3(\lambda_0) = 4$, $\widetilde{\delta}_2 = 13/2$, $\widetilde{\delta}_4 = 1/2$, $p_2(\lambda_0) = 6$, $p_4(\lambda_0) = 1$), and its two halved graphs are the complete graph K_8 (with spectrum $\{7^1, (-1)^7\}$) and the complete multipartite graph $K_{2,2,2,2}$ (with spectrum $\{6^1, 0^4, (-2)^3\}$), which are both distance-regular.

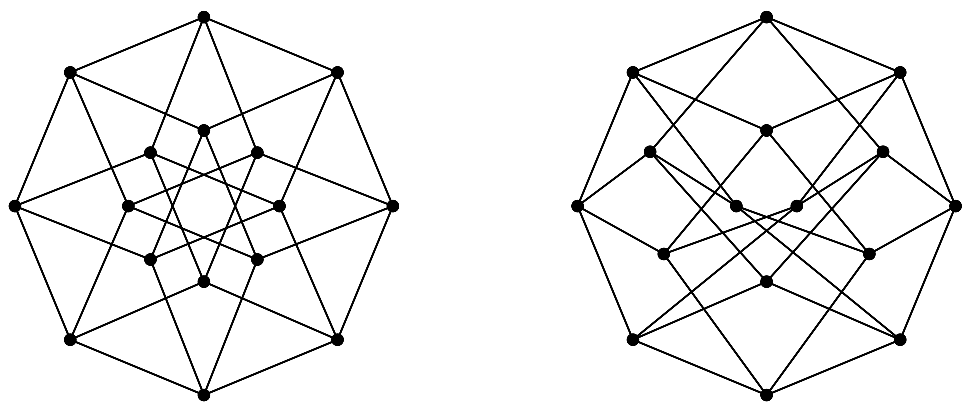

Figure 5. The Hamming 4-cube and the Hoffman graph

Example 4.12. (not weighted 2-punctually distance-regular and odd spectral diameter) Consider the graph obtained by deleting a cycle C_{10} from the complete bipartite graph $K_{5,5}$, with spectrum $\{3^1, ((\sqrt{5}+1)/2)^2, ((\sqrt{5}-1)/2)^2, ((-\sqrt{5}+1)/2)^2, ((-\sqrt{5}-1)/2)^2, (-3)^1\}$. Note that $D = 3 < 5 = d$, $p_0 = 1$, $p_1 = x$, $p_2 = 3(x^2 - 3)/5$, $p_3 = 12(x^3 - 19x/3)/49$, $p_4 = (x^4 - 48x^2/5 + 49/5)/11$, $p_5 = (x^5 - 543x^3/49 + 2820x/147)/33$, $\widetilde{A}_i = p_i(A)$ for $i \in \{0,1\}$ ($\widetilde{\delta}_0 = p_0(\lambda_0) = 1$, $\widetilde{\delta}_1 = p_1(\lambda_0) = 3$, $\widetilde{\delta}_2 = 4$, $\widetilde{\delta}_3 = 2$, $p_2(\lambda_0) = 18/5$, $p_3(\lambda_0) = 96/49$, $p_4(\lambda_0) = 2/5$, $p_5(\lambda_0) = 2/49$), and both halves graphs are the complete graphs K_5 (with spectrum $\{4^1, (-1)^4\}$), which are distance-regular.

Now we have considered three erexamples.

- Example 4.10 (weighted 2-punctually distance-regular and odd spectral diameter)

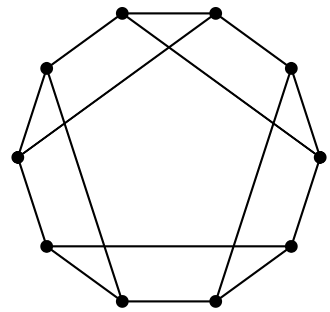

Figure 6. The graph $K_{5,5} - C_{10}$

- Example 4.11 (not weighted 2-punctually distance-regular and even spectral diameter)
- Example 4.12 (not weighted 2-punctually distance-regular and odd spectral diameter)

Note that the remaining case is that 'G is weighted 2-punctually distance-regular with even spectral diameter'. In the next section we show that, under these additional conditions, the graph will be distance-regular.

4.5 Proof of characterization

The following result is related to [6, Proposition 4.2.2] (in the case that d is even).

Theorem 4.13. *Let G be a connected bipartite graph with bipartition (X, Y) and spectral diameter d. Suppose that $\widetilde{A}_i = p_i(A)$ for even i, where $0 \leq i \leq d$. Then both halved graphs G^X and G^Y of G are distance-regular with diameter $\lfloor d/2 \rfloor$.*

Proof. By assumption, $\widetilde{A}_0 = p_0(A) = I$ and $\widetilde{A}_2 = p_2(A) = aA^2 + bI$ for some real numbers a, b with $a \neq 0$. Then G is regular and weighted 2-punctually distance-regular. By Lemma 4.5, G^X and G^Y have the same spectrum, and are of spectral diameter $\lfloor d/2 \rfloor$. Since p_{2i} is even, we can assume $p_{2i} = f_i(ax^2 + b)$ for some $f_i \in \mathbb{R}[x]$ of degree i. Thus, for $0 \leq i \leq \lfloor d/2 \rfloor$,

$$\begin{pmatrix} X_i & 0 \\ 0 & Y_i \end{pmatrix} = \widetilde{A}_{2i} = p_{2i}(A) = f_i(aA^2 + bI) = f_i(\widetilde{A}_2) = \begin{pmatrix} f_i(X_1) & 0 \\ 0 & f_i(Y_1) \end{pmatrix},$$

where X_i and Y_i are distance-i matrices of G^X and G^Y, respectively. Therefore, G^X and G^Y are distance-regular with diameter $\lfloor d/2 \rfloor$. □

Now we are ready to prove Theorem 4.1.

Theorem 4.14. *Let G be a connected bipartite graph with bipartition (X, Y) and spectral diameter d. Suppose that G is weighted 2-punctually distance-regular and both halved graphs G^X and G^Y are distance-regular with diameter $\lfloor d/2 \rfloor$. Then $\widetilde{\delta}_\ell = p_\ell(\lambda_0)$, where $\ell = d-1$ if d is odd, and $\ell = d$ otherwise. In particular, if d is even, then the result reduces to* Theorem 4.1.

Proof. First note that, by Proposition 4.2 and Proposition 3.20, G is regular. By Lemma 4.5, G^X and G^Y have the same spectrum, and are of spectral diameter $\lfloor d/2 \rfloor$. Thus G^X and G^Y have the same (pre)distance-polynomials f_i, $0 \le i \le \lfloor d/2 \rfloor$. Since G^X and G^Y are distance-regular,

$$\widetilde{A}_{2i} = \begin{pmatrix} X_i & 0 \\ 0 & Y_i \end{pmatrix} = \begin{pmatrix} f_i(X_1) & 0 \\ 0 & f_i(Y_1) \end{pmatrix} = f_i(\widetilde{A}_2) = f_i(p_2(A)) = g_{2i}(A)$$

for $0 \le i \le \lfloor d/2 \rfloor$, where X_i and Y_i are distance-i matrices of G^X and G^Y, respectively, and $g_{2i} \in \mathbb{R}[x]$ is even of degree $2i$. Since G is regular, $\widetilde{A}_\ell J = g_\ell(A)J = g_\ell(\lambda_0)J$. Then each row of $\widetilde{A}_\ell$ has exactly $g_\ell(\lambda_0)$ ones, and thus $\widetilde{\delta}_\ell = g_\ell(\lambda_0)$. Now it remains to show that $g_\ell = p_\ell$. Note that $\langle g_\ell, g_\ell \rangle_G = \langle g_\ell(A), g_\ell(A) \rangle = \langle \widetilde{A}_\ell, \widetilde{A}_\ell \rangle = \widetilde{\delta}_\ell = g_\ell(\lambda_0)$. For every polynomial $p \in \mathbb{R}_{\ell-1}[x]$, $\langle g_\ell, p \rangle_G = \langle \widetilde{A}_\ell, p(A) \rangle = 0$. By uniqueness of predistance polynomials, it follows that $g_\ell = p_\ell$. Moreover, if d is even, then by Theorem 3.10, G is distance-regular. □

4.6 A concluding remark

Putting Proposition 4.8, Theorem 3.10, Theorem 4.13 and Theorem 4.14 together, we can conclude the following theorem.

Theorem 4.15. *Let G be a connected bipartite graph with bipartition (X, Y) and spectral diameter d. Then the following conditions are equivalent.*

(*i*) $\widetilde{A}_i = p_i(A)$ *for even i, where $0 \le i \le d$;*

(*ii*) $\widetilde{\delta}_\ell = p_\ell(\lambda_0)$, *where $\ell = d-1$ if d is odd, and $\ell = d$ otherwise;*

(*iii*) *G is weighted 2-punctually distance-regular and both halved graphs G^X and G^Y are distance-regular with diameter $\lfloor d/2 \rfloor$.*

Proof. (*ii*) ⇒ (*i*) By Proposition 4.8 (if d is odd) and Theorem 3.10 (if d is even). (*i*) ⇒ (*iii*) By Theorem 4.13. (*iii*) ⇒ (*ii*) By Theorem 4.14. □

Remark 4.16. Applying a result in [1, Theorem 4.2], Theorem 4.15 (*i*) seems to be improved to the condition that only $i \in \{0, d-2\}$ is necessary when d is even. Unfortunately, there is a flaw in the proof of this result [2].

Note that the Möbius–Kantor graph (Example 4.10) with odd spectral diameter satisfies Theorem 4.15 (*i*)–(*iii*) with $D = d - 1$. The following example shows that a bipartite graph with odd spectral diameter satisfying Theorem 4.15 (*i*)–(*iii*) and $D = d$ needs not to be distance-regular.

Example 4.17. The Desargues graph is the bipartite double of the Petersen graph [38], which means that its adjacency matrix is of the form

$$\begin{pmatrix} 0 & B \\ B & 0 \end{pmatrix},$$

where B is the adjacency matrix of the Petersen graph. Consider the regular bipartite graphs on 20 vertices obtained from the Desargues graph by the Godsil–McKay switching [33], which is cospectral to the Desargues graph, but not distance-regular, with spectrum $\{3^1, 2^4, 1^5, (-1)^5, (-2)^4, (-3)^1\}$ ([38, Proposition 2.3], [21, Proposition

3]). Note that $D = d = 5$, $p_0 = 1$, $p_1 = x$, $p_2 = x^2 - 3$, $p_3 = (x^3 - 5x)/2$, $p_4 = (x^4 - 9x^2 + 12)/4$, $p_5 = (x^5 - 11x^3 + 22x)/12$, $\widetilde{A}_i = p_i(A)$ for $i \in \{0, 1, 2, 4\}$ ($\widetilde{\delta}_0 = p_0(\lambda_0) = 1$, $\widetilde{\delta}_1 = p_1(\lambda_0) = 3$, $\widetilde{\delta}_2 = p_2(\lambda_0) = 6$, $\widetilde{\delta}_4 = p_4(\lambda_0) = 3$, $\widetilde{\delta}_3 = 32/5$, $\widetilde{\delta}_5 = 3/5$, $p_3(\lambda_0) = 6$, $p_5(\lambda_0) = 1$), and both halved graphs are distance-regular with spectrum $\{6^1, 1^4, (-2)^5\}$. More precisely, its two halved graphs are strongly regular with parameter $(n, k, \lambda, \mu) = (10, 6, 3, 4)$, and both of them are isomorphic to the triangular graphs $T(5)$ [12, 41, 14, 39].

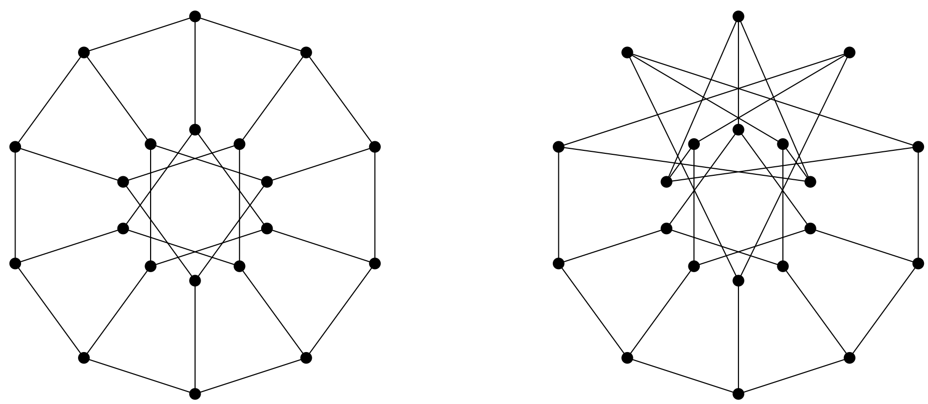

Figure 7. The Desargues graph and its cospectral mate

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