國 立 交 通 大 學

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碩 士 論 文

花 苞 圖 的 最 小 秩

The Minimum Rank of Buds

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摘 要

對一以 $[n]=\{1,2,\cdots,n\}$ 為點的簡單連通圖 G 而言，當一大小為 n 的實對稱矩陣滿足性質：此矩陣非對角的第 ij 位置非零若且唯若 i 與 j 在圖 G 上有邊，則我們稱此矩陣與 G 相對應。一張圖的最小秩為其相對應的所有矩陣之中最小的秩。在此論文中我們定義一種與一介於 1 與 $n/4$ 間的數 m 有關且點數為 n 的圖，命名為基於 $[n-m]$ 的花苞圖。花苞圖含一個 $n-m$ 點的環，其餘 m 個點之間沒有邊相連。環可以藉由切斷 m 邊將環分割成 m 段長度大於 2 的區塊，使得這 m 個點各自與不同的區塊之間至少有 3 個邊相連。在此論文中，我們將證明一個基於 $[n-m]$ 的花苞圖其最小秩為 $n-m-2$。

關鍵詞：圖、最小秩、花苞圖。

The Minimum Rank of Buds

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abstract

For a simple graph G of order n with vertex set $[n] = \{1, 2, \cdots, n\}$, an $n \times n$ real symmetric matrix A, whose ij-th entry is not zero if and only if there is an edge joined i and j in G, is said to be associated with G. The minimum rank of G is defined to be the smallest possible rank over all symmetric real matrices associated with G. A bud based on $[n-m]$ is a graph G with vertex set $V(G) = [n]$ satisfying the following axioms:

1. The subgraph of G induced on $[n-m]$ is a cycle C_{n-m}, and the subgraph induced on $[n] \setminus [n-m]$ has no edge.

2. The cycle C_{n-m} can be parted into m disjoints paths, and the length of these paths are at least 2. For all vertex v in $[n] \setminus [n-m]$, v has at least three neighbors in the same path. Any two vertices in $[n] \setminus [n-m]$ are not connected to the same path.

In the thesis we will show that a bud based on $[n-m]$ has minimum rank $n-m-2$.

Keywords: Graph, Minimum rank, Bud.

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Contents

Chapter 1

Introduction

Many mathematical theories have their combinatorial realizations and vice versa, and the study of matrices associated with a graph G gives a connection between Graph Theory and Linear Algebra. In this thesis, ranks of matrices associated with graphs and their combinatorial meaning are investigated.

All graphs considered in this thesis are simple and connected. For a graph G of order n, we use $E(G)$ as its edge set and $V(G)$ as its vertex set, usually $V(G) = [n] = \{1, 2, ..., n\}$. For an $n \times n$ real symmetric matrix A, $\Gamma(A)$ represents the graph such that $ij \in E(\Gamma(A))$ if and only if the ij-th entry of A is not zero, indicating that the matrix A is *associated* with $\Gamma(A)$. The *minimum rank* of a graph G, denoted by $m(G)$, is defined to be the integer

$$m(G) = \min\{\text{rank}(A) : \Gamma(A) = G\},$$

where the minimum is taken over all $n \times n$ symmetric matrices A.
The minimum rank of G is related to the *maximum nullity* of G, denoted by

$$M(G) = \max\{\text{nullity}(A) : \Gamma(A) = G\}.$$

It is well-known that $m(G) + M(G) = n$ for all graphs. Since $\Gamma(A) = \Gamma(A + \lambda\mathbf{I}) = G$, $M(G)$ is also the maximum multiplicity among the possible multiplicities of eigenvalues of all matrices associated with G.

The number $m(G)$ also has combinatorial meanings. Ping-Hong Wei and Chih-Wen Weng[8] showed that if G is a **tree**, that is, a connected graph satisfies $|V(G)|-1 = |E(G)|$, then $|E(G)|-m(G)$ is equal to the minimum size of edge subset S whose deletion will yield a graph with each vertex of degree 1 or 2 . The AIM Minimum Rank - Special Graphs Work Group[1] defined *Initial configuration*, *color-change rule*, and *zero-forcing set* of a graph G as described below :

1. Initial configuration

   The vertex set $V(G)$ of G is partitioned into two classes, and these two classes are colored black and white.

2. Color-change rule

   If u is a black vertex of G, and v is the unique white neighbor of u, then the color of v is changed to be black.

3. Zero-forcing set

   A subset $S \subseteq V(G)$ is a zero-forcing set if an all-black coloring is obtained from the initial configuration with S colored black followed by a sequence of color-change rules.

The minimum size of a zero-forcing set of G is denoted by $Z(G)$. Note that $M(G) \leq Z(G)$ [1].

We will compute the minimum rank of a class of graphs which are obtained by adding a vertex and some edges to a cycle C_{n-1}. We check some minimum ranks of these graphs and give two conjectures. In the end of the thesis, we define a class of graphs, called buds based on $[n-m]$ of order n, and show that such a graph G has the minimum rank $n-m-2$, and $M(G) = Z(G) = m+2$.

The thesis is organized as follows. In Chapter 2, we introduce some notations and operations for graphs and matrices used in the thesis. In Chapter 3, we introduces well-known theorems and propositions in which the relation of minimum rank between a graph and its subgraph is investigated. Also the relation between the minimum size of zero-forcing set and the maximum nullity is introduced there. There are two parts in Chapter 4. In the first part, we study the changing of minimum ranks when adding a vertex to a cycle C_{n-1}, and give two conjectures. In the second part, we compute the minimum rank and the minimum size of zero-forcing sets of a bud, and show that its maximum nullity is equal to the minimum size of its zero-forcing sets.

Chapter 2

Preliminaries

In this Chapter, we define notations and operations for graphs and matrices which we will use in this thesis.

2.1 Graphs

We consider simple and connected graphs in this thesis. For a graph G, we use $E(G)$ as its edge set and $V(G)$ as its vertex set, usually $V(G) = [n] = \{1, 2, ..., n\}$. The following table defines these three graphs K_n, P_n, C_n with vertex set $[n]$.

Graph	Notation	Edge set
Complete graph	K_n	$\{ij \| 1 \leq i < j \leq n\}$
Path	P_n	$\{i(i+1) \| 1 \leq i \leq n-1\}$
Cycle	C_n	$\{i(i+1) \| 1 \leq i \leq n-1\} \cup \{1n\}$

Let $x, y \in V(G)$. We use the following graph operations:

1. $x \sim y$ means that x is a neighbor of y.

2. $G_1(x)$ denotes the set of the neighbors of vertex $x \in G$.

3. $G - x$ denotes the induced subgraph of G with vertex set $V(G) - \{x\}$.

4. $|G|$ denotes the order of graph G.

2.2 Matrices

We use the following notation with an $n \times n$ matrix A, a column vector $x \in \mathbb{R}^n$, and subsets α, β for $\mathbb{N}$.

1. $\{e_1, e_2, ..., e_n\}$ is the standard basis of $\mathbb{R}^n$.

2. supp(x):= $\{i \in \mathbb{N} |$ the i-th entry of x is not zero $\}$.

3. $C_i(A)$ is the i-th column of A.

4. $A(\alpha|\beta)$ means the submatrix formed by deleting rows in α and columns in β. $A(\alpha) = A(\alpha|\alpha)$.

5. $A[\alpha|\beta]$ means the submatrix formed by rows in α and columns in β. $A[\alpha] = A[\alpha|\alpha]$.

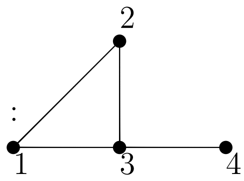
2.3 Matrices associated with graph G

Recall that for an $n \times n$ real symmetric matrix A, $\Gamma(A)$ represents the graph G such that $ij \in E(\Gamma(A))$ if and only if the ij-th entry of A is not zero. The matrix A is said to be associated with G if $\Gamma(A) = G$.

Example 2.1. The 4×4 matrix A is associated with $\Gamma(A)$.

$$A = \begin{bmatrix} 1 & 1/5 & -1 & 0 \\ 1/5 & 0 & 1 & 0 \\ -1 & 1 & -4 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

$\Gamma(A)$:



Note that the diagonal entries do not need to be 0.

Chapter 3

Known Results

Known theorems and propositions are introduced in this chapter. The relation of minimum rank between a graph and its subgraph is investigated. Also the relation between the minimum size of zero-forcing set and the maximum nullity of a graph are introduced here.

Lemma 3.1. If H is an induced subgraph of G, then we have $m(H) \leq m(G)$.

Proof. For any matrix A with $\Gamma(A) = G$, the submatrix $A[V(H)] = A[V(H)|V(H)]$ is associated with H. Thus rank$(A[V(H)])$ $\leq$rank(A). This implies $m(H) \leq m(G)$.

□

Theorem 3.2. [4, Theorem 2.8] Let A be an $n \times n$ real symmetric matrix. Then the following (i)-(ii) are equivalent.

(i) rank$(A + D) \geq n - 1$ for any diagonal matrix D.

(ii) $\Gamma(A) = P_n$. □

Example 3.3. The following matrix P satisfies $\Gamma(P) = P_n$, and $\operatorname{rank}(P) = n - 2$.

$$P = \begin{bmatrix} 1 & 1 & & & 0 \\ 1 & 2 & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & 2 & 1 \\ 0 & & & 1 & 1 \end{bmatrix}$$

Theorem 3.2 and Example 3.3 show that P_n is the unique graph with minimum rank $n - 1$ among all graphs of order n. Now we can determine the minimum rank of graphs with order n, which has an induced subgraph P_{n-1}.

Lemma 3.4. If a graph G of order n is not a path and contains an induced subgraph P_{n-1}, then $m(G) = n - 2$.

Proof. Since P_{n-1} is an induced subgraph of G, by Lemma 3.1 we have $m(G) \geq m(P_{n-1}) = n - 2$. From Theorem 3.2, because G is not a path of orde n, we know that $m(G) \leq n - 2$. Thus $m(G) = n - 2$. □

Lemma 3.5. The minimum rank of C_n is $n - 2$.

Proof. It is immediately from Lemma 3.4. □

Example 3.6. The matrix $A_t = (a_{ij})$ defined below satisfies $\Gamma(A_t) = C_t$ with $\operatorname{rank}(A_t) = t - 2$.

$$a_{ij} = \begin{cases} 2, & \text{if } i = j \text{ and } i, j \notin \{1, t-1, t\}; \\ 1, & \text{if } i = j, i, j \in \{1, t-1\}; \\ t - 2, & \text{if } i = j = t; \\ 1, & \text{if } |i - j| = 1; \\ (-1)^{t-1}, & \text{if } (i, j) = (1, t) \text{ or } (i, j) = (t, 1); \\ 0, & \text{otherwise.} \end{cases}$$

$$A_t = \begin{bmatrix} 1 & 1 & & & & (-1)^{t-1} \\ 1 & 2 & 1 & & & \\ & 1 & \ddots & \ddots & & \\ & & \ddots & 2 & 1 & \\ & & & 1 & 1 & 1 \\ (-1)^{t-1} & & & & 1 & t-2 \end{bmatrix}$$

Proposition 3.7. For a cycle C_n, the set of any two adjacent vertices is a zero-forcing set.

Proof. These two adjacent vertices form a black path. Once we change the white neighbor of the endpoint of the black path, there forms a new black path. Finally, all vertices are all black. Thus the set of any two adjacent vertices is a zero-forcing set. □

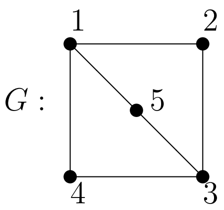
Here we introduce the relation between the minimum size $Z(G)$ of zero-forcing set and the maximum nullity $M(G)$ of a graph G.

Proposition 3.8. [1, Proposition 2.4] Let G be any graph. Then $M(G) \leq Z(G)$. □

Also there are some graphs with $Z(G) = M(G)$.

Proposition 3.9. [1, Proposition 4.3] If $|G| \leq 6$, then $M(G) = Z(G)$. □

Example 3.10. The following graph G with it's associated matrix A as an example for Proposition 3.9 has $m(G) = \text{rank}(A) = 2$, and $Z(G) = M(G) = 3$.



$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Theorem 3.11. [1, Proposition 4.10] For each of the following families of graphs, $Z(G) = M(G)$

1 Any graph G such that $|G| \leq 6$.

2 K_n, P_n, C_n.

3 Any tree T.

4 All the graphs listed in Table 1[1, Page 1630]. $\square$

Chapter 4

Main Results

There are two parts in this chapter. In the first part, we study the changing of minimum ranks when adding a vertex to a cycle C_{n-1}. In the second part, we compute minimum ranks and the minimum sizes of zero-forcing set of a class of graphs, called buds, and show that their maximum nullities is equal to the minimum sizes of zero-forcing set of buds.

4.1 Add a vertex to C_{n-1}

Let n be a vertex adding to the cycle C_{n-1} in this section.

Proposition 4.1. Suppose that there is exactly one edge which joins n to some vertex $x \in C_{n-1}$. Then the minimum rank of the new graph G' is $n-2$, and $M(G') = Z(G') = 2$.

Proof. Since G' contains an induced subgraph P_{n-1}, by deleting a neighbor of x, we know that $m(G') \geq m(P_{n-1}) = n-2$. On the other hand, G' is not a path of order n, so $m(G') < n-1$. Thus the minimum rank of G' is $n-2$ and the maximum nullity is 2. By Proposition 3.8, $M(G') \leq Z(G')$, we only need to claim that there is a zero-forcing set of G' with size 2. Let y be a neighbor of x in a clockwise direction.

Consider the set $\{x, y\}$ colored in black. From y in a clockwise direction, a white vertex can be changed to a black vertex at one time. When all vertices in C_{n-1} are all black, n is the exactly one white neighbor of x, then n can change to black. Thus the set $\{x, y\}$ is a zero-forcing set of G' with size 2. $\square$

Proposition 4.2. Let $x, y \in V(C_{n-1})$ and $x \sim y$. If there are exactly two edges incident on n such that $n \sim x$ and $n \sim y$, then the minimum rank of this new graph G'' is $n-2$, and $M(G'') = Z(G'') = 2$.

Proof. By deleting vertex y, we know that P_{n-1} is an induced subgraph of G''. Thus $m(G'') \geq m(P_{n-1}) = n-2$. Because G'' is not a path of order n, we have $m(G'') < n-1$. Therefore, the minimum rank of G'' is $n-2$, and the maximum nullity is 2. By Proposition 3.8, $M(G'') \leq Z(G'')$, we have to claim that there is a zero-forcing set of G'' with size 2. Consider the set $\{x, n\}$ colored in black, y is the only one white neighbor of n, so y can change to black. For other white vertices, using the same argument as the proof in Proposition 3.7 can color all white vertices to black. Thus the set $\{x, n\}$ is a zero-forcing set of G'' with size 2. $\square$

Let G_3 be a graph of order n. G_3 is obtained by adding a vertex n and two edges to a cycle C_{n-1}, and the vertex n is adjacent to two vertices which have distance 2. Before we discuss the situations of G_3, we define four types $n \times n$ matrices W, X, Y, Z as follows. According to the result of n mod 4, these matrices can be associated with G_3.

1. When $n = 4k+1, k \in \mathbb{N}$,
$$w_{ij} = \begin{cases} 1, & \text{if one of } i, j \text{ is } 1 \text{, and the other is } n-1 \text{ or } n \text{ ;} \\ 1, & \text{if } (i,j) = (3,n) \text{ or } (i,j) = (n,3) \text{ ;} \\ 1, & \text{if } |i-j| = 1, \forall i,j < n \text{ ;} \\ 0, & \text{otherwise.} \end{cases}$$

$$W=\begin{bmatrix} 0 & 1 & & & 1 & 1 \\ 1 & 0 & 1 & & & 0 \\ & 1 & \ddots & \ddots & & 1 \\ & & \ddots & \ddots & 1 & \\ 1 & & & 1 & 0 & 0 \\ 1 & 0 & 1 & & 0 & 0 \end{bmatrix}.$$

2. When $n=4k+2, k\in\mathbb{N}$,

$$x_{ij}=\begin{cases} 1, & \text{if one of } i,j \text{ is 1, and the other is } n-1 \text{ or } n \text{ ;} \\ 1, & \text{if } (i,j)=(3,n) \text{ or } (i,j)=(n,3) \text{ ;} \\ 1, & \text{if } |i-j|=1, \forall i,j<n-1 \text{ ;} \\ 1, & \text{if } i=j, i,j\in\{1,n-2,n-1\} \text{ ;} \\ -1, & (i,j)=(n-2,n-1) \text{ or } (i,j)=(n-1,n-2) \text{ ;} \\ 0, & \text{otherwise.} \end{cases}$$

$$X=\begin{bmatrix} 1 & 1 & & & & 1 & 1 \\ 1 & 0 & 1 & & & & 0 \\ & 1 & \ddots & \ddots & & & 1 \\ & & \ddots & 0 & 1 & & \\ & & & 1 & 1 & -1 & \\ 1 & & & & -1 & 1 & 0 \\ 1 & 0 & 1 & & & 0 & 0 \end{bmatrix}.$$

3. When $n = 4k+3, k \in \mathbb{N}$,

$$y_{ij} = \begin{cases} 1, & \text{if one of } i,j \text{ is } 1, \text{ and the other is } n-1 \text{ or } n \ ; \\ 1, & \text{if } (i,j) = (3,n) \text{ or } (i,j) = (n,3) \ ; \\ 1, & \text{if } |i-j| = 1, \forall i,j < n-1 \ ; \\ -1, & (i,j) = (n-2,n-1) \text{ or } (i,j) = (n-1,n-2) \ ; \\ 0, & \text{otherwise.} \end{cases}$$

$$Y = \begin{bmatrix} 0 & 1 & & & & 1 & 1 \\ 1 & & 1 & & & & 0 \\ & 1 & \ddots & \ddots & & & 1 \\ & & \ddots & \ddots & 1 & & \\ & & & 1 & & -1 & \\ 1 & & & & -1 & 0 & 0 \\ 1 & 0 & 1 & & & 0 & 0 \end{bmatrix}.$$

4. When $n = 4k+4, k \in \mathbb{N}$,

$$z_{ij} = \begin{cases} 1, & \text{if one of } i,j \text{ is } 1, \text{ and the other is } n-1 \text{ or } n \ ; \\ 1, & \text{if } (i,j) = (3,n) \text{ or } (i,j) = (n,3) \ ; \\ 1, & \text{if } |i-j| = 1, \forall i,j < n-1 \text{ except } i+j = 9 \ ; \\ 1, & \text{if } i = j, i,j \in \{1,4,n-1\} \ ; \\ -1, & (i,j) = (n-2,n-1) \text{ or } (i,j) = (n-1,n-2) \ ; \\ -1, & (i,j) = (4,5) \text{ or } (i,j) = (5,4) \ ; \\ 0, & \text{otherwise.} \end{cases}$$

$$
Z = \begin{bmatrix}
1 & 1 & & & & & & & 1 & 1 \\
1 & 0 & 1 & & & & & & & 0 \\
 & 1 & 0 & 1 & & & & & & 1 \\
 & & 1 & 1 & -1 & & & & & \\
 & & & -1 & 0 & 1 & & & & \\
 & & & & 1 & \ddots & \ddots & & & \\
 & & & & & \ddots & & 1 & & \\
 & & & & & & 1 & 0 & -1 & \\
1 & & & & & & & -1 & 1 & 0 \\
1 & 0 & 1 & & & & & & 0 & 0
\end{bmatrix}.
$$

Lemma 4.3. The rank of the above-mentioned four $n \times n$ matrices W, X, Y, Z are at most $n-3$.

Proof. For each W, X, Y, Z, express its 3-th, $(n-1)$-th and n-th columns as linear combination of other columns.

(1) For W,

$$
\begin{aligned}
C_3(W) =& e_2 + e_4 + e_n \\
=& e_2 + e_4 + e_n + \sum_{i=3}^{2k} e_{2i} - \sum_{i=3}^{2k} e_{2i} \\
=& e_2 + e_{n-1} + e_n + \sum_{i=2}^{2k-1} e_{2i} - \sum_{i=3}^{2k} e_{2i} \\
=& C_1(w) + \sum_{i=1}^{k-1} C_{4i+1}(W) - \sum_{i=1}^{k-1} C_{4i+3}(W),
\end{aligned}
$$

$$
\begin{aligned}
C_{n-1}(W) =& e_1 + e_{n-2} \\
=& e_1 + e_{4k-1} + \sum_{i=1}^{2k-2} e_{2i+1} - \sum_{i=1}^{2k-2} e_{2i+1} \\
=& \sum_{i=0}^{2k-1} e_{2i+1} - \sum_{i=1}^{2k-2} e_{2i+1} \\
=& \sum_{i=0}^{k-1} C_{4i+2}(W) - \sum_{i=1}^{k-1} C_{4i}(W),
\end{aligned}
$$

and

$$
C_n(W) = e_1 + e_3 = C_2(W).
$$

(2) For X,

$$
\begin{aligned}
C_3(X) =& e_2 + e_4 + e_n \\
=& e_2 + e_4 + e_n + \sum_{i=5}^{n-4} e_i - \sum_{i=5}^{n-4} e_i \\
=& e_1 + e_2 + e_{n-1} + e_n - e_1 - e_3 - e_{n-1} + \sum_{i=3}^{4k-2} e_i - \sum_{i=5}^{4k-2} e_i \\
=& C_1(X) - C_2(X) + \sum_{i=1}^{k-1} (C_{4i}(X) + C_{4i+1}(X)) - \\
& \sum_{i=1}^{k-1} (C_{4i+2}(X) + C_{4i+3}(X)) + C_{n-2}(X),
\end{aligned}
$$

$$
\begin{aligned}
C_{n-1}(X) =& e_1 - e_{n-2} + e_{n-1} \\
=& e_1 + \sum_{i=1}^{2k-1} e_{2i+1} - \sum_{i=1}^{2k-1} e_{2i+1} - e_{4k} + e_{n-1} \\
=& \sum_{i=0}^{2k-1} e_{2i+1} - \sum_{i=1}^{2k-2} e_{2i+1} - e_{4k-1} - e_{4k} + e_{n-1} \\
=& \sum_{i=0}^{k-1} C_{4i+2}(X) - \sum_{i=1}^{k-1} C_{4i}(X) - C_{4k}(X),
\end{aligned}
$$

and

$$
C_n(X) = e_1 + e_3 = C_2(X).
$$

(3) For Y,

$$\begin{aligned}
C_3(Y) =& e_2 + e_4 + e_n \\
=& e_2 + e_4 + e_n + \sum_{i=3}^{2k+1} e_{2i} - \sum_{i=3}^{2k+1} e_{2i} \\
=& e_2 + e_{n-1} + e_n + \sum_{i=2}^{2k} e_{2i} - \sum_{i=3}^{2k+1} e_{2i} \\
=& C_1(Y) + \sum_{i=1}^{k-1} C_{4i+1}(Y) - \sum_{i=1}^{k-1} C_{4i+3}(Y) + e_{4k} - e_{4k+2} \\
=& C_1(Y) + \sum_{i=1}^{k-1} C_{4i+1}(Y) - \sum_{i=1}^{k-1} C_{4i+3}(Y) + C_{n-2}(Y),
\end{aligned}$$

$$\begin{aligned}
C_{n-1}(Y) =& e_1 - e_{n-2} \\
=& e_1 - e_{4k+1} + \sum_{i=1}^{2k-1} e_{2i+1} - \sum_{i=1}^{2k-1} e_{2i+1} \\
=& \sum_{i=0}^{2k-1} e_{2i+1} - \sum_{i=1}^{2k} e_{2i+1} \\
=& \sum_{i=0}^{k-1} C_{4i+2}(Y) - \sum_{i=1}^{k} C_{4i}(Y),
\end{aligned}$$

and

$$C_n(Y) = e_1 + e_3 = C_2(Y).$$

(4) For Z,

$$\begin{aligned}
C_3(Z) =& e_2 + e_4 + e_n \\
=& e_2 + e_4 + e_n + \sum_{i=0}^{2k+1} e_{2i+1} - \sum_{i=0}^{2k+1} e_{2i+1} \\
=& e_1 + e_2 + e_{n-1} + e_n - e_1 - e_3 + e_3 + e_4 - e_5 + \sum_{i=2}^{2k} e_{2i+1} - \sum_{i=3}^{2k+1} e_{2i+1} \\
=& C_1(Z) - C_2(Z) + C_4(Z) + \sum_{i=2}^{k} C_{4i-2}(Z) - \sum_{i=2}^{k} C_{4i}(Z) + e_{4k+1} - e_{4k+3} \\
=& C_1(Z) - C_2(Z) + C_4(Z) + \sum_{i=2}^{k} C_{4i-2}(Z) - \sum_{i=2}^{k} C_{4i}(Z) + C_{n-2}(Z),
\end{aligned}$$

$$\begin{aligned}
C_{n-1}(Z) =& e_1 - e_{n-2} + e_{n-1} \\
=& e_1 - e_{4k+2} + e_{4k+3} + \sum_{i=3}^{4k+1} e_i - \sum_{i=3}^{4k+1} e_i \\
=& e_1 + e_3 - e_3 - e_4 + e_5 + e_4 - e_6 + \sum_{i=6}^{4k+1} e_i - e_5 - e_7 - \\
& \sum_{i=8}^{4k-1} e_i - e_{4k} - e_{4k+2} - e_{4k+1} + e_{4k+3} \\
=& C_2(Z) - C_4(Z) - C_5(Z) + \sum_{i=1}^{k-1} (C_{4i+3}(Z) + C_{4i+4}(Z)) - \\
& \sum_{i=1}^{k-1} (C_{4i+2}(Z) + C_{4i+5}(Z)) - C_{n-2}(Z),
\end{aligned}$$

and

$$C_n(Z) = e_1 + e_3 = C_2(Z).$$

□

Theorem 4.4. The minimum rank of the graph G_3 is $n-3$, and $M(G_3) = Z(G_3) = 3$.

Proof. Let $x, y, z \in V(C_{n-1})$ with $x \sim n$ and $y \sim n$. Since C_{n-1} is an induced subgraph of G_3, by Lemma 3.1 we have that $m(G_3) \geq m(C_{n-1}) = n-3$. These four types matrices W, X, Y, Z can be associated with G_3. From Lemma 4.3 we know that the rank of W, X, Y, Z are less than or equal to $n-3$. Therefore, we have $m(G_3) \leq n-3$. Thus the minimum rank of G_3 is $n-3$, and the maximum nullity is 3. By Proposition 3.8, $M(G_3) \leq Z(G_3)$, so we have to claim that there is a zero-forcing set of G_3 with size 3. Consider the set $\{x, z, n\}$ colored in black. y is the only one white neighbor of n, so y can change to black. For other white vertices, using the same argument as the proof in Proposition 3.7 can color all white vertices to black. Thus the set $\{x, z, n\}$ is a zero-forcing set of G_3 with size 3. □

Conjecture 4.5. Let $x, y \in V(C_{n-1})$ with $x \nsim y$. If there are exactly two edges incident on n such that $n \sim x$ and $n \sim y$, then the minimum rank of this new graph is $n-3$, and the maximum nullity and the minimum size of zero-forcing set are equal to 3.

Lemma 4.6. For all $n \in \mathbb{N}$, let A_n be the $n \times n$ symmetric matrix associated with a cycle defined as the matrix in Example 3.6. Thus for any subset $S \subseteq [n]$ with $|S| > 2$, there exists a vector $u \in \mathbb{R}^n$ such that $\text{supp}(u) \subseteq [\max(S) - 1]$ and $\text{supp}(A_n u) = S$.

Proof. For integers $1 \leq i < j \leq n$, define

$$\begin{aligned} b_i =& (-1)^0 e_i + (-1)^1 e_{i-1} + \cdots + (-1)^{i-1} e_1, \text{ and} \\ C_{ij} =& (-1)^0 b_i + (-1)^1 b_{i+1} + \cdots + (-1)^{j-i-1} b_{j-1}. \end{aligned}$$

Then we have

$$A_n b_i = \begin{cases} e_i + e_{i+1} + (-1)^{n-1} e - n, & i = [n-2]; \\ 0, & i = n-1; \\ (-1)^{n-1} e_1 + e_{n-1} + (n-2) e_n, & i = n. \end{cases}$$

$$A_n C_{ij} = e_i + (-1)^{j-i-1} e_j + (-1)^{n-i} (j-i) e_n.$$

Now suppose $S = \{t_1, t_2, ..., t_k\} \subseteq [n], k \geq 3$, and $t_1 < t_2 < \cdots < t_k$.

Case 1: $t_k = n$: Choose $u = C_{t_1,t_2} + C_{t_1,t_3} + \cdots + C_{t_1,t_{k-1}}$.

Then

$$\begin{aligned} A_n u =& \sum_{i=1}^{k-1} A_n C_{t_1 t_j} \\ =& \sum_{j=2}^{k-1} [e_{t_1} + (-1)^{t_j - t_1 - 1} e_{t_j} + (-1)^{n-t_1} (t_j - t_1) e_n] \\ =& (k-2) e_{t_1} + \sum_{j=2}^{k-1} (-1)^{t_j - t_1 - 1} e_{t_j} + (-1)^{n-t_1} (\sum_{j=2}^{k-1} j - (k-2) t_1) e_n. \end{aligned}$$

Thus $\mathrm{supp}(A_n u) = S$ and $\mathrm{supp}(u) \subseteq [\max(S) - 1]$.

Case 2: $t_k \neq n$: Let $a = \sum_{i=3}^{k} t_i - (k-2)t_1$, $d = lcm((t_2 - t_1), a)$.

Choose

$$u = \frac{d}{t_2 - t_1} C_{t_1,t_2} - \frac{d}{a}(\sum_{i=3}^{k} C_{t_1,t_i}).$$

Then

$$\begin{aligned} A_n u =& \frac{d}{t_2 - t_1} A_n C_{t_1,t_2} - \frac{d}{a}(\sum_{i=3}^{k} A_n C_{t_1,t_i}) \\ =& \frac{d}{t_2 - t_1} e_{t_1} + \frac{d}{t_2 - t_1}(-1)^{t_2 - t_1 - 1} e_{t_2} + d(-1)^{n-t_1} e_n - \frac{d}{a}(k-2)e_{t_1} - \\ & \frac{d}{a}[\sum_{i=3}^{k}(-1)^{t_i - t_1 - 1} e_{t_i}] + d(-1)^{n-t_1} e_n \\ =& \frac{ad - d(k-2)(t_2 - t_1)}{a(t_2 - t_1)} e_{t_1} + \frac{d}{t_2 - t_1}(-1)^{t_2 - t_1 - 1} e_{t_2} - \\ & \frac{d}{a}\sum_{i=3}^{k}(-1)^{t_i - t_1 - 1} e_{t_i}. \end{aligned}$$

To check $ad - d(k-2)(t_2 - t_1) > 0$, we claim $a - (k-2)(t_2 - t_1) > 0$.

$$\begin{aligned} a - (k-2)(t_2 - t_1) =& \sum_{3}^{k} t_i - (k-2)t_1 - (k-2)(t_2 - t_1) \\ =& \sum_{3}^{k} t_i - (k-2)t_2 \\ >& (k-2)t_3 - (k-2)t_2 > 0. \end{aligned}$$

Thus $\mathrm{supp}(A_n u) = S$ and $\mathrm{supp}(u) \subseteq [\max(S) - 1]$. □

Theorem 4.7. If G is a graph of order n which are obtained by adding a vertex n and at least three edges to a cycle C_{n-1}, then the minimum rank of G is $n - 3$, and $M(G) = Z(G) = 3$.

Proof. Let $S = G_1(V) = [t_1, t_2, ..., t_k]$, where $t_i \in [n-1], k \geq 3$; and A_{n-1} be the $(n-1) \times (n-1)$ matrix defined as the matrix in Example 3.6. By Lemma 4.6, we know that there exists a vector $u \in \mathbb{R}^{n-1}$ such that $\operatorname{supp}(A_{n-1}u) = S$. Thus the following matrix B satisfies $\operatorname{rank}(B) = n-3$ and $\Gamma(B) = G$.

$$B = \begin{bmatrix} A_{n-1} & A_{n-1}u \\ u^T A_{n-1} & u^T A_{n-1} u \end{bmatrix}_{n \times n}$$

This implies that the maximum nullity is 3. By proposition 3.8, $M(G) \leq Z(G)$, so we have to claim that there is a zero-forcing set of G with size 3. Let $x, y \in [n-1]$ and $x \sim y$. Consider the set $\{x, y, n\}$ colored in black. For other white vertices, we can color all white vertices to black by the same argument as the proof in Proposition 3.7. Thus the set $\{x, y, n\}$ is a zero-forcing set of G with size 3. $\square$

Conjecture 4.8. If G is a graph obtained by adding a vertex and some edges to a cycle C_{n-1}, then the maximum nullity of G is equal to the minimum size of a zero-forcing set of G.

4.2 Buds

Here we use notation $[i, j]$ to mean $\{i, i+1, \cdots, j-1, j\}$.

Definition 4.9. For integers $m < n$, let $B_{n,m}$ be a class of graphs G with vertex set $V(G) = [n]$ satisfying the following axioms:

1. The subgraph of G induced on $[n-m]$ is a cycle C_{n-m}, and the subgraph induced on $[n] \setminus [n-m]$ has no edge.
2. Let $1 = t_0 < t_1 < t_2 < \cdots < t_m = n-m+1$, and $t_j - t_{j-1} > 2$, for all $j \in [m]$. Let $S_i = G_1(n-m+i)$, where $i \in [m]$. Then $|S_i| \geq 3$ and $S_i \subseteq [t_{i-1}, t_i - 1]$.

The graph $G \in B_{n,m}$ is called a bud based on $[n-m]$.

Theorem 4.10. If $G \in B_{n,m}$, then $m(G) = n - m - 2$.

Proof. Since G is in $B_{n,m}$, G has an induced subgraph C_{n-m}. By Lemma 3.1, we have $m(G) \geq m(C_{n-m}) = n-m-2$. Now we claim that there exists a symmetric matrix B associated with G, and the rank of B is $n-m-2$. Let $1 = t_0 < t_1 < t_2 < \cdots < t_m = n-m+1$, and $t_j - t_{j-1} > 2$, for all $j \in [n-m]$. Let $S_i = G_1(n-m+i)$, where $i \in [m]$. Then $|S_i| \geq 3$ and $S_i \subseteq [t_{i-1}, t_i - 1]$. Let $A = (a_{ij})$ be the matrix associated with cycle C_{n-m} defined as the matrix in Example 3.6. By Lemma 4.6, for any $i \in [m]$, we can choose a vector $u_i \in \mathbb{R}^{n-m}$ such that

$$\operatorname{supp}(u_i) \subseteq [\max(S_i) - 1] \subseteq [t_i - 2] \text{ and } \operatorname{supp}(Au_i) = S_i \subseteq [t_{i-1}, t_i - 1]. \tag{4.1}$$

Notice that from the construction $u_j^T A u_i = 0$, if $j < i$, and indeed for $i \neq j$ since $u_j^T A u_i = u_i^T A u_j$. Hence

$$u_j^T A u_i = 0, \text{ for } i \neq j. \tag{4.2}$$

Now define the $n \times n$ symmetric matrix $B = (b_{ij})$ by:

$$b_{ij} = \begin{cases} a_{ij}, & \text{if } 1 \leq i \leq j \leq n-m; \\ u_{j-n+m}^T A u_{i-n+m}, & \text{if } n-m+1 \leq i \leq j \leq n; \\ e_i^T A u_{j-n+m}, & \text{if } 1 \leq i \leq n \text{ and } n-m+1 \leq j \leq n; \\ u_{i-n+m}^T A e_j, & \text{if } n-m+1 \leq i \leq n \text{ and } 1 \leq j \leq n-m. \end{cases} \tag{4.3}$$

Let $C = [u_1 u_2 \cdots u_m]$, then

$$B = \begin{bmatrix} A & AC \\ C^T A & C^T AC \end{bmatrix}.$$

From (4.1)(4.2)(4.3), we can easily check that $\Gamma(B) \in B_{n,m}$. For $n-m+1 \leq i \leq n$, the i-th column of $B[[n-m]|[n]]$ is a linear combination of columns of $B[[n-m]] = A$. Thus $\operatorname{rank}(B[[n-m]|[n]]) = \operatorname{rank}(B[[n-m]]) = \operatorname{rank}(A) = n-m-2$. For $n-m+1 \leq i \leq n$, the i-th row of B is a linear combination of the first $n-m$ rows of B. Hence $\text{rank}(B) = \text{rank}(B[[n-m]|[n]]) = n-m-2$.

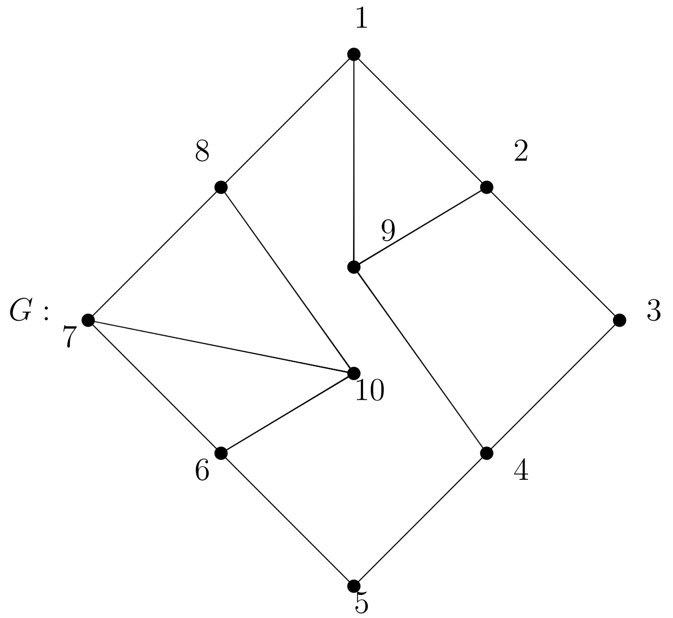
□

Corollary 4.11. If $G \in B_{n,m}$, then $M(G) = Z(G) = m+2$.

Proof. By Proposition 3.8, we know that $M(G) \leq Z(G)$. From Theorem 4.10, we have $m(G) = n-m-2$. Thus $M(G) = n-(n-m-2) = m+2$. Hence $Z(G) \geq m+2$. By coloring the set $S = [n] \setminus [n-m-2]$ in black, we can check that by using color-change rule, all vertices can color to black. Therefore, $Z(G) \leq n-(n-m-2) = m+2$. Hence $M(G) = Z(G) = m+2$.

□

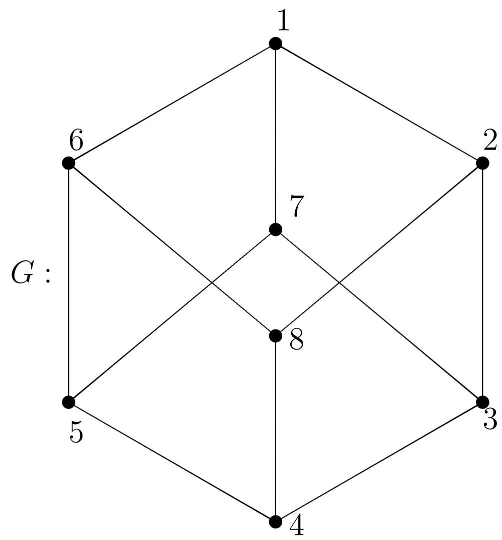
Example 4.12. Let G be a graph in $B_{10,2}$ base on [8] such that $G_1(9) = \{1, 2, 4\}$, $G_1(10) = \{6, 7, 8\}$ as in the following figure.



From Corollary 4.11, we know $M(G) = 4$ and then $m(G) = 6$. here we precisely give a matrix B associated with G and the rank of B is 6. Let $S_1 = \{1, 2, 4\}, S_2 = \{6, 7, 8\}$ and A_8 be the matrix defined in Example 3.6. Choose $u_1 = (0, 2, -1, 0, 0, 0, 0, 0)^T$, $u_2 = (-1, 1, -1, 1, -1, 1, 0, 0)^T$. Then the following matrix B is associated with G and $\text{rank}(B) = 6$.

$$B = \begin{bmatrix} A_8 & A_8u_1 & A_8u_2 \\ u_1^TA_8 & u_1^TA_8u_1 & 0 \\ u_2^TA_8 & 0 & u_2^TA_8u_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 \\ -1 & 1 & 0 & 0 & 0 & 0 & 1 & 6 & 0 & 1 \\ 2 & 3 & 0 & -1 & 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \end{bmatrix}$$

Example 4.13. Let G be a graph of order 8 such that $G_1(7) = \{1, 3, 5\}$, $G_1(8) = \{2, 4, 6\}$ as in the following figure.



Here we precisely give a matrix B associated with G and the rank of B is 4. Let $S_1 = \{1, 3, 5\}, S_2 = \{2, 4, 6\}$ and A_6 be the matrix defined in Example 3.6. Choose $u_1 = (0, 1, -2, 1, 0, 0)^T$, $u_2 = (-2, 2, -1, 0, 0, 0)^T$. Then the following matrix B is associated with G and $\mathrm{rank}(B) = 4$.

$$B = \begin{bmatrix} A_6 & A_6 u_1 & A_6 u_2 \\ u_1^T A_6 & u_1^T A_6 u_1 & 0 \\ u_2^T A_6 & 0 & u_2^T A_6 u_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & -1 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 1 & 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 & 4 & 0 & 2 \\ 1 & 0 & -2 & 0 & 1 & 0 & 4 & 0 \\ 0 & 1 & 0 & -1 & 0 & 2 & 0 & 2 \end{bmatrix}$$

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