國立陽明交通大學
應用數學系
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National Yang Ming Chiao Tung University
Master Thesis

圖的點邊度和邊點度
On ve-degrees and ev-degrees in graphs

研究生：李家豪（Li, Jia-Hai）
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中華民國一一〇年六月
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經學位考試委員會審查通過，特此證明。

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李家豪 謹誌於

國立陽明交通大學應用數學系

中華民國一百一十年六月

圖的點邊度和邊點度

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摘要

設 G 為簡單無向圖。一個點 u 的點邊度為 u 的所有鄰居所連出之邊的總數，表示為 $\mathrm{deg_{ve}(u)}$。一個邊 e 的邊點度為 e 的兩頂點及其所連出之點的總數，表示為 $\mathrm{deg_{ev}(e)}$。如果一個圖的所有點 (或邊) 都具有相同的度數 k(或點邊度 k、邊點度 k)，我們稱它為 k-正則 (或 k-點邊正則、k-邊點正則)；如果一個圖的所有邊都具有相同的邊點度數 k，我們稱它為 k-邊點正則。本論文主要討論點邊度與邊點度在一些圖上所具有的性質。我們有以下的結果。

- 設 η_e 表示包含邊 e 的三角形個數。若圖 G 裡的邊 e 有 η_e 為一大於 0 的常數 c，且 t 為某個自然數。則 G 是 t-邊點正則若且為若 $t+c$ 是偶數且 G 是 $\frac{t+c}{2}$-正則。
- 若 G 是一個直徑為 2、邊數為 m 且沒有五邊形的圖，則 G 是 m-點邊正則。
- 若 G 是一個參數為 n, k, λ, μ 的強正則圖，則 $k\lambda$ 是偶數且 G 是 $(2k-\lambda)$-邊點正則和 $(k^2 - \frac{k\lambda}{2})$-點邊正則。
- 設 G 是沒有三角形、無孤立點、且點數為 n 的圖。若對於 G 的每個邊 e 我們有 $\mathrm{deg_{ev}}(e) \leq \lfloor \frac{n}{2} \rfloor$，則 G 的補圖 $\overline{G}$ 是一個漢彌爾頓圖。
- 我們整理一些未解的問題做為未來研究的方向。

關鍵字: 點邊度, 邊點度, 點邊正則, 邊點正則, 正則圖, 強正則圖, 漢彌爾頓圖

On ve-degrees and ev-degrees in graphs

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abstract

Let $G = (V, E)$ be a simple undirected graph with vertex set V and edge set E. The **ve-degree** of a vertex u in G is the number of edges, denoted by $\deg_{\mathrm{ve}}(u)$, that are incident with a neighbour of u. The **ev-degree** of an edge e is the number of vertices, denoted by $\deg_{\mathrm{ev}}(e)$, that are incident with an endpoint of e. We call a graph **k-regular** (resp. **k-ve-regular**, **k-ev-regular**) if all of its vertices (edges) have the same degree (resp. ve-degree, ev-degree) k. The thesis mainly discusses the properties of ve-degrees and ev-degrees of graphs. We have the following results.

- Let η_e denote the number of triangles containing edge e. Assume that $\eta_e = c > 0$ is a scalar for $e \in E(G)$, and $t \in \mathbb{N}$. Then G is t-ev-regular if and only if $t + c$ is even and G is $\frac{t+c}{2}$-regular.
- If G has diameter 2, with size m and without pentagon, then G is m-ve-regular.
- If $G = (V, E)$ is a strongly regular graph with parameters n, k, λ, μ, then $k\lambda$ is even and G is $(2k - \lambda)$-ev-regular and $(k^2 - \frac{k\lambda}{2})$-ve-regular.
- Let G be a triangle-free graph of order n without isolated vertices. If $\deg_{\mathrm{ev}}(e) \leq \lfloor \frac{n}{2} \rfloor$ for every edge $e \in G$, then the complement $\overline{G}$ of G is Hamiltonian.
- We list some unresolved problems as the direction for future research.

Keywords: vertex-edge degree, edge-vertex degree, ve-regular, ev-regular, regular graphs, strongly regular graphs, Hamiltonian graphs

Contents

List of Figures

1 Introduction

Let $G = (V, E)$ be a simple graph with vertex set V and edge set E. We are curious about the number of edges incident with neighbours of a vertex. A vertex u **ve-dominates** every edge that is incident with a neighbour of u. The **ve-degree** of a vertex u, denoted by $\deg_{\text{ve}}(u)$, is the number of edges that are ve-dominated by u in G. On the other hand, an edge $e = uw$ **ev-dominates** the neighbours of u and the neighbours of w. The **ev-degree** of an edge e, denoted by $\deg_{\text{ev}}(e)$, is the number of vertices **ev-dominated** by e in G. Let k and m be positive integers. We call a graph k**-regular** (or **regular** for short) if all of its vertices have the same degree k; calling a graph m-**ve-regular** (or **ve-regular** for short) if all of its vertices share the same ve-degree m; and call a graph k-**ev-regular** (or **ev-regular** for short) if all of its edges are with ev-degree k. In Figure 1, the graph is both 8-ve-regular and 6-ev-regular. The label on a vertex is its ve-degree and the label on an edge is its ev-degree.

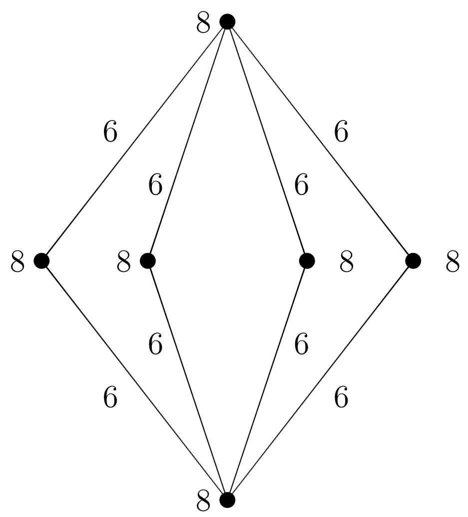


Figure 1. An 8-ve-regular and 6-ev-regular

A graph is **ve-irregular** (resp. **ev-irregular**) if the ve-degrees (resp. ev-degrees) of vertices (resp. edges) are all different. Recently, many people have studied ve-irregular graphs and ev-irregular graphs. In [2, 2017], Mustapha Chellali , Teresa W. Haynes , Stephen T. Hedetniemi ,Thomas M. Lewis provided many ve-irregular graphs, and then proved that if G is a graph with girth at least 5, then G is not ve-irregular. They also showed the nonexistent of a connected ev-irregular graph of order at least 3. In the paper [1, 2019], Batmend Horoldagva, Kinkar Ch. Das, Tsend-Ayush Selenge discussed the relationship between regular, ve-regular, and ev-regular. They further proved that there exists a ve-irregular graphs of every order greater than 7; and there exists an ev-irregular graph of every order greater than 5. They provided a special method to construct irregular graphs.

This paper discusses the relationship between ve-regular, ev-regular and regular properties of graphs that extends the previous study. Moreover, we prove that strongly regular graphs must be ve-regular and ev-regular.

In Section 3, we use the number of triangles containing a vertex (or an edge) to calculate the values of ve-degree and ev-degree. Let G be a connected graph. Assume that $\eta_e = c > 0$ is a scalar for $e \in E(G)$, and $t \in \mathbb{N}$. In Section 4, we prove that G is t-ev-regular if and only if G is $\frac{t+c}{2}$-regular. In Section 5, we prove that if G has diameter 2, size m and without pentagon, then G is m-ve-regular. In Section 6, we prove that if G is strongly regular, then G is ev-regular and ve-regular. In Section 7, we prove that if G is an undirected triangle-free graph of order n without isolated vertices and $\deg_{\mathrm{ev}}(e) \leq \lfloor \frac{n}{2} \rfloor$, for every edge $e \in G$, then the complement $\overline{G}$ of G is Hamiltonian. In Section 8, we list some unresolved problems as the direction for future research.

2 Notations

Our graph $G = (V, E)$ is always simple and undirected without loops. Sometimes, we will write $V(G)$ and $E(G)$ for V and E respectively to emphasize the underlying graph G. The number $|V(G)|$ is called the **order** of G and the number $|E(G)|$ is called the **size** of

G. For an edge $e = uv$, the following terminologies are used interchangeably: the vertices u and v are **endpoints** of the edge e; and u and v are **adjacent**; u is a **neighbour** of v; u is **incident** with e; e is **incident** with u. The **complement** of a graph G is a graph $\overline{G}$ on the same vertices such that two distinct vertices of $\overline{G}$ are adjacent if and only if they are not adjacent in G. A **walk** is a sequence of edges such that two consecutive edges share an endpoint. We call G a **Hamiltonian graph** if there exists a closed walk that visits every vertex of the graph exactly once and the starting point is the same as the ending point. The **distance** between two vertices u and v in a graph is the smallest number of edges in a walk that u and v are the endpoints of the first edge and the last edge respectively. We called a graph has diameter 2 if the maximum distance of two vertices in the graph is equal to 2. For $v \in V(G)$ and $i \in \mathbb{N}$, let $G_i(v)$ denote the set of vertices at distance i to v. Define $N(v) = G_1(v)$, the set of neighbours of v. The number $|N(v)|$ is called the **degree** of vertex v in G, denoted by $\deg(v)$.

A **cycle** of **size** k in a graph is a walk with k edges in which the endpoints of edges are all different except one in the first edge and in the last edge. Denote C_k as a cycle with size k. In the rest of this thesis, the cycle C_3 is called a **triangle**. A **subgraph** of G is a graph H whose vertex set and edge set satisfy $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. We refer to as H **in** G if H is a subgraph of G. The **girth** of G is the smallest size of a cycle in G. For a graph G, let $\eta(G)$ denote the number of triangles in G. For $v \in V(G)$ and $e \in E(G)$, let η_v (respectively η_e) denote the number of triangles containing vertex v (respectively edge e).

We call a graph G **strongly regular** with parameters n, k, λ, μ if G is k-regular of order n and every two adjacent vertices have λ common neighbours, and every two non-adjacent vertices have μ common neighbours. For convenience, we usually denote strongly regular graphs by $\mathrm{srg}(n, k, \lambda, \mu)$. Strongly regular graphs were introduced by [3, 1963].

3 Preliminary

We shall introduce some known results on ve-degree and ev-degree in this section. For completeness, we also include the proofs.

Lemma 3.1. *Let $G = (V, E)$ be a simple graph and $v \in V(G)$. Then*

$$\deg_{\mathrm{ve}}(v) = \sum_{u \in N(v)} \deg(u) - \eta_v.$$

Proof. By the definition of ve-degree, if all of neighbours of vertex v are not adjacent to each other, then we have $\deg_{\mathrm{ve}}(v) = \sum_{u \in N(v)} \deg(u)$. However, an edge uw has been counted twice in $\deg_{\mathrm{ve}}(v)$ as an incident edge of neighbours of v if and only if vertices u, v, w will form a triangle. Since there are η_v such edges uw, we have $\deg_{\mathrm{ve}}(v) = \sum_{u \in N(v)} \deg(u) - \eta_v$.

■

Lemma 3.2. *Let $G = (V, E)$ be a simple graph and $e = uv \in E(G)$. Then*

$$\deg_{\mathrm{ev}}(e) = \deg(u) + \deg(v) - \eta_e.$$

Proof. Similar to the proof of Lemma 3.1, for an edge $e = uv$, a vertex w has been counted twice in $\deg_{\mathrm{ev}}(e)$ if and only if vertices u, v, w form a triangle. Hence $\deg_{\mathrm{ev}}(e) = \deg(u) + \deg(v) - \eta_e$. ■

Lemma 3.3. *Let $G = (V, E)$ be a simple graph and $v \in V(G)$. Then*

$$\eta_v = \frac{1}{2} \sum_{e = vu \in E(G)} \eta_e.$$

Proof. For a vertex $v \in V(G)$, every triangle that contains v must have two edges adjacent to v. Hence we have this Lemma. ■

The following result in [2, 2017] is the relation of the ve-degree, ev-degree and the sum of squares of degree.

Theorem 3.4. *For any graph $G = (V, E)$, we have*

$$\sum_{v\in V} \deg_{\text{ve}}(v) = \sum_{e\in E} \deg_{\text{ev}}(e) = \big(\sum_{v\in V} \deg^2(v)\big) - 3\eta(G).$$

Proof. A vertex $v \in V$ ve-dominates edge e if and only if e ev-dominates v, hence $\sum_{v\in V} \deg_{\text{ve}}(v) = \sum_{e\in E} \deg_{\text{ev}}(e)$. By Lemma 3.2, for an edge $e = uv$, we have

$$\deg_{\text{ev}}(e) = \deg(u) + \deg(v) - \eta_e.$$

Therefore,

$$\sum_{e\in E} \deg_{\text{ev}}(e) = \sum_{e=uv\in E} \deg(u) + \deg(v) - \sum_{e\in E} \eta_e.$$

Observe that $\sum_{e\in E} \eta_e = 3\eta(G)$, since a triangle contain 3 edges. On the other hand, each vertex $w \in V$ will be counted $\deg(w)$ times in the sum $\sum_{e\in E} \deg(u) + \deg(v)$, hence

$$\sum_{e\in E} \deg(u) + \deg(v) = \sum_{v\in V} \deg^2(v).$$

Finally, we have

$$\sum_{v\in V} \deg_{\text{ve}}(v) = \sum_{e\in E} \deg_{\text{ev}}(e) = \big(\sum_{v\in V} \deg^2(v)\big) - 3\eta(G).$$

The proof is completed. ■

4 Ev-regular graphs

In this section, we are going to find some necessary conditions of ev-regular graphs. The following Lemma is a simple result of using ev-regular condition to count ve-degree.

Lemma 4.1. *If G is a t-ev-regular graph, then for every vertex $v \in V(G)$ we have*

$$\deg_{\text{ve}}(v) = \frac{\deg(v)t + \sum_{u\in N(v)}(\deg(u) - \deg(v))}{2}.$$

Proof. Fix a vertex $v \in G$. Then for any edge $e = vu$, we have $\eta_e = \deg(v) + \deg(u) - t$. By Lemma 3.3, we have

$$\eta_v = \frac{1}{2} \sum_{e=vu \in E(G)} \eta_e = \frac{\deg(v)(\deg(v) - t)}{2} + \frac{1}{2} \sum_{u \in N(v)} \deg(u).$$

By Lemma 3.1, we have

$$\deg_{\mathrm{ve}}(v) = -\eta_v + \sum_{u \in N(v)} \deg(u)$$

$$= \frac{\deg(v)(t - \deg(v))}{2} + \frac{1}{2} \sum_{u \in N(v)} \deg(u)$$

$$= \frac{\deg(v)t + \sum_{u \in N(v)} (\deg(u) - \deg(v))}{2}.$$

■

Corollary 4.2. *If G is k-regular graph and t-ev-regular graph, then kt is even and G is $\frac{kt}{2}$-ve-regular.*

Proof. Fix a vertex v in G. By Lemma 3.3, we have

$$\eta_v = \frac{1}{2} \sum_{e=vu \in E(G)} \eta_e$$

By Lemma 3.2, we have $\eta_e = \deg(v) + \deg(u) - t = 2k - t$. Then

$$\eta_v = \frac{1}{2} \sum_{e=vu \in E(G)} \eta_e = \frac{k(2k - t)}{2} = k^2 - \frac{kt}{2}.$$

By Lemma 3.1, we have

$$\deg_{\mathrm{ve}}(v) = \sum_{u \in N(v)} \deg(u) - \eta_v = k^2 - (k^2 - \frac{kt}{2}) = \frac{kt}{2}.$$

The proof is completed.

■

A bipartite graph is called (t, s)-**semiregular** if all of the degrees of the vertices on one side are t; and all of the degrees of the vertices on other side are s.

The following result is proved by Batmend Horoldagva, Kinkar Ch. Das, Tsend-Ayush Selenge [1, 2019].

Lemma 4.3. *Let G be a connected triangle-free graph and $k \in \mathbb{N}$. Then G is t-ev-regular if and only if*

(i) t is even and G is $\frac{t}{2}$-regular; or

(ii) G is bipartite and (i, j)-semiregular for some positive integers i, j satisfying $i+j = t$.

Proof. ($\Rightarrow$) Fix an edge $e = uv \in E(G)$. Since G is triangle-free t-ev-regular, we have $t = \deg_{\mathrm{ev}}(e) = \deg(u) + \deg(v)$. Therefore the neighbours of vertex u must have the same degree, $t - \deg(u)$. Moreover, every neighbour of v has degree $\deg(u)$. Since G is connected, we have that the degree of every vertex of G is either $\deg(u)$ or $t - \deg(u)$. If G contains an odd cycle, it follows that $\deg(u) = t - \deg(u)$. Then t is even and G is $\frac{t}{2}$-regular. If G does not contain an odd cycle, then G is bipartite. Furthermore, the vertices in one partite set have degree $i := \deg(u)$ and the other partite set have degree $j := t - \deg(u)$. Therefore G is (i, j)-semiregular bipartite satisfying $i + j = t$.

($\Leftarrow$) It is easy to see that triangle-free t-regular graphs are $2t$-ev-regular, and (i, j)-semiregular bipartite graphs are $(i + j)$-ev-regular.

■

The following is our generalization of lemma 4.3.

Corollary 4.4. *Let G be a connected graph. Assume that $\eta_e = c > 0$ is a scalar for every $e \in E(G)$, and $t \in \mathbb{N}$. Then the following (i)-(ii) are equivalent.*

(i) G is t-ev-regular;

(ii) $t+c$ is even and G is $\frac{t+c}{2}$-regular.

Proof. ((i) $\Rightarrow$ (ii)) Fix an edge $e = uv \in E(G)$. By Lemma 3.2, we have $t = \deg_{\text{ev}}(e) = \deg(u) + \deg(v) - c$. Since G is t-ev-regular and $\eta_e = c$ is scalar, the neighbours of vertex u must have the same degree, $t + c - \deg(u)$. Moreover, every neighbour of v has degree $\deg(u)$. Since G is has triangle, we have that $\deg(u) = t + c - \deg(u)$. So G is $\frac{t+c}{2}$-regular.

((ii) $\Rightarrow$ (i)) If G is $\frac{t+c}{2}$-regular and $\eta_e = c > 0$ is a scalar, then for any edge e, we have $\deg_{\text{ev}}(e) = \frac{t+c}{2} + \frac{t+c}{2} - c = t$. So G is t-ev-regular. ∎

To generalize Corollary 4.4 further, it is natural to ask if there is an ev-regular graph whose η_e is not a constant. The following example denies this question.

Example 4.5. Let G be the 6-ev-regular graph of order 6 illustrated in Figure 2. Observe that the η_e of the middle edge e has $\eta_e = 4$, and all of the other edges $e' \neq e$ has $\eta_{e'} = 2$.

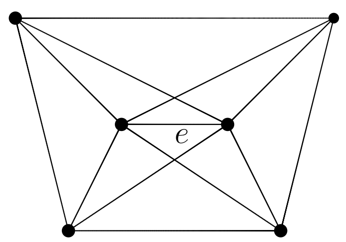


Figure 2. This is 6-ev-regular graph with different η_e.

5 Graphs with diameter 2

We shall study graphs G of diameter 2 in this section. Motivated by the definition of strongly regular graphs, the following definition is given.

Definition 5.1. For two vertices u, v at distance k and $i, j \in \mathbb{N}$, let $p^k_{ij}(u, v)$ denote the number of vertices at distance i to u and at distance j to v. Define $a_i(u, v) := p^i_{i1}(u, v)$.

Base on Definition 5.1, we obtain a formula for $\deg_{\mathrm{ve}}(v)$ on graphs with diameter 2.

Lemma 5.2. *If G has diameter 2 with m edges, then for vertex $v \in G$, we have*

$$\deg_{\mathrm{ve}}(v) = m - \sum_{w \in G_2(v)} \frac{a_2(v,w)}{2} = \sum_{u \in N(v)} \deg(u) - \frac{a_1(v,u)}{2}.$$

Proof. Since G has diameter 2, the number of edges not counted in the definition of $\deg_{\mathrm{ve}}(v)$ is $\sum_{w \in G_2(v)} \frac{a_2(v,w)}{2}$. We have the first equality. By Lemma 3.1, we have that $\deg_{\mathrm{ve}}(v) = \sum_{u \in N(v)} \deg(u) - \eta_v$. Since every triangle containing v is counted twice in $a_1(v,u)$, we have that $\eta_v = \sum_{u \in N(v)} \frac{a_1(v,u)}{2}$. The proof is completed. ∎

Similarly, we also obtain a formula for $\deg_{\mathrm{ev}}(e)$.

Lemma 5.3. *If G has diameter 2 of order n, then for each edge $e = uv \in E$ we have*

$$\deg_{\mathrm{ev}}(e) = n - p^1_{22}(u,v) = |N(u)| + |N(v)| - a_1(u,v).$$

Proof. Fix an edge $e = uv$. Since G has diameter 2, $p^1_{22}(u,v)$ is the number of vertices not counted in the definition of $\deg_{\mathrm{ev}}(e)$. Hence $\deg_{\mathrm{ev}}(e) = n - p^1_{22}(u,v)$. By lemma 3.2, we have $\deg_{\mathrm{ev}}(e) = \deg(u) + \deg(v) - \eta_e$. Since $\eta_e = a_1(u,v)$, we have

$$\deg(u) + \deg(v) - \eta_e = |N(u)| + |N(v)| - a_1(u,v).$$

The proof is completed. ∎

Finally, we have the following result.

Theorem 5.4. *If G has diameter 2, size m and without pentagon, then G is m-ve-regular.*

Proof. By Lemma 5.2 we have that $\deg_{\mathrm{ve}}(u) = m - \sum_{v \in G_2(u)} \frac{a_2(u,v)}{2}$. Since G has no pentagon, $\sum_{v \in G_2(u)} a_2(u,v) = 0$. The proof is completed. ∎

6 Strongly regular graphs

Among regular graphs, strongly regular graphs need stronger conditions and have good properties. This leads to many properties being necessary conditions for strongly regular, such as regular, ev-regular and ve-regular. The following is an example to help us understand strongly regular graphs better.

Example 6.1. The following is the well known Petersen graph. We can check that Petersen graph is 3-regular and any two adjacent vertices have no common neighbours; any two non-adjacent vertices have 1 common neighbour. So Petersen graph is $\mathrm{srg}(10, 3, 0, 1)$.

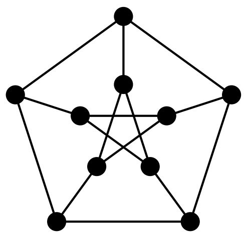


Figure 3. Petersen graph is $\mathrm{srg}(10, 3, 0, 1)$.

The following is a known result regarding the parameters of strongly regular graph.

Lemma 6.2. *The four parameters in an* $\mathrm{srg}(n, k, \lambda, \mu)$ *have the following relation:*

$$(n - k - 1)\mu = k(k - \lambda - 1).$$

Proof. If $\mu = 0$, then G is a disjoint union of complete graph. Moreover, each complete graph has order $k + 1$. Hence $k = \lambda + 1$, so the left and right sides of the equation are 0. The formula naturally holds.

If $\mu \neq 0$, then G is a connected graph with diameter 2. Choosing a vertex $w \in G$ arbitrarily, we can divide the vertices except w into two sets: $G_1(w)$ and $G_2(w)$. We will prove this formula by counting the number of edges which are incident with $G_1(w)$ and $G_2(w)$ in two different methods.

(i) For each vertex in $G_2(w)$, they have μ common neightbors with w, and these neightbors must in $G_1(w)$. Hence, for each vertex in $G_2(w)$, the number of edges which are incident with $G_2(w)$ and $G_1(w)$ is μ. Since G is k-regular, $|G_1(w)| = k$ and $|G_2(w)| = n - k - 1$. Hence the number of edges which are incident with $G_2(w)$ and $G_1(w)$ is $(n - k - 1)\mu$.

(ii) The degree of each vertex in $G_1(w)$ is k. Since these vertices and w are adjacent, they must have λ other neighbours in common with w. (These common neighbours must be in $G_1(w)$, too.) Hence, for each vertex in $G_1(w)$, there are $(k - \lambda - 1)$ edges are incident with vertices in $G_2(w)$. Therefore there are $k(k - \lambda - 1)$ edges are incident with $G_1(w)$ and $G_2(w)$.

Combining (i) and (ii), the proof is completed. ∎

The following is a known result.

Lemma 6.3. *If G is* $\mathrm{srg}(n, k, \lambda, \mu)$ *satisfying* $1 \le k < n - 1$*, then the complement* $\overline{G}$ *is*

$$\mathrm{srg}(n, n - k - 1, n - 2 - 2k + \mu, n - 2k + \lambda).$$

Proof. Assume $\overline{G}$ is $\mathrm{srg}(n', k', \lambda', \mu')$. In $\overline{G}$, we have $n' = n$ and $k' = n - k - 1$ obviously. We consider λ' and μ'.

If $\mu = 0$, then G is a disjoint union of complete graph. Moreover, each complete graph has order $k + 1$, hence $\lambda = k - 1$. $\overline{G}$ consists of several parts of order $k + 1$. In $\overline{G}$, any two vertices in same part are not adjacent; and any two vertices in different part are adjacent. For any two adjacent vertices, the number of vertices that are not adjacent to both vertices is $2(k + 1)$. Hence $\lambda' = n - 2(k + 1) = n - 2 - 2k + 0 = n - 2 - 2k + \mu$. For any two non-adjacent vertices, the number of common neighbours of these two vertices is $n - (k + 1)$. Hence $\mu' = n - k - 1 = n - 2k + (k - 1) = n - 2k + \lambda$.

If $\mu \neq 0$, then G is a connected graph with diameter 2. Choosing a vertex $w \in G$ arbitrarily, we can divide the vertices except w into two sets: $G_1(w)$ and $G_2(w)$. In $\overline{G}$, w

and vertices in $G_2(w)$ are adjacent; w and vertices in $G_1(w)$ are not adjacent. For a vertex $v \in G_2(w)$, the number of common neighbours of w and v in $\overline{G}$ equals to the number of vertices that are not adjacent to w and v in G. In G, vertices that are not adjacent to w and v must be in $G_2(w)$, where $|G_2(w)| = n - k - 1$. Since $\deg(v) = k$ and v is adjacent to μ vertices in $G_1(w)$, the number of vertices in $G_2(w)$ that are not adjacent to v is $n - k - 1 - (k - \mu - 1) = n - 2 - 2k + \mu$. So we have $\lambda' = n - 2 - 2k + \mu$.

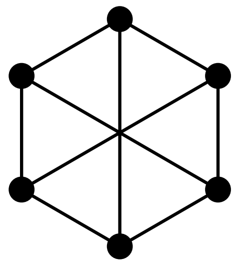
By similar method, for a vertex $v \in G_1(w)$, the number of common neighbours of w and v in $\overline{G}$ equals to the number of vertices that are not adjacent to w and v in G. In G, vertices that are not adjacent to w and v must be in $G_2(w)$, where $|G_2(w)| = n - k - 1$. Since $\deg(v) = k$ and v is adjacent to w and λ vertices in $G_1(w)$, the number of neighbours of v in $G_2(w)$ is $k - \lambda - 1$. Hence the number of vertices in $G_2(w)$ that are not adjacent to v is $n - k - 1 - (k - \lambda - 1) = n - 2k + \lambda$. So we have $\mu' = n - 2k + \lambda$.

■

The following is an example of verifying Lemma 6.3.

Example 6.4. Let G be the 3-regular graph of order 6 illustrated in Figure 4. Observe that any two adjacent vertices have no common neighbourin in G; any two non-adjacent vertices have 3 common neighbours in G. Hence G is $\mathrm{srg}(6, 3, 0, 3)$. And any two adjacent vertices have 1 common neighbour in $\overline{G}$; any two non-adjacent vertices have no common neighbour in $\overline{G}$. Hence $\overline{G}$ is $\mathrm{srg}(6, 2, 1, 0)$. So this is an example that the complement of a strongly regular graph is also a strongly regular graph.

G :



$\overline{G}$:

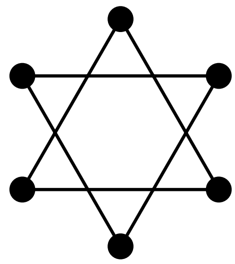


Figure 4. G is srg$(6,3,0,3)$, 6-ev-regular and 9-ve-regular; $\overline{G}$ is srg$(6,2,1,0)$, 3-ev-regular and 3-ve-regular.

Now we are ready to show strongly regular graphs are ve-regular and ev-regular. The following are some results we obtain.

Proposition 6.5. *If $G = (V, E)$ is* srg(n, k, λ, μ)*, then $k\lambda$ is even and G is $(2k - \lambda)$-ev-regular and $(k^2 - \frac{k\lambda}{2})$-ve-regular.*

Proof. For any edge $e = uv \in E(G)$, by Lemma 3.2, we have

$$\deg_{\text{ev}}(e) = \deg(u) + \deg(v) - \eta_e = 2k - \lambda.$$

For any vertex $v \in V(G)$, by Lemma 3.1, we have

$$\deg_{\text{ve}}(v) = \sum_{u \in N(v)} \deg(u) - \eta_v = k^2 - \eta_v = k^2 - \frac{k\lambda}{2},$$

since $\eta_v = \frac{1}{2}\sum_{e=vu \in E(G)} \eta_e$. ∎

Using Lemma 6.3, we have the following result.

Proposition 6.6. *If G is* srg(n, k, λ, μ) *satisfying $1 \leq k < n-1$, then $(n-k-1)(n-\mu)$ is even and the complement $\overline{G}$ is $(n-\mu)$-ev-regular and $((n-k-1)(n-\mu)/2)$-ve-regular.*

Proof. Assume the complement $\overline{G}$ of G is srg(n', k', λ', μ'). By Lemma 6.3, $\overline{G}$ is srg$(n, n-k-1, n-2-2k+\mu, n-2k+\lambda)$. Applying Proposition 6.5 with the following computations

$$\begin{aligned} 2k' - \lambda' &= 2(n-k-1) - (n-2-2k+\mu) \\ &= 2n - 2k - 2 - n + 2 + 2k - \mu \\ &= n - \mu; \end{aligned}$$

$$
\begin{aligned}
k'^2 - \frac{k'\lambda'}{2} &= (n-k-1)^2 - \frac{1}{2}(n-k-1)(n-2-2k+\mu) \\
&= (n-k-1)\frac{n-\mu}{2},
\end{aligned}
$$

the complement $\overline{G}$ of G is $(n-\mu)$-ev-regular and $((n-k-1)(n-\mu)/2)$-ve-regular. ■

In the following example, we show that G is regular graph and ev-regular can not imply the complement graph $\overline{G}$ of G is ev-regular.

Example 6.7. Let G be the 2-regular and 4-ev-regular graph illustrated in Figure 5. Observe that the complement $\overline{G}$ of G is 3-regular, and the η_e of the three vertical edges e in $\overline{G}$ has $\eta_e = 0$. But the other edges e' has $\eta_{e'} = 1$, so $\overline{G}$ is not ev-regular.

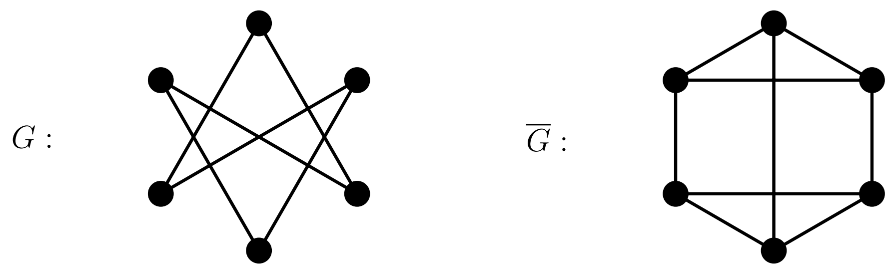


Figure 5. G is 2-regular and 4-ev-regular and 4-ve-regular; $\overline{G}$ is 3-regular and 8-ve-regular, but $\overline{G}$ is not ev-regular.

Now we remove some conditions of strongly regular graphs, and see whether graphs under these remaining conditions could still imply strongly regular graph. The following are three examples which are not strongly regular.

Example 6.8. Let G be the 4-ev-regular graph of order 4 illustrated in Figure 6. The complement $\overline{G}$ of G has only one edge, so $\overline{G}$ is 2-ev-regular. Observe that both G and $\overline{G}$ are not regular graph. So there exist graphs G and $\overline{G}$ that both are ev-regular but both graphs are not strongly regular.

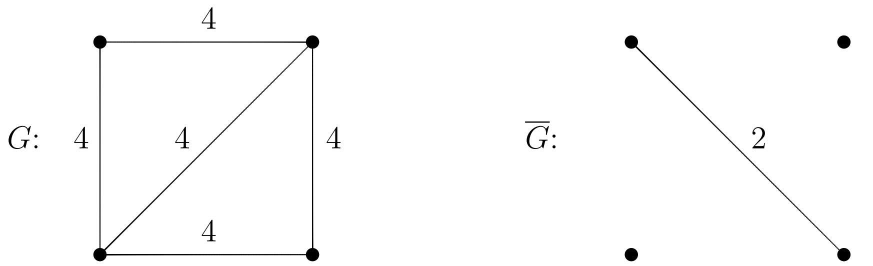


Figure 6. G is 4-ev-regular, and $\overline{G}$ is 2-ev-regular.

Example 6.9. Figure 5 gives an example that G is 4-ve-regular and $\overline{G}$ is 8-ve-regular. Since $\overline{G}$ is not ev-regular, $\overline{G}$ is not strongly regular. So there exist graphs G and $\overline{G}$ that are ve-regular, but both graphs are not strongly regular.

Example 6.10. Figure 7 gives a graph which is 3-regular, 6-ev-regular, 9-ve-regular and has diameter 2. Observe that the distance between a and b is 2, and so is a and c, but $p^2_{11}(a,b) = 2$; $p^2_{11}(a,c) = 1$. Hence G is not a strongly regular graph. So there exists a graph G which is regular, ev-regular and ve-regular with diameter 2, but G is not strongly regular.

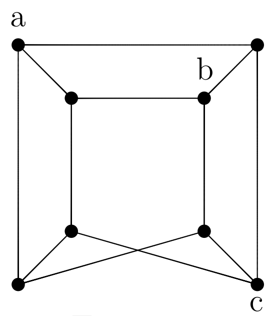


Figure 7.

7 Hamitonian graphs

In this section, we will discuss the application of ev-degree and ve-degree on the Hamiltonian graphs. First we introduce an old result about Hamiltonian property. The following is Dirac's Theorem [4, 1952].

Theorem 7.1. *Let G be an undirected simple graph of order $n \geq 3$. If* $\deg(u)+\deg(v) \geq n$ *for any two vertices $u, v \in V(G)$, then G is Hamiltonian.*

Since we can increase the degrees of vertices of a graph to a Hamiltonian graph, we thought of increasing the ev-degree of a graph G to obtain that a Hamiltonian graph. The following result is obtained by using Theorem 7.1.

Proposition 7.2. *Let G be a regular graph of order n. If* $\deg_{\mathrm{ev}}(e) = n$ *for some $e \in E(G)$, then G is Hamiltonian.*

Proof. Let G be k-regular. Choosing an edge $e = uv$ with $\deg_{\mathrm{ev}}(e) = n$, we have

$$n = \deg_{\mathrm{ev}}(e) = \deg(u) + \deg(v) - \eta_e = 2k - \eta_e,$$

implying $k \geq \frac{n}{2}$. By Theorem 7.1, G is Hamiltonian. ■

Remark 7.3. The complete bipartite graph $K_{i,j}$ of order n is $(i+j)$-ev-regular (n-ev-regular), and $K_{i,j}$ is regular (respectively Hamiltonian) only if $i = j$. Hence the Hamiltonicity of a general graph is not related to its minimum ev-degree.

Since a graph G of order n with $\deg_{\mathrm{ev}}(e) = n$ for all edge e may not be Hamiltonian, we turn to consider the complement of the graph.

Theorem 7.4. *Let G be a triangle-free graph of order n without isolated vertices. If* $\deg_{\mathrm{ev}}(e) \leq \lfloor \frac{n}{2} \rfloor$ *for every edge $e \in G$, then $\overline{G}$ is Hamiltonian.*

Proof. For any vertex $u \in V(G)$, there is an edge $e = uv \in E(G)$. Then

$$\lfloor \frac{n}{2} \rfloor \geq \deg_{\mathrm{ev}}(e) = \deg(u) + \deg(v).$$

Since $\deg(v) \geq 1$, we have$\lfloor \frac{n}{2} \rfloor - 1 \geq \deg(u)$. Hence $n \geq 4$, and for the same vertex $\overline{u} \in \overline{G}$, we have

$$\deg(\overline{u}) = n - \deg(u) - 1 \geq n - \lfloor \frac{n}{2} \rfloor \geq \frac{n}{2}.$$

By Theorem 7.1, $\overline{G}$ is Hamiltonian. ■

Now we conjecture if the Hamiltonian property is related to the ve-degree of the graph. The following Proposition is a simple result based on bipartite property.

Proposition 7.5. *If G is bipartite graph with order n, and there is an vertex v satisfying $\deg_{\text{ve}}(v) \geq \lceil \frac{n}{2} \rceil^2$, then n is even and G is Hamiltonian, moreover G is complete bipartite graph.*

Proof. We assume that G has i vertices in part A and j vertices in part B. Note that $i + j = n$ is the order of G, and the maximum ve-degree of G is ij. Since $ij \geq \lceil \frac{i+j}{2} \rceil^2$ only if $i = j = \frac{n}{2}$, we have that every vertex is adjacent to all of vertices in the other part. So G is complete bipartite graph with order $(\frac{n}{2}, \frac{n}{2})$. ■

The following example shows the bipartite assumption in Proposition 7.5 is necessary.

Example 7.6. Figure 8 gives a graph G which is 9-ve-regular of order 6. Observe that $9 \geq \lceil \frac{6}{2} \rceil^2 = 9$. Let u and v be the vertices whose degree is 5 in G. Observe that upper middle vertex and lower middle vertex have degree 2. If there exist Hamiltonian cycle in G, the adjacent edges of these two vertices must be included. But the four edges form a C_4, hence G is not Hamiltonian.

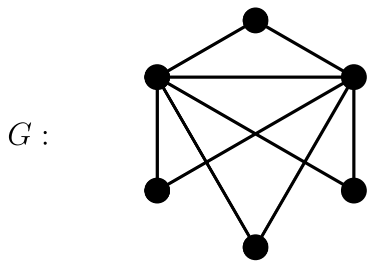


Figure 8. G is a non-Hamiltonian 9-ve-regular graph of order 6.

Moreover, we find a non-Hamiltonian graph whose minimum ve-degree is $\frac{3n^2}{8} - \frac{n}{2} + \frac{1}{8}$.

Example 7.7. Let G be obtained from a complete bipartite graph of bi-order $(i, i+1)$ by adding $\binom{i}{2}$ edges in the part of order i. Then G is a non-Hamiltonian $(i(i+1) + \binom{i}{2})$-ve-regular graph, where

$$i(i+1) + \binom{i}{2} = \frac{(3n-1)(n-1)}{8} = \frac{3n^2}{8} - \frac{n}{2} + \frac{1}{8} \approx \frac{3n^2}{8}.$$

By Example 7.7, we know that even if the minimum ve-degree of a graph G is $\frac{3n^2}{8} - \frac{n}{2} + \frac{1}{8}$, it can not imply G is Hamiltonian. Now we try to find the lower bound of ve-degree of a graph such that this graph satisfies Hamiltonian property. So there is the following conjecture.

Conjecture 7.8. If $\deg_{\mathrm{ve}}(v) > \frac{3n^2}{8} - \frac{n}{2} + \frac{1}{8}$ for every vertex $v \in G$, then G is Hamiltonian.

We have not found an example to deny this conjecture.

8 Open problems

In this section, we list some unresolved problems, which can be used as the direction of future research.

Problem 8.1. If G is regular and ve-regular, is $\overline{G}$ ve-regular?

We guess the answer to this problem is wrong, but we have not found a counterexample. Maybe in the future we can try to prove the positive direction.

Problem 8.2. Find the lower bound of ve-degree of a graph such that this graph satisfies Hamiltonian property.

Given an graph G of order n, we know that the maximum ve-degree number of G is $\binom{n}{2}$. In this condition, G is a complete graph and must be Hamiltonian obviously (if $n \geq 3$). If Conjecture 7.8 holds then the range of x is as follows :

$$\frac{3n^2}{8} - \frac{n}{2} + \frac{1}{8} < x \leq \frac{4n^2}{8} - \frac{n}{2} = \binom{n}{2}$$

Problem 8.3. If both graphs G and $\overline{G}$ are regular, ev-regular and ve-regular with diameter 2, is G strongly regular?

In Example 6.8 and Example 6.9, we know that a graph G that satisfies that G and $\overline{G}$ are ev-regular and ve-regular may not be strongly regular. So we guess at what strengthening conditions can make the hypothesis true. But we have not found counterexamples in our research. So we have not yet determined the answer to this problem.

Problem 8.4. Does the condition exist such that the graph satisfies the condition, regular, ve-regular, ev-regular and diameter 2 can be strongly regular?

We know that a strongly regular graph must satisfy the above conditions if it is connected. In Figure 7, we have a graph satisfying regular, ve-regular, ev-regular and diameter 2, but it is not strongly regular.

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