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碩 士 論 文

Combinatorial Identities from Lagrange's Interpolation Polynomial

拉 格 朗 日 插 值 多 項 式 與 組 合 恆 等 式

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摘 要

對任意實多項式 $g(x)$ 及相異數所組成的無窮數列 $\mathbf{a} = (a_0, a_1, \ldots)$，本論文定義一個實數列 $L_{\mathbf{a}}(g(x), n)$。本文研究發現 $L_{\mathbf{a}}(x^k, n)$ 與某種拉格朗日插值多項式的係數有關，同時也是第二類斯特靈數的推廣。本文進行與數列 $L_{\mathbf{a}}(g(x), n)$ 有關的恆等式及組合結構之研究。

關鍵詞：拉格朗日插值多項式、第二類斯特靈數。

Combinatorial Identities from Lagrange Interpolation Polynomial

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Abstract

For a given real polynomial $g(x)$ and infinite sequence $\mathbf{a} = (a_0, a_1, \ldots)$ of distinct real numbers, we define the sequence $L_{\mathbf{a}}(g(x), n)$. We find that $L_{\mathbf{a}}(x^k, n)$ appears in coefficient of a term of some Lagrange's interpolation polynomial, and is also a generalization of the Stirling number of the second kind. Further properties of $L_{\mathbf{a}}(g(x), n)$ related to identities and combinatorial structure are given.

Keywords: Lagrange's interpolation polynomial, Stirling number of the second kind.

Contents

Chapter 1

Introduction

This thesis is motivated from [8] in which identities obtained from Lagrange's interpoation polynomial are given. The Lagrange's Interpolation polynomial was first published by Waring in 1779, rediscovered by Euler in 1783, and published by Lagrange in 1795[7]. Here, the definition was given as follows:

Definition 1.1. Given $n+1$ distinct points (a_0,y_0), ..., (a_n,y_n) in a plane, not two of them in a vertical line, the **Lagrange's interpolation polynomial** with respect to these points, is defined as

$$\sum_{i=0}^{n} y_i \cdot \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{x-a_j}{a_i-a_j}.$$

Note that the above polynomial is the unique polynomial of degree $\leq n$ that goes through these $n+1$ points.

Throughout the thesis, let $\mathbf{a} = (a_0, a_1, \ldots)$ denote a given infinite sequence of some numbers. According to Lagrange's interpolation polynomial and Euclidean division algorithm, for any natural number $n \in \mathbb{N}$ and any polynomial $g(x)$, there exists a polynomial $q(x)$ such that

$$g(x) = (x-a_0)(x-a_1)\cdots(x-a_n)q(x) + \sum_{i=0}^{n} g(a_i) \cdot \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{x-a_j}{a_i-a_j}. \tag{1.1}$$

Note that the $n+1$ points $(a_0, g(a_0)), \ldots, (a_n, g(a_n))$ are in $g(x)$.

In this thesis, $g(x)$ is used to construct a new number $L_{\mathbf{a}}(g(x), n)$. In chapter 2, we find that when $g(x) = x^k$ and $\mathbf{a} = (0, 1, 2, \ldots)$, $L_{\mathbf{a}}(x^k, n)$ is the Stirling number of the second kind. Then we derive a recurrence relation for $L_{\mathbf{a}}(g(x), n)$. In chapter 3, we modify the application of the Stirling number of the second kind to find our application of $L_{\mathbf{a}}(x^k, n)$. In chapter 4, we find that a q-analogue of Stirling number of the second kind is a special case of our $L_{\mathbf{a}}(x^k, n)$. Finally, conclusion is in chapter 5.

Chapter 2

Generalized Stirling number of the second kind

Recall that $\mathbf{a} = (a_0, a_1, \ldots)$ is an infinite sequence of some numbers. The following is the main definition in the thesis.

Definition 2.1. Given a real polynomial $g(x)$, define the sequence $\{L_{\mathbf{a}}(g(x), n)\}_{n=0}^{\infty}$ as

$$L_{\mathbf{a}}(g(x), n) := \sum_{i=0}^{n} \frac{g(a_i)}{\prod_{j \neq i}(a_i - a_j)}.$$

Note that $L_{\mathbf{a}}(g(x), 0) = g(a_0)$.

Lemma 2.2. *The number*

$$L_{\mathbf{a}}(g(x), n) := \sum_{i=0}^{n} \frac{g(a_i)}{\prod_{j \neq i}(a_i - a_j)}$$

appears in the coefficient of x^n of the remainder when $g(x)$ is divided by $(x - a_0)(x - a_1) \cdots (x - a_n)$.

Proof. This is clear from (1.1). □

Lemma 2.3. *If $g(x)$ is a polynomial of degree n then*

$$L_{\mathbf{a}}(g(x), n) = \text{ leading coefficient of } g(x).$$

Proof. By comparing the coefficients of x^n in both sides from (1.1). □

2.1 Identities

The following theorem is from [8], which provide many identities.

Theorem 2.4. *If $g(x)$ is a polynomial of degree at most $n-1$ then*

$$L_{\mathbf{a}}(g(x), n) = 0.$$

Proof. This is clear from Lemma 2.2. $\square$

Taking $n = 2$, $\deg(g(x)) = 1$, $a_i = \pi^i$, we have the following nontrivial identity.

$$\frac{1}{(1-\pi)(1-\pi^2)} + \frac{\pi}{(\pi-1)(\pi-\pi^2)} + \frac{\pi^2}{(\pi^2-1)(\pi^2-\pi)} = 0.$$

2.2 Three-term recurrence relation

The following lemma says that $L_{\mathbf{a}}(g(x), n)$ is linear on $g(x)$.

Lemma 2.5.

$$L_{\mathbf{a}}(g(x) + c \cdot h(x), n) = L_{\mathbf{a}}(g(x), n) + cL_{\mathbf{a}}(h(x), n)$$

for $c \in \mathbb{R}$ and polynomials $g(x), h(x) \in \mathbb{R}[x]$.

Proof. By Definition 2.1,

$$\begin{aligned} L_{\mathbf{a}}(g(x) + c \cdot h(x), n) &= \sum_{i=0}^{n} \frac{g(a_i) + ch(a_i)}{\prod_{j\neq i}(a_i - a_j)} \\ &= \sum_{i=0}^{n} \frac{g(a_i)}{\prod_{j\neq i}(a_i - a_j)} + \sum_{i=0}^{n} \frac{ch(a_i)}{\prod_{j\neq i}(a_i - a_j)} \\ &= L_{\mathbf{a}}(g(x), n) + cL_{\mathbf{a}}(h(x), n). \end{aligned}$$

$\square$

The following lemma shows that recurrence relations of $L_{\mathbf{a}}(g(x), n)$ and $L_{\mathbf{a}}(x^k, n)$, which let us receive some special cases.

Lemma 2.6. *For integers $k, n \geq 1$, the following (i)-(ii) hold.*

(i)

$$L_{\mathbf{a}}((x - a_n)g(x), n) = L_{\mathbf{a}}(g(x), n-1); \tag{2.1}$$

(ii)

$$L_{\mathbf{a}}(x^k, n) = L_{\mathbf{a}}(x^{k-1}, n-1) + a_n L_{\mathbf{a}}(x^{k-1}, n). \tag{2.2}$$

Proof. (i) Note that

$$\begin{aligned} L_{\mathbf{a}}(xg(x),n): &= \sum_{i=0}^{n} \frac{a_i g(a_i)}{\prod_{j\neq i}(a_i-a_j)} \\ &= \sum_{i=0}^{n} \frac{(a_i-a_n)g(a_i)+a_n g(a_i)}{\prod_{j\neq i}(a_i-a_j)} \\ &= \sum_{i=0}^{n} \frac{(a_i-a_n)g(a_i)}{\prod_{j\neq i}(a_i-a_j)} + a_n \sum_{i=0}^{n} \frac{g(a_i)}{\prod_{j\neq i}(a_i-a_j)} \\ &= L_{\mathbf{a}}(g(x),n-1) + a_n L_{\mathbf{a}}(g(x),n). \end{aligned}$$

Hence

$$L_{\mathbf{a}}(xg(x),n) - a_n L_{\mathbf{a}}(g(x),n) = L_{\mathbf{a}}(g(x),n-1).$$

By Lemma 2.5,

$$L_{\mathbf{a}}((x-a_n)g(x),n) = L_{\mathbf{a}}(g(x),n-1).$$

(ii) Set $g(x) = x^{k-1}$ in (i). □

2.3 Three bases

This section studies three bases of $\mathbb{R}[x]_n$, the vector spaces of real polynomials of degree at most n. The first one $\{x^i\}_{i=0}^n$ is clear to be a basis of $\mathbb{R}[x]_n$. Define $E_i(x)$ and $[x_{\mathbf{a}}]_i$ as follows:

$$E_i(x) := \prod_{\substack{j=0 \\ j\neq i}}^{n} (x-a_j),$$

$$[x_{\mathbf{a}}]_i := (x-a_0)(x-a_1)\cdots(x-a_{i-1}) \qquad (\text{here } [x_{\mathbf{a}}]_0 := 1).$$

The following lemma shows that they are bases.

Lemma 2.7.

$$\{E_i(x)\}_{i=0}^n, \quad \{[x_{\mathbf{a}}]_i\}_{i=0}^n$$

are bases of the vector space $\mathbb{R}[x]_n$.

Proof. The set of polynomials $\{[x_{\mathbf{a}}]_i\}_{i=0}^n$ is a base because $[x_{\mathbf{a}}]_i$ is degree of i for all i. $\{[x_{\mathbf{a}}]_i\}_{i=0}^n$ contains degree 1 to degree n. In order to show $\{E_i(x)\}_{i=0}^n$ is a base of $\mathbb{R}[x]_n$, show $\sum_{i=0}^n b_i E_i(x) = 0$ is needed where $b_i = 0$ for all i. First, set $x = a_0$ in $\sum_{i=0}^n b_i E_i(x) = 0$. Then $b_0 \prod_{\substack{j=0 \\ j\neq 0}}^{n} (a_0 - a_j) = 0$. Because $\prod_{\substack{j=0 \\ j\neq 0}}^{n} (a_0 - a_j) \neq 0$, $b_0 = 0$. Set $x = a_1, \ldots, a_n$ in $\sum_{i=0}^n b_i E_i(x) = 0$. Hence $b_2, \ldots, b_n = 0$. □

Note that in the new notation,

$$L_{\mathbf{a}}(g(x),n) = \sum_{i=0}^{n} \frac{g(a_i)}{E_i(a_i)},$$

and

$$x^k = \sum_{i=0}^{n} \frac{a_i^k}{E_i(a_i)} E_i(x) = \sum_{i=0}^{n} T_{\mathbf{a}}(k,i) E_i(x), \tag{2.3}$$

where

$$T_{\mathbf{a}}(k,i) := \frac{a_i^k}{E_i(a_i)}.$$

If $a_i = i$ in (2.3), then $E_i(i) = (-1)^{n-i} i!(n-i)!$ and

$$x^k = \sum_{i=0}^{n} \frac{i^k}{(-1)^{n-i} i!(n-i)!} E_i(x) = \sum_{i=0}^{n} T(k,i) E_i(x),$$

where the number

$$T(k,i) := \frac{(-1)^{n-i} i^k}{i!(n-i)!} = \binom{n}{i} \frac{i^k}{n!}$$

does not have a name to our knowledge.

Problem 2.8. The following two problems are for further study.

(i) Find a combinatorial interpretation of $T_{\mathbf{a}}(k,i)$.

(ii) Find a combinatorial interpretation of $L_{\mathbf{a}}(x^k,n)$ in general.

This study will answer (ii) of the above problem in the special case $a_i = i$ in the Section 2.5.

2.4 Generalized Stirling number of the second kind

As both $\{x^k\}_{k=0}^n$ and $\{[x_{\mathbf{a}}]_k\}_{k=0}^n$ are bases of $\mathbb{R}[x]_n$,

$$x^k = \sum_{i=0}^{n} S_{\mathbf{a}}(k,i)[x_{\mathbf{a}}]_i \tag{2.4}$$

for some scalars $S_{\mathbf{a}}(k,i) \in \mathbb{R}$.

Definition 2.9. The number $S_{\mathbf{a}}(k,n)$ is called the **generalized Stirling number of the second kind**.

In the special case $a_i = i$ of Definition 2.9, the number $S_{\mathbf{a}}(k,n)$, denoted by $S(k,n)$, is called **the Stirling number of the second kind** [2].

Stirling number is named after James Stirling and defined in the 18th century. There are two kinds of Stirling number: Stirling number of the first kind and Stirling number of the second kind. These two numbers are well-known in combinatorics. In [5], $S(k,n)$ is defined as the number of all partitions of an k-set into n nonempty subsets. One can use Principle of Inclusion and Exclusion to explain it. Note that

$$\begin{aligned} n!S(k,n) &= \sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}i^k \\ &= \sum_{i=0}^{n}(-1)^{i}\binom{n}{i}(n-i)^k, \end{aligned}$$

is the number of surjective functions from $[k] := \{1,2,\ldots,k\}$ into $[n]$. Hence $S(k,n)$ is the number of partitions of $\{1,2,\ldots,k\}$ into n indistinguishable nonempty boxes.

We close this section by showing that the sequence $S_{\mathbf{a}}(k,n)$ also satisfies a three-term recurrence relation.

Theorem 2.10.

$$S_{\mathbf{a}}(k,n) = S_{\mathbf{a}}(k-1,n-1) + a_n S_{\mathbf{a}}(k-1,n). \tag{2.5}$$

Proof. Compare coefficients of $[x_{\mathbf{a}}]_i$ in (2.2). Hence

$$\sum_{i=0}^{n} S_{\mathbf{a}}(k,i)[x_{\mathbf{a}}]_i = x^k = x^{k-1}x.$$

Use (2.4) and take k as $k-1$. x can be replace with $x - a_{i-1} + a_{i-1}$. Then

$$\sum_{i=0}^{n} S_{\mathbf{a}}(k,i)[x_{\mathbf{a}}]_i = \sum_{i=1}^{n} S_{\mathbf{a}}(k-1,i-1)[x_{\mathbf{a}}]_{i-1}(x - a_{i-1} + a_{i-1}).$$

Write two summations as

$$\sum_{i=0}^{n} S_{\mathbf{a}}(k,i)[x_{\mathbf{a}}]_i = \sum_{i=1}^{n} S_{\mathbf{a}}(k-1,i-1)[x_{\mathbf{a}}]_{i-1}(x-a_{i-1}) + \sum_{i=1}^{n} S_{\mathbf{a}}(k-1,i-1)[x_{\mathbf{a}}]_{i-1}a_{i-1}.$$

Change initial number of i from 1 to 0. Then

$$\sum_{i=0}^{n} S_{\mathbf{a}}(k,i)[x_{\mathbf{a}}]_i = \sum_{i=0}^{n}(S_{\mathbf{a}}(k-1,i-1) + a_i S_{\mathbf{a}}(k-1,i))[x_{\mathbf{a}}]_i.$$

□

2.5 An interpretation of $L_{\mathbf{a}}(x^k, n)$

Lemma 2.11. *For $k < n$, $L_{\mathbf{a}}(x^k, n) = S_{\mathbf{a}}(k, n)$.*

Proof. Note that $S_{\mathbf{a}}(k, n) = 0$ by (2.4), and $L_{\mathbf{a}}(x^k, n) = 0$ by Theorem 2.4. □

We will prove that this is also true for $k > i$ later. Note that $S_{\mathbf{a}}(k, 0) = a_0^k$. The following theorem shows that $S_{\mathbf{a}}(k, n)$ is a special case of $L_{\mathbf{a}}(g(x), n)$. In this thesis, we focus on $L_{\mathbf{a}}(x^k, i)$.

Theorem 2.12.

$$L_{\mathbf{a}}(x^k, n) = S_{\mathbf{a}}(k, n).$$

Proof. We have proved the case for $k < n$, indeed both being zero. It is clear from their definitions $L_{\mathbf{a}}(x^k, k) = 1 = S_{\mathbf{a}}(k, k)$. Now the theorem follows from Theorem 2.10 since both sequences satisfy the same pattern of three-term recurrence relation. □

Corollary 2.13.

$$\frac{a_i^k}{\prod_{j \neq i}(a_i - a_j)} = S_{\mathbf{a}}(k, i) - S_{\mathbf{a}}(k, i-1).$$

Proof. This corollary is received by Theorem 2.12 and Definition 2.1. □

2.6 A conjecture

In this section, we will give a conjecture which is related to generalized unsigned Stirling number of the first kind. First, recall the definition of the unsigned Stirling number of the first kind.

Definition 2.14. In [5], $c(k, i)$ is the number of permutations $\pi \in S_k$ with exactly i cycles. Then $c(k, i)$ is called **signless Stirling number of the first kind**.

Corollary 2.15. *In [5], there is the result:*

$$\sum_{i=0}^{k} c(k, i) x^i = x(x+1)\cdots(x+k-1).$$

It is also equals to

$$\sum_{i=0}^{k} (-1)^{k-i} c(k, i) x^i = x(x-1)\cdots(x-k+1).$$

The following definition is generalized signless Stirling number of the first kind by Corollary 2.15. This is similar to the definition of generalized Stirling number of the second kind.

Definition 2.16. $c_{\mathbf{a}}(k,i)$ is the **generalized signless Stirling number of the first kind**:

$$\sum_{i=0}^{n}(-1)^{n-i}c_{\mathbf{a}}(k,i)x^i = [x_{\mathbf{a}}]_k. \tag{2.6}$$

Definition 2.17. Choose $t_{\mathbf{a}}(n,k,i) \in \mathbb{R}$ such that

$$E_k(x) = \sum_{i=0}^{n}(-1)^{n-i}t_{\mathbf{a}}(n,k,i)x^i. \tag{2.7}$$

Note that

$$t_{\mathbf{a}}(n,k,0) = a_0a_1\cdots\widehat{a_k}\cdots a_n,$$

and

$$t_{\mathbf{a}}(n,k,i) = \sum_{\substack{|S|=n-i \\ S\subseteq\{0,1,\ldots,\widehat{k},\ldots,n\}}} \prod_{j\in S} a_j,$$

where $\widehat{a_k}$ means that a_k is not contained.

Conjecture 2.18.

$$\sum_{j=0}^{i} t_{\mathbf{a}}(n,k,j) = c_{\mathbf{a}}(k,i), \qquad t_{\mathbf{a}}(n,k,i) = c_{\mathbf{a}}(k,i) - c_{\mathbf{a}}(k,i-1).$$

The easiest way is comparing their coefficient in (2.6) and (2.7). Then we may receive their relations.

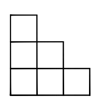
Chapter 3

Some recurrence relations

Here are two well-known questions and their recurrence relations. The first section introduces triangle boards question [1]. This question tells that counting how many ways we put k rooks on triangle board of size m such that these k rooks are non-attacking. Then we use the result of [1] to connected to $L_{\mathbf{a}}(x^k, n)$. Second question tells that using difference table to get Euler's formula [3]. Then we will get Euler's formula is related to $L_{\mathbf{a}}(x^k, n)$.

3.1 Triangle boards

A triangle board of size m is a board consisting of m layers of unit squares and the i-th layer from the top has i consecutive unit squares from the left. See below for a triangle board of size 3.



We want to place k non-attacking rooks on a triangle board of size m and $k \le m$. By placing of rooks on a triangle board of size m, we mean a subset p of $\{1, 2, \ldots, m\} \times \{1, 2, \ldots, m\}$ such that if $(i_1, j_1), (i_2, j_2)$ are distinct pairs in p then $i_2 \ne i_1, j_2 \ne j_1$ and $j_1 \le i_1$. Let $R_p := \{i \mid (i, j) \in p\}$ for a placing p. Fix a sequence $\mathbf{w} = (w_1, w_2, \ldots, w_m)$ of row weights w_i. For a placing $p = \{(p_1, q_1), (p_2, q_2), \ldots, (p_k, q_k)\}$, where $p_1 < p_2 < \cdots < p_k$, define the placing weight $w_p := \prod_{i \in R_p} w_{p_i - i + 1}$ of p, and total weight $W_{k,m} := \sum_p w_p$, where the sum is over all placing p of size $|R_p| = k$. We want to compute the total weight $W_{k,m}$ of k non-attacking rooks on a triangle board of size m.
Note that if $w_i = 1$ then $W_{k,m}$ is the number of ways to place k non-attacking rooks

on a triangle board of size m. The following theorem shows the recurrence relation of $W_{k,m}$.

Theorem 3.1.

$$W_{k,m} = W_{k,m-1} + w_{m-k+1}(m-k+1)W_{k-1,m-1}.$$

Proof. To place k non-attacking rooks on a triangle board of size m, one can either place them all on the first $m-1$ layers or place $k-1$ of them on the first $m-1$ layers and the last one on the bottom. The first case contributes to number $W_{k,m-1}$ to the total weight. In the latter case, after that $k-1$ rooks are placed on the first $m-1$ layers, there are $m-(k-1)$ positions on the bottom can be chosen to place the last rook. □

3.1.1 Compare with Section 2.2

The following theorem shows the relation of $S_{\mathbf{a}}(k,n)$ and $W_{k,m}$.

Theorem 3.2. *Given row weights w_i, let $a_i = iw_i$. Then*

$$S_{\mathbf{a}}(m+1, m+1-k) = W_{k,m},$$

where $0 \leq k \leq m$.

Proof. By induction on $k+m$. If $k+m=0$ then $m=k=0$ and $S_{\mathbf{a}}(1,1) = W_{0,1} = 1$. In general

$$W_{k,m} = W_{k,m-1} + w_{m-k+1}(m-k+1)W_{k-1,m-1}.$$

By induction hypothesis, $S_{\mathbf{a}}(m, m-k) = W_{k,m-1}$ and $S_{\mathbf{a}}(m, m-k+1) = W_{k-1,m-1}$. Hence, write $W_{k,m}$ as

$$W_{k,m} = S_{\mathbf{a}}(m, m-k) + a_{m-k+1}S_{\mathbf{a}}(m, m-k+1).$$

By Theorem 2.10, write $W_{k,m}$ as

$$W_{k,m} = S_{\mathbf{a}}(m+1, m+1-k).$$

□

By Theorem 3.2 and Theorem 2.12, we will get $W_{k,m}$ and $L_{\mathbf{a}}(x^k, n)$ whose relations are below.

Proposition 3.3.

$$W_{k,m} = L_{\mathbf{a}}(x^{m+1}, m+1-k).$$

Thenthe total weight $W_{k,m}$ of k non-attacking rooks on a triangle board of size m is equal to

$$L_{\mathbf{a}}(x^{m+1}, m+1-k) = \sum_{m+1-k=0}^{m+1} \frac{a_i^{m+1}}{\prod_{j\neq i}(a_i - a_j)},$$

where $a_i = w_i i$.

3.2 Euler's formula for the n-th differences of powers

Consider $g(x)$ be a polynomial. Let $b_0 = g(0), \ldots, b_i = g(i), \ldots$ be a sequence. Now, we define a new sequence $\Delta b_0, \ldots, \Delta b_i, \ldots$ where $\Delta b_i = b_{i+1} - b_i$ $(i \geq 0)$. This sequence called first difference. We can also define sequence $\Delta^j b_0, \ldots, \Delta^j b_i, \ldots$ as jth difference where $\Delta^j b_i = \Delta(\Delta^{j-1} b_i)$.

The difference table is obtained from sequences $b_0, \ldots, b_i, \ldots$ laid on 0th row and $\Delta^j b_0, \ldots, \Delta^j b_i, \ldots$ laid on jth row.

For example, $g(x) = x^4$ difference table as follows.

$$
\begin{array}{c}
0, \quad 1, \quad 16, \quad 81, \quad 256, \quad 625, \quad 1296, \quad 2401, \quad 4096, \quad 6561 \cdots \\
1, \quad 15, \quad 65, \quad 175, \quad 369, \quad 671, \quad 1105, \quad 1695, \quad 2465 \cdots \\
14, \quad 50, \quad 110, \quad 194, \quad 302, \quad 434, \quad 590, \quad 770 \cdots \\
36, \quad 60, \quad 84, \quad 108, \quad 132, \quad 156, \quad 180 \cdots \\
24, \quad 24, \quad 24, \quad 24, \quad 24, \quad 24 \cdots \\
0, \quad 0, \quad 0, \quad 0, \quad 0 \cdots
\end{array}
$$

Next, we define $h(g(x), n)$ as the difference table of the first number of nth row.

Lemma 3.4.

$$h(g(x), n) := \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} g(i).$$

When $g(x) = x^k$,

$$h(x^k, n) = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} i^k.$$

Lemma 3.5.

$$h(x^k, n) = \begin{cases} 0, & \textit{if } 0 \leq k < n; \\ n!, & \textit{if } k = n. \end{cases}$$

There are two cases of k. Both of them are proved by induction in [3].

3.2.1 Compare with Definition 2.1

The following proposition shows the relation of $h(x^k, n)$ and $L_{\mathbf{a}}(x^k, n)$.

Proposition 3.6.

$$h(x^k, n) = n! \cdot L_{\mathbf{a}}(x^k, n),$$

where $\mathbf{a} = (0, 1, \ldots, n, \ldots)$.

Proof.

$$\begin{aligned} h(x^k, n) &= \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} i^k \\ &= \sum_{i=0}^{n} \frac{(-1)^{n-i} n! i^k}{i!(n-i)!} \\ &= n! \sum_{i=0}^{n} \frac{(-1)^{n-i} i^k}{i!(n-i)!} \\ &= n! S(k,n) = n! L_{\mathbf{a}}(x^k, n), \end{aligned}$$

where $\mathbf{a} = (0, 1, \ldots, n, \ldots)$. □

Hence, $h(x^k, n)$ is the number of surjective functions from $[k] := \{1, 2, \ldots, k\}$ into $[n]$.

Lemma 3.7. *In [2], let $g(x)$ be a polynomial of degree p*

$$g(x) = a_p x^p + a_{p-1} x^{p-1} + \cdots + a_1 x + a_0.$$

Then $\Delta^{p+1} g(i) = 0$ for all $i \geq 0$.

Proof. By induction on p. If $p = 0$ then $g(x) = a_0$ is a constant. Hence $\Delta g(i) = g(i+1) - g(i) = 0$. In general

$$\begin{aligned} \Delta g(i) &= g(i+1) - g(i) \\ &= (a_p(i+1)^p + a_{p-1}(i+1)^{p-1} + \cdots + a_1(i+1) + a_0) \\ &\quad -(a_p(i)^p + a_{p-1}(i)^{p-1} + \cdots + a_1(i) + a_0). \end{aligned}$$

By binomial theorem,

$$\begin{aligned} a_p(i+1)^p - a_p(i)^p &= a_p(i^p + \binom{p}{1} i^{p-1} + \cdots + 1) - a_p(i)^p \\ &= a_p(\binom{p}{1} i^{p-1} + \cdots + 1). \end{aligned}$$

Because $\Delta g(i)$'s degree is at most $p-1$, it is know that $\Delta^{p+1} g(i) = \Delta^p(\Delta g(i)) = 0$ by induction hypothesis. □

Because of Lemma 2.3 and Theorem 2.4, we try to derive a Theorem by using Lemma 3.7 as follows. Then we will get whether $h(g(x), n)$ is related to $L_{\mathbf{a}}(g(x), n)$.

Theorem 3.8.

$$h(g(x), n) = \begin{cases} 0, & \text{if } 0 \leq \deg(g(x)) < n; \\ \text{leading coefficient of } g(x) \cdot n!, & \text{if } \deg(g(x)) = n. \end{cases}$$

Proof. By Lemma 3.5, we can find that when $0 \leq \deg(g(x)) < n$, $h(g(x), n)$ is 0. Next, we consider the case for $\deg(g(x)) = n$. In Lemma 3.5, we can separate $g(x)$ as $a_n x^n$, ..., a_0. Then we write down difference table of x^i for all i. Next, we combine all the difference tables of $a_n x^n$, ..., a_0 so this difference table is also the difference table of $g(x)$. Then by Lemma 3.7 we will get leading coefficient of $g(x) \cdot n!$. □

The following corollary is received by Theorem 3.8, Lemma 2.3 and Theorem 2.4.

Corollary 3.9. *When* $\deg(g(x)) \leq n$,

$$h(g(x), n) = n! \cdot L_{\mathbf{a}}(g(x), n).$$

Chapter 4

q-analogue

In this chapter, the first section introduces n-dimensional subspace in the beginning. In this section, we derive recurrence relation of n-dimensional subspace. Then we use previous result to get $L_{\mathbf{a}}(x^k, n)$ is related to n-dimensional subspace problem. The second section introduces q-Stirling number of the second kind. It is q-analogue of Stirling number of the second kind. Then the result is that it is relatd to $L_{\mathbf{a}}(x^k, n)$.

4.1 n-dimensional subspace

Definition 4.1. Let $A(1)$, ... ,$A(n)$ be a sequence of n distinct numbers. If $i < j$ and $A(i) > A(j)$, then the pair (i, j) is called an **inversion** of A.
The inversion number $inv(A)$ is defined to be the number of inversions.

$$inv(A) = \#\{(i, j)|i < j \text{ and } A(i) > A(j)\}.$$

Definition 4.2.

$$\begin{aligned}[n]_q =& 1 + q + \cdots + q^{n-1},\\ [n]_q! =& (1 + q)(1 + q + q^2) \cdots (1 + q + \cdots + q^{n-1}).\end{aligned}$$

Lemma 4.3.

$$\sum_{\alpha \in S_n} q^{inv(\alpha)} = [n]_q!$$

Proof. By induction on n. If $n = 1$ then $\sum_{\alpha \in S_1} q^{inv(\alpha)} = [1]_q! = 1$. In general

$$[k]_q! = (1 + q + \cdots + q^{k-1})[k - 1]_q!.$$

By induction hypothesis, replace $[k - 1]_q!$ with $\sum_{\alpha \in S_{k-1}} q^{inv(\alpha)}$. Then

$$[k]_q! = (1 + q + \cdots + q^{k-1}) \sum_{\alpha \in S_{k-1}} q^{inv(\alpha)}.$$

Hence

$$[k]_q! = \sum_{\alpha \in S_{k-1}} (1 + q + \cdots + q^{k-1}) q^{inv(\alpha)}.$$

$(1 + q + \cdots + q^{k-1})$ means that we put k into sequence α. Then the inversion will increase from 0 to $k-1$. Suppose sequence β is sequence α joined k. Hence

$$[k]_q! = \sum_{\beta \in S_k} q^{inv(\beta)}.$$

□

4.1.1 Compare with Section 2.2

Let $\binom{k}{n}_q$ be the number of n-dimensional subspaces in a k-dimensional space over F_q. We shall explain that this number satisfies the previous $L_{\mathbf{a}}(x^k, n) = L_{\mathbf{a}}(x^{k-1}, n-1) + a_n L_{\mathbf{a}}(x^{k-1}, n)$ with $a_i = q^i$ as

$$\binom{k}{n}_q = \binom{k-1}{n-1}_q + q^n \binom{k-1}{n}_q.$$

Fix a vector $u \in GF^k(q)$. The number of n-dimensional subspaces in $GF^k(q)$ containing u is $\binom{k-1}{n-1}_q$. Let B be the set of n-dimensional subspaces that intersect 0 with u. Then the join of u and an element in B is an $(n+1)$-dimensional subspace containing u. Note that there are $\binom{k-1}{n}_q$ subspaces, each of them containing $\binom{n+1}{n}_q$ n-dimensional subspaces, among which there are $\binom{n}{n-1}_q$ containing u. Hence

$$|B| = \binom{k-1}{n}_q \left(\binom{n+1}{n}_q - \binom{n}{n-1}_q \right) = q^n \binom{k-1}{n}_q.$$

According to proof of this study, $\binom{k}{n}_q$ has the same pattern of three-term recurrence relation with $L_{\mathbf{a}}(x^k, n)$ when $a_i = q^i$. Hence $\binom{k}{n}_q = L_{\mathbf{a}}(x^k, n)$ where $a_i = q^i$. Let $a_i = q^i$. Then

$$\begin{aligned}
L_{\mathbf{a}}(x^k, n) : &= \sum_{i=0}^{n} \frac{a_i^k}{\prod_{j \neq i}(a_i - a_j)} \\
&= \sum_{i=0}^{n} \frac{q^{ik}}{\prod_{j \neq i}(q^i - q^j)} \\
&= \sum_{i=0}^{n} \frac{q^{ik}}{\prod_{j>i}(q^i - q^j) \prod_{j<i}(q^i - q^j)} \\
&= \sum_{i=0}^{n} \frac{q^{ik}}{(-1)^{n-i} q^{ni-i^2} q^{(i^2-i)/2} \prod_{j>i}(q^{j-i} - 1) \prod_{j<i}(q^{i-j} - 1)} \\
&= \sum_{i=0}^{n} \frac{q^{ik}}{(-1)^{n-i} q^{-i(2n-i-1)/2} (q-1)^n [i]_q! [n-i]_q!} = \binom{k}{n}_q.
\end{aligned}$$

By Lemma 4.3, we can replace $[i]_q!$ with $\sum_{\alpha\in S_i} q^{inv(\alpha)}$ and replace $[n-i]_q!$ with $\sum_{\beta\in S_{n-i}} q^{inv(\beta)}$. It would become more easier to compute.

4.2 q-Stirling number of the second kind

We already know that $S(k,n)$ is Stirling number of the second kind when $a_i = i$. q-Stirling number of the second kind was first defined by Carlitz in 1948. After Carlitz's paper, many combinatorial papers have introduced the q-analogue. Now, we define $S_q(k,n)$ be **q-Stirling number of the second kind**. In [6], we define $S[k,n]$ is the set of all partitions of $\{1,\ldots,k\}$ into n nonempty subsets $B_1, B_2, \ldots, B_n$. That is $\pi = B_1/B_2/\cdots/B_n$ where B_i are all increasing sequences and $\min B_1 < \min B_2 < \cdots < \min B_n$. Here, $inv(\pi)$ is the number of the pair (b, B_j) such that $b \in B_i$ where $i<j$ and $b > \min B_j$.

Definition 4.4.

$$S_q(k,n) = \sum_{\pi\in S[k,n]} q^{inv(\pi)}.$$

4.2.1 Three term recurrence relation

The following proposition shows three term recurrence relation of $S_q(k,n)$.

Proposition 4.5.

$$S_q(k,n) = S_q(k-1,n-1) + [n]_q S_q(k-1,n).$$

Proof. Use combinatorial argument. In the first case, let $\{k\}$ be a subset of $\{1,\ldots,k\}$. There are $S_q(k-1,n-1)$ ways. In the second case, $\{k\}$ isn't a subset of $\{1,\ldots,k\}$. First, we consider $S_q(k-1,n)$. Finally, we put k into any subset. Hence the inversion will increase from 0 to $n-1$ after adding k into $S_q(k-1,n)$. There are $(1+q+\cdots+q^{n-1})S_q(k-1,n) = [n]_q S_q(k-1,n)$ ways. □

The following corollary shows the relation of $S_q(k,n)$ and $L_{\mathbf{a}}(x^k,n)$.

Corollary 4.6.

$$S_q(k,n) = L_{\mathbf{a}}(x^k,n),$$

for $\mathbf{a}$=$([1]_q,[2]_q,\ldots,[n]_q,\ldots)$.

Proof. By Proposition 4.5 and (2.2) then $S_q(k,n) = L_{\mathbf{a}}(x^k,n)$ when $a_n = [n]_q$. □

Next, we will reprove Corollary 4.6 by direct computing. First, consider $[i]_q - [j]_q$. There are two cases.

1. $j < i$:

$$[i]_q - [j]_q = \frac{q^i - q^j}{q-1} = q^j [i-j]_q;$$

2. $j > i$:

$$[i]_q - [j]_q = \frac{q^i - q^j}{q-1} = -q^i [j-i]_q.$$

Hence

$$\prod_{\substack{j \neq i \\ 0 \leq j \leq n}} ([i]_q - [j]_q) = q^{i(2n-i-1)/2} [i]_q! [n-i]_q!.$$

Let $a_i = [i]_q$, where $q \neq 0$. Then

$$\begin{aligned} L_{\mathbf{a}}(x^k, n) &= \sum_{i=0}^{n} \frac{a_i^k}{\prod_{j \neq i}(a_i - a_j)} \\ &= \sum_{i=0}^{n} \frac{[i]_q^k}{\prod_{j \neq i}(a_i - a_j)} \\ &= \sum_{i=0}^{n} \frac{[i]_q^k}{\prod_{j \neq i}([i]_q - [j]_q)} \\ &= \sum_{i=0}^{n} (-1)^{n-i} q^{-i(2n-i-1)/2} \frac{[i]_q^k}{[i]_q! [n-i]_q!} = S_q(k, n). \end{aligned}$$

Chapter 5

Conclusions

In this thesis, we get two important recurrence relations which are

$$L_{\mathbf{a}}((x-a_n)g(x),n) = L_{\mathbf{a}}(g(x),n-1)$$

and

$$L_{\mathbf{a}}(x^k,n) = L_{\mathbf{a}}(x^{k-1},n-1) + a_n L_{\mathbf{a}}(x^{k-1},n).$$

We also proved

$$L_{\mathbf{a}}(x^k,i) = \sum_{j=0}^{i} \frac{a_j^k}{\prod_{t\neq j}(a_j-a_t)} = S_{\mathbf{a}}(k,i).$$

This thesis focus on the recurrence relation $L_{\mathbf{a}}(x^k,n) = L_{\mathbf{a}}(x^{k-1},n-1)+a_n L_{\mathbf{a}}(x^{k-1},n)$. When $a_n = 1$, it is a well-known recurrence relation which is the same as $\binom{k}{n} = \binom{k-1}{n-1} + \binom{k-1}{n}$. Then we get

$$L_{\mathbf{a}}(x^k,n) = \binom{k}{n}.$$

In chapter 2, consider $a_n = n$. Then we get

$$L_{\mathbf{a}}(x^k,n) = S(k,n).$$

In chapter 4, consider $a_n = q^n$ and $a_n = [n]_q$. Then we get

$$L_{\mathbf{a}}(x^k,n) = \binom{k}{n}_q$$

and

$$L_{\mathbf{a}}(x^k,n) = S_q(k,n)$$

respectively. All of them are special cases of $L_{\mathbf{a}}(x^k,n)$.

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