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強乘積圖的漢米爾頓性和強韌性

Hamiltonicity and Toughness of Strong Product Graphs

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強乘積圖的漢米爾頓性和強韌性

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摘　要

在本論文中，我們研究了強乘積圖的漢米爾頓性質。具體來說，在 n 的些許限制下，我們證明了 $P_n \boxtimes G$ 是漢米爾頓圖若且唯若圖 G 含有 $\{K_{1,1}, K_{1,2}, K_{1,3}, K_{1,4}\}$-因子，此文 P_n 和 $K_{1,t}$ 分別是點數爲 n 的路徑圖和點數爲 $t+1$ 的星圖。此外，我們探討了圖的強韌性與漢米爾頓迴圈的存在性之間的關係。特別地，我們證明了若一個圖 $P_n \boxtimes G$ 是 $\frac{3}{2}$-強韌的，那麼它是漢米爾頓圖。此外，我們還證明了如果 G 是一個包含 $\{K_{1,1}, K_{1,3}\}$-因子的樹狀圖，$P_n \boxtimes G$ 是漢米爾頓圖若且唯若 $P_n \boxtimes G$ 是 1-強韌的。此外，我們介紹一個圖族新概念，稱爲漢米爾頓強韌性，並證明圖族 $P_n \boxtimes G$ 的漢米爾頓強韌性恰好爲 $\frac{3}{2}$。

關鍵詞: 漢米爾頓迴圈、強乘積圖、因子、強韌性。

Hamiltonicity and Toughness of Strong Product Graphs

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Abstract

In this thesis, we investigate the Hamiltonian properties of strong product graphs. Specifically, with a mild restriction on n, we prove that $P_n \boxtimes G$ is Hamiltonian if and only if G contains a $\{K_{1,1}, K_{1,2}, K_{1,3}, K_{1,4}\}$-factor, where P_n and $K_{1,t}$ are the path graph of order n and star graph of order $t+1$, respectively. Additionally, we explore the relationship between toughness and the existence of Hamiltonian cycles. In particular, we demonstrate that $P_n \boxtimes G$ is Hamiltonian if it is $\frac{3}{2}$-tough. Moreover, we show that if G is a tree with a $\{K_{1,1}, K_{1,3}\}$-factor, then $P_n \boxtimes G$ is Hamiltonian if and only if $P_n \boxtimes G$ is 1-tough. In addition, we introduce a new concept for a family of graphs called Hamiltonian toughness and show that the Hamiltonian toughness for the family of graphs $P_n \boxtimes G$ is exactly $\frac{3}{2}$.

Keyword: Hamiltonian cycle, strong product, factor, toughness

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Chapter 1

Introduction

In the realm of graph theory, the quest for understanding and unraveling the intricate structures of graphs has been a persistent endeavor. Among the myriad of graph-theoretic problems, the exploration of Hamiltonian cycles stands as a cornerstone, offering profound insights into the connectivity and traversal properties of graphs. The Hamiltonian cycle problem garnered considerable attention due to its fundamental nature and wide-ranging applications in diverse fields such as computer science [3, 6], circuit design [14], transportation [13], and DNA sequencing [2] etc. The well-known Dirac's theorem and Ore's theorem provide sufficient conditions of the existence of a Hamiltonian cycle. However, these theorems hinge on assumptions that demand a high degree of connectivity within the graph structure, rendering them applicable only under restrictive conditions.

In [7], Chvatal introduces the concept of toughness and applies it to establish a necessary condition for a graph containing Hamiltonian cycles. Specifically, he proved that every Hamiltonian graph is 1-tough. This technique will be used to furnish a criterion for identifying graphs devoid of Hamiltonian cycles. Furthermore, Chvatal conjectured that there exists a real number t such that every t-tough graph is Hamiltonian. However, Chvátal's conjecture remains unresolved. As noted in [5], there are examples of 2-tough graphs that are not Hamiltonian. Conversely, for certain specific classes of graphs, there may exist a toughness bound that guarantees Hamiltonicity. For instance, [9] demonstrates that every 10-tough chordal graph is Hamiltonian. Motivated by Chvatal's conjecture, we introduce a new concept called Hamiltonian toughness for a family of graphs, defined as the infimum of t such that every t-tough graph in the family is Hamiltonian. Applying this definition, the work of Kabela and Kaiser [9] indicates that the Hamiltonian toughness for the family of chordal graphs is at most 10. In addition, under specific constraints, [12] proves that the Hamiltonian toughness for the family of the Cartesian product of a path and a graph is exactly 1.

In general, the task of determining whether or not a graph contains a Hamiltonian cycle has been proven to be an NP-complete problem. Consequently, several researchers have endeavored to formulate a concise equivalence condition for the existence of Hamiltonian cycles within specific classes of graphs, aiming to mitigate the computational complexity associated with this fundamental problem. For example, Kao presents an equivalent statement regarding the existence of a Hamiltonian cycle on the Cartesian product of a path and a graph [12]. Additionally, in [4], Batagelj explores analogous cases concerning the Cartesian product of a cycle and a graph. Motivated by these investigations, we mainly focus on the strong product of a path and a graph in this thesis. Strong product graphs generate a new topology that inherits properties from both of its constituent graphs. This amalgamation allows researchers to explore novel structural characteristics and behaviors, offering insights into diverse fields such as network design [17] and chemistry [8].

In this thesis, we provide an equivalent condition for the graph G so that the strong product of a path and G is Hamiltonian. More precisely, we prove the following theorem.

Theorem 1.1. *Let G be a connected graph and $n > \frac{4}{3}(\Delta(G)+2)$. Then $P_n \boxtimes G$ is Hamiltonian if and only if G has a $\{K_{1,1}, K_{1,2}, K_{1,3}, K_{1,4}\}$-factor, where P_n denotes the path graph of order n, $K_{1,t}$ denotes the star graph of order $t+1$, $\Delta(G)$ denotes the maximum degree of G, and $P_n \boxtimes G$ denotes the strong product of P_n and G.*

We also prove that the graphs of the form $P_n \boxtimes G$ are Hamiltonian if they are $\frac{3}{2}$-tough. Under certain conditions, we obtain even stronger results, showing that 1-tough is equivalent to Hamiltonicity. Specifically, we present the following results.

Corollary 1.2. *Let G be a connected graph and $n > \frac{4}{3}(\Delta(G)+2)$. If $P_n \boxtimes G$ is $\frac{3}{2}$-tough, then $P_n \boxtimes G$ is Hamiltonian.*

Corollary 1.3. *Let T be a tree with a $\{K_{1,1}, K_{1,3}\}$-factor and n be a positive integer. Then the following statements are equivalent.*

(i) $P_n \boxtimes T$ is Hamiltonian.

(ii) $P_n \boxtimes T$ *is* 1*-tough.*

(iii) $n \geq \Delta(T)$.

Our results thereby support Chvátal's conjecture within this particular class of graphs, offering new insights into the relationship between graph toughness and the existence of Hamiltonian cycles. Furthermore, we explore the Hamiltonian toughness for a specific class of graphs.

Theorem 1.4. *Let* $\mathcal{G} = \{P_n \boxtimes G \mid n > \frac{4}{3}(\Delta(G) + 2)\}$ *be a family of graphs. Then* $\mathrm{Ht}(\mathcal{G}) = \frac{3}{2}$*, where* $\mathrm{Ht}(\mathcal{G})$ *denotes the Hamiltonian toughness of* $\mathcal{G}$

Our approach bears resemblance to that of Kao and Weng [12]. While ensuring the existence of the Hamiltonian cycle, we employ a constructive method for proof. Conversely, in establishing non-existence, we leverage the concept of graph factors. Particularly, we capitalize on the notion of a component factor, a spanning subgraph with specified components. In this area, numerous results have been established by Tutte [15, 16] and Kano [1, 10, 11]. For example, in [1], Amahashi and Kano characterize the equivalent conditions of the existence of certain star factors and tree factors. We also utilize this result in proving the non-existence of the Hamiltonian cycle.

The remaining parts of this thesis are organized as follows. Section 2 introduces some basic notations and definitions. Section 3 provides preliminaries of some known results concerning graph factors, which is a crucial concept of our results. Section 4 presents our primary research outcomes concerning the existence of Hamiltonian cycles within the strong product of a path and a graph. Section 5 discusses the toughness of such strong product graphs and provides the exact value of Hamiltonian toughness for this family of graphs.

Chapter 2

Notation

In this section, we give the notations and definitions that will be used in this thesis.

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. The number $|V(G)|$ is called the **order** of G and the number $|E(G)|$ is called the **size** of G. Let $c(G)$ denote the number of connected components of G. For a vertex $v \in V(G)$ and a subset $S \subseteq V(G)$, we use $N_G(v)$ and $N_G(S)$ to denote the set of neighbor of v in G and the set of vertices adjacent to at least one of vertex of S, respectively. The number $|N_G(v)|$ is called the **degree** of v, denoted by $\deg_G(v)$. Sometimes, we abbreviate $\deg_G(v)$ and $N_G(v)$ as $\deg(v)$ and $N(v)$, respectively. We use $\Delta(G) = \max_{v \in V(G)} \deg_G(v)$ and $\delta(G) = \min_{v \in V(G)} \deg_G(v)$ to denote the maximum degree and minimum degree of G, respectively. A vertex of degree 0 is called an **isolated vertex**, and a vertex of degree 1 is called a **leaf**. We denote $i(G)$ the number of isolated vertices in G. A **path** of length n in G is a sequence of distinct vertices $(v_0, v_1, \ldots, v_n)$ such that $v_i v_{i+1} \in E(G)$ for $i = 0, 1, \ldots, n-1$. A **cycle** of length n in G is a sequence of vertices $(v_0, v_1, \ldots, v_{n-1}, v_0)$ such that $v_i v_{i+1} \in E(G)$ for $i = 0, 1, \ldots, n-2$ and $v_0, v_1, \ldots, v_{n-1}$ are distinct. Sometimes, we use the edge set $E(C) = \{v_i v_{i+1} \mid i = 0, 1, \ldots, n-2\} \cup \{v_{n-1} v_0\}$ for the above cycle C. A **Hamiltonian cycle** is a cycle that contains all the vertices of G. We say G is Hamiltonian if it has a Hamiltonian cycle. Following the conventions, we use P_n, C_n, K_n, and $K_{m,n}$ to denote the path graph of order n, the cycle graph of order n, the complete graph of order n, and the complete bipartite graph with partitions of size m and n, respectively. A star graph of order n is a complete bipartite graph $K_{1,n-1}$.

A graph H is a **subgraph** of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A **spanning subgraph** of G is a subgraph of G if it contains all the vertices of G. Let $S \subseteq V(G)$ be a subset of vertices of G. The **induced subgraph** $G[S]$ is the graph whose vertex set is S and edge set $E = \{uv \in E(G) \mid u, v \in S\}$. We use $G - S$ to denote the subgraph of G induced by $V(G) \setminus S$.

Let H be a subgraph of G. For $e_1 \in E(H)$ and $e_2 \in E(G) \setminus E(H)$, we denote by $H - e_1$ and $H + e_2$ the edge-induced subgraphs of G whose edge sets are $E(H) \setminus \{e_1\}$ and $E(H) \cup \{e_2\}$, respectively.

Let G be a connected graph. A vertex set $S \subseteq V(G)$ is called a **vertex cut** if $c(G-S) \geq 2$. For a real number t, G is said to be t-**tough** if $|S| \geq t \cdot c(G-S)$ for any vertex cut S. If G is not complete, we define the **toughness** of G, denoted by $t(G)$, to be the maximum value t for which G is t-tough. For convenience, we define $t(K_n) = \infty$. Next, we define the Hamiltonian toughness for a family of graphs.

Definition 2.1. Let $\mathcal{G}$ be a family of graphs. The Hamiltonian toughness of $\mathcal{G}$, denoted as $\mathrm{Ht}(\mathcal{G})$, is defined to be

$$\inf\{t \mid \text{every } t\text{-tough graph } G \in \mathcal{G} \text{ is Hamiltonian}\}.$$

Notice that this definition is equivalent to

$$\sup\{t(G) \mid G \in \mathcal{G} \text{ is not Hamiltonian}\}.$$

If every graph $G \in \mathcal{G}$ is Hamiltonian, then we vacuously define $\mathrm{Ht}(\mathcal{G}) = 1$.

From the definition above, it is obvious that $\mathrm{Ht}(\mathcal{G}) \geq 1$ for any $\mathcal{G}$. In addition, we have that $\mathrm{Ht}(\mathcal{G}) \leq \mathrm{Ht}(\mathcal{H})$ if $\mathcal{G} \subseteq \mathcal{H}$.

In this thesis, we discuss a type of graphs called strong product, which is defined as follows:

Definition 2.2. The strong product of two graphs G_1 and G_2 is a graph, denoted by $G_1 \boxtimes G_2$, with vertex

$$V(G_1 \boxtimes G_2) = V(G_1) \times V(G_2),$$

where $V(G_1) \times V(G_2) = \{v_u \mid v \in V(G_1),\, u \in V(G_2)\}$, and edge set

$$\begin{aligned} E(G_1 \boxtimes G_2) = &\{v_u v_w \mid v \in V(G_1),\, uw \in E(G_2)\} \\ &\cup \{v_u w_u \mid u \in V(G_2),\, vw \in E(G_1)\} \\ &\cup \{v_u w_x \mid vw \in E(G_1),\, ux \in E(G_2)\}. \end{aligned}$$

Let G be a graph and $\{G_1, G_2, \ldots, G_n\}$ be a set of graphs. A $\{G_1, G_2, \ldots, G_n\}$-subgraph of G is a subgraph H of G such that each component of H is isomorphic to one of the graphs in $\{G_1, G_2, \ldots, G_n\}$. A $\{G_1, G_2, \ldots, G_n\}$-**factor** of G is a spanning $\{G_1, G_2, \ldots, G_n\}$-subgraph of G. A $\{G_1, G_2, \ldots, G_n\}$-subgraph H of G is said to be **maximum** if G has no $\{G_1, G_2, \ldots, G_n\}$-subgraph H' such that $|V(H')| > |V(H)|$.

Let G be a graph and H be its subgraph. An H-**alternating path** of G is a path whose edges are alternately in $E(H)$ and not in $E(H)$. It can either start from an edge in $E(H)$ or an edge not in $E(H)$. For $u \in V(G)$, we denote by $A(u)$ the set of vertices w of G such that there exists an H-alternating path from u to w. Notice here that the set $A(u)$ depends on H, but we abbreviate the notation without explicitly mentioning H. Furthermore, we define two subsets $OA(u)$ and $EA(u)$ of $A(u)$ to be the sets of vertices w of G such that there exists an H-alternating path from u to w of odd length and even length, respectively. For convenience, let $u \in A(u)$ and $u \notin EA(u)$. In the following, we give an example of the above definition.

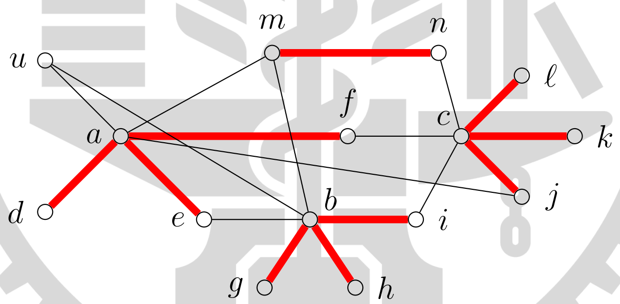

Figure 2.1

Example 2.3. Let G be the graph in Figure 2.1 and H be a subgraph of G where $V(H) = V(G) \setminus \{u\}$ and the edges of H are labeled in red. Then (u, a, e, b, i, c) and (g, b, m, n, c, k) are H-alternating paths. We also have that $A(u) = \{a, b, c, d, e, f, g, h, i, j, k, \ell, u\}$, $EA(u) = \{d, e, f, g, h, i, j, k, \ell\}$, and $OA(u) = \{a, b, c\}$.

Chapter 3

Preliminary

In this section, we give some known results on the Hamiltonian cycle.

Proposition 3.1. [7] *If a graph G has a Hamiltonian cycle, then G is 1-tough.*

Proof. Let n be the order of G. Notice that a cycle graph C_n is 1-tough. That is, $c(C_n - S) \leq |S|$ for any nonempty subset S of $V(C_n)$. If G contains a Hamiltonian cycle, then G can be obtained by adding edges on C_n. Thus, we have that $c(G - S) \leq c(C_n - S) \leq |S|$ for any nonempty subset S of $V(G)$. Hence, G is 1-tough. □

This result gives an important necessary condition for a graph being Hamiltonian.

Next, we will characterize the relationship between the existence of a particular star factor and the number of isolated vertices in any induced subgraph.

Proposition 3.2. [1] *Let G be a graph and $t \geq 2$ be an integer. Then G has a $\{K_{1,1}, K_{1,2}, \ldots, K_{1,t}\}$-factor if and only if $i(G - S) \leq t|S|$ for every $S \subseteq V(G)$.*

To prove Proposition 3.2, we use the standard technique of alternating paths. Note that this result is proposed by Amahashi and Kano [1]. For the completeness of the thesis, we provide the proof here. First, we prove some properties of the alternating paths of a $\{K_{1,1}, K_{1,2}, \ldots, K_{1,t}\}$-subgraph.

Lemma 3.3. [1] *For $t \geq 2$, let G be a graph having no $\{K_{1,1}, K_{1,2}, \ldots, K_{1,t}\}$-factors and let H be a maximum $\{K_{1,1}, K_{1,2}, \ldots, K_{1,t}\}$-subgraph of G. If u is a vertex of G not contained in H, then the following statements hold:*

(i) If $ux_1y_1x_2y_2 \cdots x_ry_r$ is an H-alternating path, then $\deg_H(x_i) = t$ and $\deg_H(y_i) = 1$ for every i.

(ii) $A(u) \setminus \{u\} \subseteq V(H)$, and $A(u)$ is a disjoint union of $\{u\}$, $EA(u)$, and $OA(u)$.

(iii) We have $|EA(u)| = t|OA(u)|$.

(iv) If a vertex w *of* G *is adjacent in* G *to some vertices of* $EA(u)$*, then* w *is contained in* $OA(u)$*. That is,* $N_G(EA(u)) \subseteq OA(u)$.

Proof. Notice that $x_iy_i \in E(H)$ and $y_ix_{i+1} \notin E(H)$ for all i. We first prove deg$(x_1) = t$ and deg$(y_1) = 1$. If deg$_H(y_1) = 1$ and deg$_H(x_1) < t$, then $H + ux_1$ is a $\{K_{1,1}, K_{1,2}, \ldots, K_{1,t}\}$-subgraph with vertex set $V(H) \cup \{u\}$, which contradicts to H is maximum. If deg$_H(y_1) > 1$, then deg$_H(x_1) = 1$ and $H + ux_1 - x_1y_1$ is a $\{K_{1,1}, K_{1,2}, \ldots, K_{1,t}\}$-subgraph with vertex set $V(H) \cup \{u\}$, a contradiction. Hence, we have deg$_H(x_1) = t$ and deg$_H(y_1) = 1$. Similarly, if deg$_H(y_2) = 1$ and deg$_H(x_2) < t$, then we obtain a contradiction by considering $H + y_1x_2 - x_1y_1 + ux_1$. If deg$_H(y_2) > 1$, then deg$_H(x_2) = 1$ and $H - x_2y_2 + y_1x_2 - x_1y_1 + ux_1$ produces a larger $\{K_{1,1}, K_{1,2}, \ldots, K_{1,t}\}$-subgraph. We can repeat this process to prove deg$_H(x_i) = t$ and deg$_H(y_i) = 1$ for all i.

It is immediate that $A(u) \setminus \{u\} \subseteq V(H)$. Furthermore, from the definition, it is clear that

$$A(u) = \{u\} \cup EA(u) \cup OA(u).$$

By (i), we have deg$_H(x) = t$ for $x \in OA(u)$ and deg$_H(y) = 1$ for $y \in EA(u)$. Hence, $EA(u) \cap OA(u) = \emptyset$ as $t \geq 2$.

For each vertex $x \in OA(u)$, by (i), we have deg$_H(x) = t$. Then there exist exactly t vertices adjacent to x in H. Notice that these t vertices are in $EA(u)$. Conversely, each vertex $y \in EA(u)$ has degree 1 in H. Therefore, y must be adjacent to exactly one vertex in $OA(u)$. Thus, we have $|EA(u)| = t|OA(u)|$.

Suppose that a vertex $w \in N_G(EA(u))$ is adjacent to a vertex $v \in EA(u)$. Let $ux_1y_1 \cdots x_tv$ be an H-alternating path of even length. Notice that $x_tv \in E(H)$ and deg$_H(v) = 1$. If w is not in the path, then $ux_1y_1 \cdots x_tvw$ is an H-alternating path of odd length and $w \in OA(u)$. If $w = x_j$ for some j, then $w \in OA(u)$ as $x_j \in OA(u)$. If $w = y_j$ for some j, then $ux_1y_1 \cdots x_jy_jv$ is an H-alternating path of odd length, which implies $v \in OA(u)$. However, from (ii), we must have $EA(u) \cap OA(u) = \emptyset$, which is a contradiction. □

Now, we shall prove Proposition 3.2.

Proof of Proposition 3.2. Suppose that G has a $\{K_{1,1}, K_{1,2}, \ldots, K_{1,t}\}$-factor F. Let $H_1, \ldots, H_r$ be the components of F. Notice that for each component H_j, we must have $i(H_j - S') \leq t|S'|$ for any $S' \subseteq V(H_j)$. Then, for any $S \subseteq V(G)$, we have

$$i(G - S) \leq i(F - S) = \sum_{j=1}^{r} i(H_j - (S \cap V(H_j))) \leq \sum_{j=1}^{r} t|S \cap V(H_j)| = t|S|.$$

Conversely, suppose that G has no $\{K_{1,1}, K_{1,2}, \ldots, K_{1,t}\}$-factor. Let H be a maximum $\{K_{1,1}, K_{1,2}, \ldots, K_{1,t}\}$-subgraph of G and $u \in V(G) \setminus V(H)$. By Lemma 3.3(iv), every vertex in $EA(u)$ is isolated in $G - OA(u)$, and it is clear that u is also an isolated vertex in $G - OA(u)$. By Lemma 3.3(iii), it follows that

$$i(G - OA(u)) \geq |EA(u) \cup \{u\}| = t|OA(u)| + 1 > t|OA(u)|.$$

This completes the proof. □

Chapter 4

Strong product of a path and a graph

In this section, we discuss the Hamiltonian property of $P_n \boxtimes G$ where G is a connected graph. For convenience, we assume

$$V(P_n) = \{1, 2, \ldots, n\}, \text{ and}$$
$$E(P_n) = \{i(i+1) \mid i = 1, 2, \ldots, n-1\}.$$

For $v \in V(G)$, we define

$$E_v = \{i_v(i+1)_v \mid i = 1, 2, \ldots, n-1\} \subseteq E(P_n \boxtimes G).$$

We first leverage Chvátal's result, presented as Proposition 3.1, to establish a condition for the non-existence of Hamiltonian cycles in strong product graphs. Using this proposition, we derive the following theorem.

Theorem 4.1. *Let G and H be connected graphs. Let $S \subseteq V(G)$ and $S' \subseteq V(H)$ with $c(G - S) = r$ and $c(H - S') = t$. Suppose that one of the following statements (i)-(iii) holds. Then $G \boxtimes H$ is not 1-tough. In particular, $G \boxtimes H$ is not Hamiltonian.*

(i) $|S| \cdot |V(H)| < r$.

(ii) $|S'| \cdot |V(G)| < t$.

(iii) $|S| \cdot |V(H)| + |S'| \cdot |V(G)| - |S| \cdot |S'| < rt$.

Proof. Suppose that statement (i) holds. Consider the set

$$S_1 = S \times V(H) \subseteq V(G \boxtimes H).$$

Observe that

$$(G \boxtimes H) - S_1 = (G - S) \boxtimes H,$$

so we have $c((G \boxtimes H) - S_1) = r$ with $|S_1| = |S| \cdot |V(H)| < r$. By definition, $G \boxtimes H$ is not 1-tough. Notice that statement (ii) is the same as statement (i) by considering the product $H \boxtimes G$.

Now, suppose that statement (iii) holds. Let

$$S_2 = S \times V(H) \cup S' \times V(G) \subseteq V(G \boxtimes H).$$

Then we have

$$(G \boxtimes H) - S_2 = (G - S) \boxtimes (H - S').$$

Thus, we have that $c((G \boxtimes H) - S_2) = rt$ with $|S_2| = |S| \cdot |V(H)| + |S'| \cdot |V(G)| - |S| \cdot |S'| < rt$. Hence, it follows that $G \boxtimes H$ is not 1-tough. $\square$

For a tree T, if we delete a vertex with maximum degree, then we obtain $\Delta(T)$ connected components. Hence, by Theorem 4.1, we have the following results.

Corollary 4.2. *Let G be a connected graph and T be a tree. If $\Delta(T) > |V(G)|$, then $G \boxtimes T$ is not Hamiltonian.*

Corollary 4.3. *Let T_1 and T_2 be trees. If $|V(T_1)| + |V(T_2)| - 1 < \Delta(T_1)\Delta(T_2)$, then $T_1 \boxtimes T_2$ is not Hamiltonian.*

By Corollary 4.2, we know that $P_n \boxtimes T$ can be Hamiltonian only when $n \geq \Delta(T)$. Next, we give another constraint of $P_n \boxtimes G$ being Hamiltonian by using Proposition 3.2.

Theorem 4.4. *If G is a graph that does not contain a $\{K_{1,1}, K_{1,2}, K_{1,3}, K_{1,4}\}$-factor and H is a graph with $\delta(H) = 1$, then $H \boxtimes G$ is not Hamiltonian.*

Proof. Let $x \in V(H)$ be a vertex with $\deg_H(x) = 1$ and $y \in V(H)$ be its neighbor. Suppose that $H \boxtimes G$ contains a Hamiltonian cycle C. Then $\deg_C(v) = 2$ for all $v \in V(H \boxtimes G)$. Since G does not contain a $\{K_{1,1}, K_{1,2}, K_{1,3}, K_{1,4}\}$-factor, by Proposition 3.2, there exists $S \subseteq V(G)$ such that

$i(G-S) > 4|S|$. Let I be the set of isolated vertices in $G-S$ and $z_S = \{z_s \mid s \in S\} \subseteq V(H \boxtimes G)$ for $z \in V(H)$. Observe that for each $u \in I$, we have $N_G(u) \subseteq S$ and

$$N_{H\boxtimes G}(x_u) \subseteq \{y_u\} \cup x_S \cup y_S.$$

This means that for each $u \in I$, x_u must be adjacent in C to at least one of the vertices in $x_S \cup y_S$. Notice that $|I| = i(G-S) > 4|S| = 2|x_S \cup y_S|$, by pigeonhole principle, there exists a vertex $z \in x_S \cup y_S$ such that $\deg_C(z) \geq 3$, which is a contradiction. Therefore, $H \boxtimes G$ has no Hamiltonian cycle. □

Theorem 4.4 is generalized as follows.

Theorem 4.5. *Let H be a graph and $S' \subseteq V(H)$. Let I' be the set of isolated vertices in $H - S'$ and $S_1 \subseteq S'$ be the set of vertices in S' with only one neighbor in I'. Assume that $2|I'| - 2|S'| + |S_1| > 0$ and let $t = \left\lceil \frac{2|I'|+2|S'|}{2|I'|-2|S'|+|S_1|} \right\rceil \geq 2$. If G does not contain a $\{K_{1,1}, K_{1,2}, \ldots, K_{1,t}\}$-factor, then $H \boxtimes G$ is not Hamiltonian.*

Proof. The idea of the proof is similar to Theorem 4.4. Since G does not contain a $\{K_{1,1}, K_{1,2}, \ldots, K_{1,t}\}$-factor, by Proposition 3.2, there exists $S \subseteq V(G)$ such that $i(G-S) > t|S|$. Let $I \subseteq V(G)$ be the set of isolated vertices in $G - S$. Suppose $H \boxtimes G$ has a Hamiltonian cycle C. Then we have

$$\sum_{x_u \in I' \times I} \deg_C(x_u) = 2|I' \times I|.$$

Notice that

$$N_{H\boxtimes G}(I' \times I) \subseteq I' \times S \cup S' \times S \cup S' \times I.$$

Since each $y \in S_1$ has only one neighbor in I', it follows that each $y_u \in S_1 \times I$ has only one neighbor in $I' \times I$. Hence, there are at most

$$2|I' \times S| + 2|S' \times S| + 2|(S' \setminus S_1) \times I| + |S_1 \times I|$$

edges in C between $I' \times I$ and $I' \times S \cup S' \times S \cup S' \times I$. This means

$$2|I' \times I| \leq 2|I' \times S| + 2|S' \times S| + 2|(S' \setminus S_1) \times I| + |S_1 \times I|.$$

By rearranging the above formula, we have

$$(2|I'| - 2|S'| + |S_1|)\,|I| \leq (2|I'| + 2|S'|)\,|S|.$$

However, this will lead to

$$t < \frac{|I|}{|S|} \leq \frac{2|I'| + 2|S'|}{2|I'| - 2|S'| + |S_1|} \leq \left\lceil \frac{2|I'| + 2|S'|}{2|I'| - 2|S'| + |S_1|} \right\rceil = t,$$

which is a contradiction. Therefore, $H \boxtimes G$ is not Hamiltonian. □

Notice that Theorem 4.4 is just a special case of Theorem 4.5 with $|I'| = |S'| = |S_1| = 1$.

Next, we discuss the Hamiltonicity of $P_n \boxtimes G$. Since $\delta(P_n) = 1$ for all $n \geq 2$, by Theorem 4.4, it follows that $P_n \boxtimes G$ has no Hamiltonian cycle if G does not contain a $\{K_{1,1}, K_{1,2}, K_{1,3}, K_{1,4}\}$-factor. Now, a natural question to ask is that whether or not $P_n \boxtimes G$ contains a Hamiltonian cycle provided that G has a $\{K_{1,1}, K_{1,2}, K_{1,3}, K_{1,4}\}$-factor. First, we construct a Hamiltonian cycle for $P_n \boxtimes K_{1,1}$, $P_n \boxtimes K_{1,2}$, $P_n \boxtimes K_{1,3}$, and $P_n \boxtimes K_{1,4}$.

In our construction, we want as more edges of the cycle contained in E_u where u is the vertex in $K_{1,2}$ or $K_{1,4}$ with maximum degree. Thus, we consider the following results.

Lemma 4.6. *Suppose* $V(K_{1,2}) = \{u, v, w\}$ *with* $\deg(u) = 2$. *If* C *is a Hamiltonian cycle of* $P_n \boxtimes K_{1,2}$*, then* C *does not contain a path of order* 5 *in* E_u.

Proof. Suppose that C contains a path of order 5 in E_u, says $(j-2)_u$, $(j-1)_u$, j_u, $(j+1)_u$, $(j+2)_u$. Notice that

$$N_{P_n \boxtimes K_{1,2}}(j_v) = \{(j-1)_v, (j+1)_v, (j-1)_u, j_u, (j+1)_u\},$$

and

$$N_{P_n \boxtimes K_{1,2}}(j_w) = \{(j-1)_w, (j+1)_w, (j-1)_u, j_u, (j+1)_u\}.$$

Since $(j-1)_u$, j_u, and $(j+1)_u$ has degree 2 with the 4 edges in E_u, C must also contains two paths $(j-1)_v, j_v, (j+1)_v$, and $(j-1)_w, j_w, (j+1)_w$. Now, observe that the three sets of vertices

$$\{(j-1)_v, j_v, (j+1)_v\}, \{(j-2)_u, (j-1)_u, j_u, (j+1)_u, (j+2)_u\}, \{(j-1)_w, j_w, (j+1)_w\}$$

cut $P_n \boxtimes K_{1,2}$ into 2 components. Since 3 is odd, it is impossible that a Hamiltonian cycle contains all these edges. Therefore, C does not contain a path of order 5 in E_u. $\square$

By the same technique, we have a similar result for $K_{1,4}$.

Lemma 4.7. *Suppose $V(K_{1,4}) = \{u, v, w, x, y\}$ with $\deg(u) = 4$. If C is a Hamiltonian cycle of $P_n \boxtimes K_{1,4}$, then C does not contain a path of order 5 in E_u.*

Based on Lemma 4.6 and Lemma 4.7, we construct the Hamiltonian cycle for $P_n \boxtimes K_{1,2}$ and $P_n \boxtimes K_{1,4}$ with 3 consecutive edges in E_u where u is the vertex in $K_{1,2}$ and $K_{1,4}$ with maximum degree.

Construction 4.8. The Hamiltonian cycles we choose are as follows

- Let $V(K_{1,1}) = \{u, v\}$. Then

  $$E_u \cup E_v \cup \{1_u 1_v, n_u n_v\}$$

  forms a Hamiltonian cycle for $P_n \boxtimes K_{1,1}$.

- Let $V(K_{1,2}) = \{u, v, w\}$ with $\deg(u) = 2$ and

$$\begin{aligned} E =& \{i_u(i+1)_u \mid i \equiv 0, 2, 3 \pmod 4, i < n-1\} \\ & \cup \{i_v(i+1)_v \mid i \equiv 0, 1, 2, 4, 5, 6, 7 \pmod 8\} \\ & \cup \{i_w(i+1)_w \mid i \equiv 0, 1, 2, 3, 4, 5, 6 \pmod 8\} \\ & \cup \{i_u(i+1)_v \mid i \equiv 2 \pmod 8, i < n\} \\ & \cup \{i_u(i-1)_v \mid i \equiv 5 \pmod 8, i < n\} \\ & \cup \{i_u(i+1)_w \mid i \equiv 6 \pmod 8, i < n\} \\ & \cup \{i_u(i-1)_w \mid i \equiv 1 \pmod 8, 1 < i < n\} \\ & \cup \{1_u 1_v, 1_u 1_w\} \subseteq E(P_n \boxtimes K_{1,2}). \end{aligned}$$

Then, the following set forms a Hamiltonian cycle for $P_n \boxtimes K_{1,2}$.

$$\begin{cases} E \cup \{(n-1)_u n_w, n_u n_v, n_u n_w\} & \text{if } n \equiv 0 \pmod 8 \\ E \cup \{(n-1)_u (n-1)_w, n_u n_v, n_u n_w\} & \text{if } n \equiv 1 \pmod 8 \\ E \cup \{n_u n_v, n_u n_w\} & \text{if } n \equiv 2 \pmod 8 \\ E \cup \{n_u n_w\} & \text{if } n \equiv 3 \pmod 8 \\ E \cup \{(n-1)_u n_v, n_u n_v, n_u n_w\} & \text{if } n \equiv 4 \pmod 8 \\ E \cup \{(n-1)_u (n-1)_v, n_u n_v, n_u n_w\} & \text{if } n \equiv 5 \pmod 8 \\ E \cup \{n_u n_v, n_u n_w\} & \text{if } n \equiv 6 \pmod 8 \\ E \cup \{n_u n_v\} & \text{if } n \equiv 7 \pmod 8 \end{cases}$$

- Let $V(K_{1,3}) = \{u, v, w, x\}$ with $\deg(u) = 3$. Then

$$\begin{aligned} & \{i_u(i+1)_u \mid i = 2, 3, \ldots, n-2\} \cup E_v \cup E_w \cup E_x \\ & \cup \{1_u 1_v, 1_u 1_w, 2_u 1_x, (n-1)_u n_w, n_u n_v, n_u n_x\} \end{aligned}$$

forms a Hamiltonian cycle for $P_n \boxtimes K_{1,3}$.

- Let $V(K_{1,4}) = \{u, v, w, x, y\}$ with $\deg(u) = 4$ and

$$\begin{aligned}
E =& \{i_u(i+1)_u \mid i \equiv 0, 1, 3 \pmod 4), 2 < i < n-2\} \\
& \cup \{i_v(i+1)_v \mid i \equiv 0, 1, 2, 3, 5, 6, 7 \pmod 8), i < n-2\} \\
& \cup \{i_w(i+1)_w \mid i \equiv 1, 2, 3, 4, 5, 6, 7 \pmod 8), i < n-2\} \\
& \cup \{i_u(i+1)_v \mid i \equiv 3 \pmod 8), i < n-1\} \\
& \cup \{i_u(i-1)_v \mid i \equiv 6 \pmod 8), i < n-1\} \\
& \cup \{i_u(i+1)_w \mid i \equiv 7 \pmod 8), i < n-1\} \\
& \cup \{i_u(i-1)_w \mid i \equiv 2 \pmod 8), 2 < i < n-1\} \\
& \cup \{2_u1_x, 1_u1_y, 2_u1_v, 1_u1_w, (n-1)_vn_v, (n-1)_wn_w, n_un_x, n_un_w, (n-1)_un_y, (n-1)_un_v\} \\
& \cup E_x \cup E_y \subseteq E(P_n \boxtimes K_{1,4})
\end{aligned}$$

  Then, the following set forms a Hamiltonian cycle for $P_n \boxtimes K_{1,4}$.

$$\begin{cases}
E \cup \{(n-2)_v(n-1)_v, (n-2)_w(n-1)_w\} & \text{if } n \equiv 0 \pmod 8 \\
E \cup \{(n-2)_u(n-2)_w, (n-2)_v(n-1)_v\} & \text{if } n \equiv 1 \pmod 8 \\
E \cup \{(n-2)_u(n-1)_w, (n-2)_v(n-1)_v\} & \text{if } n \equiv 2 \pmod 8 \\
E \cup \{(n-2)_u(n-2)_w, (n-2)_w(n-1)_w, (n-2)_v(n-1)_v\} & \text{if } n \equiv 3 \pmod 8 \\
E \cup \{(n-2)_v(n-1)_v, (n-2)_w(n-1)_w\} & \text{if } n \equiv 4 \pmod 8 \\
E \cup \{(n-2)_u(n-2)_v, (n-2)_w(n-1)_w\} & \text{if } n \equiv 5 \pmod 8 \\
E \cup \{(n-2)_u(n-1)_v, (n-2)_w(n-1)_w\} & \text{if } n \equiv 6 \pmod 8 \\
E \cup \{(n-2)_u(n-2)_v, (n-2)_v(n-1)_v, (n-2)_w(n-1)_w\} & \text{if } n \equiv 7 \pmod 8
\end{cases}$$

Below, We give an example of the Hamiltonian cycle we construct.

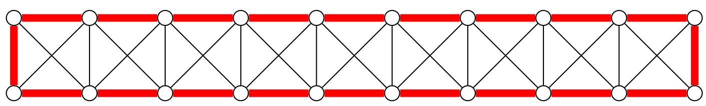

Figure 4.1: $P_{10} \boxtimes K_{1,1}$

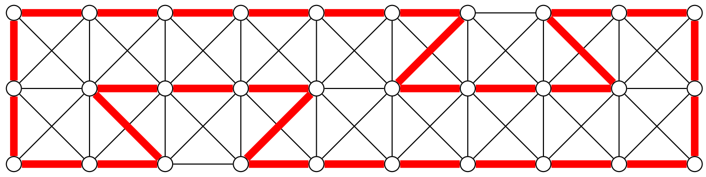

Figure 4.2: $P_{10} \boxtimes K_{1,2}$

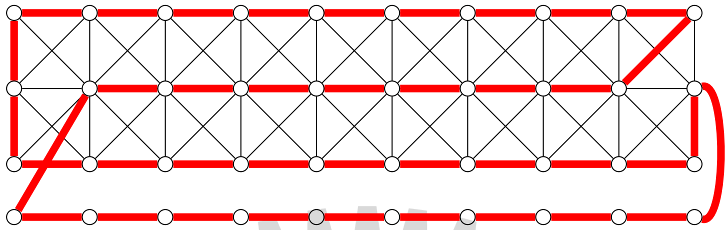

Figure 4.3: $P_{10} \boxtimes K_{1,3}$

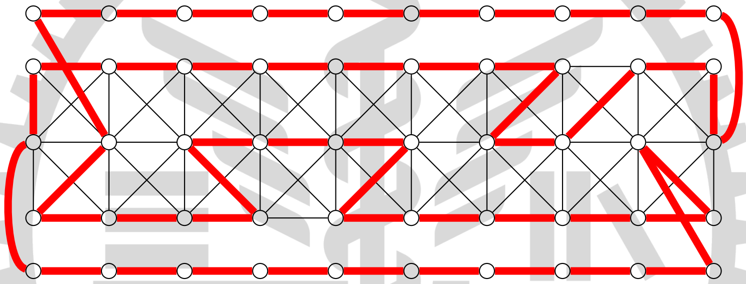

Figure 4.4: $P_{10} \boxtimes K_{1,4}$

Lemma 4.9. *Let T be a tree that has a $\{G_1, G_2, \ldots, G_k\}$-factor F with at least 2 components. Then there exists a component L of F such that $T - V(L)$ is a tree with a $\{G_1, G_2, \ldots, G_k\}$-factor $F - V(L)$.*

Proof. We consider a graph H whose vertex set consists of the components of F. Two components L_1, L_2 of F are adjacent if there exist vertices $u_1 \in V(L_1)$ and $u_2 \in V(L_2)$ such that $u_1u_2 \in E(T)$. It is clear that H is connected and a cycle in H will yield a cycle in T. Hence, H must be a tree. Let L be a leaf of H. Then deleting the corresponding component L in F gives a $\{G_1, G_2, \ldots, G_k\}$-subgraph of T. Therefore, $F - V(L)$ is a $\{G_1, G_2, \ldots, G_k\}$-factor of $T - V(L)$. □

Now, we prove our main result of the existence of the Hamiltonian cycle of $P_n \boxtimes G$ where G is a graph containing $\{K_{1,1}, K_{1,2}, K_{1,3}, K_{1,4}\}$-factor.

Theorem 4.10. *Let T be a tree with a $\{K_{1,1}, K_{1,3}\}$-factor. If $n \geq \Delta(T)$, then there exists a Hamiltonian cycle of $P_n \boxtimes T$ which contains exactly $n - \deg_T(v)$ of edges from the set E_v for any $v \in V(T)$.*

Proof. The proof is by induction on $|V(T)|$. If $T = K_{1,1}$ or $T = K_{1,3}$, then the Hamiltonian cycles of $P_n \boxtimes K_{1,1}$ and $P_n \boxtimes K_{1,3}$ in Construction 4.8 satisfies the requirements.

Now, suppose T has a $\{K_{1,1}, K_{1,3}\}$-factor F with more than one component. By Lemma 4.9, there exists a component L of F such that $T' = T - V(L)$ is a tree with a $\{K_{1,1}, K_{1,3}\}$-factor $F - V(L)$. Let $u_1 \in V(L)$ and $u_2 \in V(T')$ with $u_1u_2 \in E(T)$. Notice that $\Delta(T') \leq \Delta(T) \leq n$. Since $|V(T')| < |V(T)|$, by induction hypothesis, it follows that $P_n \boxtimes T'$ has a Hamiltonian cycle C' which contains exactly $n - \deg_{T'}(v)$ of edges from the set E_v for any $v \in V(T')$. Since $u_1 \notin V(T')$, we have

$$\deg_{T'}(u_2) = \deg_T(u_2) - 1 \leq \Delta(T) - 1 \leq n - 1.$$

Hence, C' must contain at least one edge from E_{u_2}, namely, $j_{u_2}(j+1)_{u_2}$. Furthermore, since L is a $K_{1,1}$ or $K_{1,3}$, we have a Hamiltonian cycle C of $P_n \boxtimes L$ by Construction 4.8. Observe that in our construction, C contain at least one of the edge in $\{(j-1)_{u_1}j_{u_1}, j_{u_1}(j+1)_{u_1}, (j+1)_{u_1}(j+2)_{u_1}\}$. Let $\ell_{u_1}(\ell+1)_{u_1}$ be the edge in C where ℓ is one of the $j-1, j, j+1$. Then we can construct a new Hamiltonian cycle $\hat{C}$ of $P_n \boxtimes T$ with edges

$$E(\hat{C}) = E(C') \cup E(C) \cup \{j_{u_2}\ell_{u_1}, (j+1)_{u_2}(\ell+1)_{u_1}\} \setminus \{j_{u_2}(j+1)_{u_2}, \ell_{u_1}(\ell+1)_{u_1}\}.$$

Finally, the remaining part is to verify $\hat{C}$ satisfies the requirements of containing exactly $n - \deg_T(v)$ of edges from E_v for any $v \in V(T)$. From the induction hypothesis, we know that for any $v \neq u_1, u_2$ meets the requirements. For u_1, $\hat{C}$ contains $n - \deg_L(u_1) - 1 = n - \deg_T(u_1)$ edges from E_{u_1}. Similarly, $\hat{C}$ contains $n - \deg_{T'}(u_2) - 1 = \deg_T(u_2)$ edges from E_{u_2}. This completes the proof. □

Theorem 4.11. *Let T be a tree with a $\{K_{1,1}, K_{1,2}, K_{1,3}, K_{1,4}\}$-factor. If $n > \frac{4}{3}(\Delta(T) + 2)$, then* *there exists a Hamiltonian cycle of $P_n \boxtimes T$ which contains at least $\frac{3}{4}n - \deg_T(v)$ of edges from the set E_v for any $v \in V(T)$.*

Proof. The proof is similar to Theorem 4.10 and is also by induction on $|V(T)|$. If $T = K_{1,1}, K_{1,2}, K_{1,3}$, or $K_{1,4}$, then the Hamiltonian cycles in Construction 4.8 satisfies the requirements. Notice that for $K_{1,2}$, the Hamiltonian cycle contains

$$\begin{cases} \frac{3}{4}n - 2, & n \equiv 0 \pmod 4 \\ \frac{3}{4}n - \frac{7}{4}, & n \equiv 1 \pmod 4 \\ \frac{3}{4}n - \frac{3}{2}, & n \equiv 2 \pmod 4 \\ \frac{3}{4}n - \frac{5}{4}, & n \equiv 3 \pmod 4 \end{cases}$$

edges in E_u where $u \in V(K_{1,2})$ with $\deg_{K_{1,2}}(u) = 2$. Meanwhile, for $K_{1,4}$, the Hamiltonian cycle contains

$$\begin{cases} \frac{3}{4}n - 3, & n \equiv 0 \pmod 4 \\ \frac{3}{4}n - \frac{15}{4}, & n \equiv 1 \pmod 4 \\ \frac{3}{4}n - \frac{7}{2}, & n \equiv 2 \pmod 4 \\ \frac{3}{4}n - \frac{13}{4}, & n \equiv 3 \pmod 4 \end{cases}$$

edges in E_u where $u \in V(K_{1,4})$ with $\deg_{K_{1,4}}(u) = 4$.

Now, suppose that T has a $\{K_{1,1}, K_{1,2}, K_{1,3}, K_{1,4}\}$-factor F with more than one component. By Lemma 4.9, there exists a component L of F such that $T' = T - V(L)$ is a tree with $\{K_{1,1}, K_{1,2}, K_{1,3}, K_{1,4}\}$-factor $F - V(L)$. Let $u_1 \in V(c)$ and $u_2 \in V(T')$ with $u_1u_2 \in E(T)$. It is clear that $n \geq \frac{4}{3}\Delta(T') + 2$ since $\Delta(T') \leq \Delta(T)$. By induction hypothesis, $P_n \boxtimes T'$ has a Hamiltonian cycle C' which contains at least $\frac{3}{4}n - \deg_{T'}(v)$ of edges from the set E_v for any $v \in V(T')$. Since $u_1 \notin V(T')$, we have $\deg_{T'}(u_2) = \deg_T(u_2) - 1 \leq \Delta(T) - 1$ and

$$\frac{3}{4}n - \deg_{T'}(u_2) > \frac{3}{4} \times \frac{4}{3}(\Delta(T) + 2) - \deg_{T'}(u_2) \geq 3.$$

This means that C' must contain at least 4 edges of E_{u_2}. By excluding the first and last two edges, C' must contain at least one edge from $E_{u_2} \setminus \{1_{u_2}2_{u_2}, (n-2)_{u_2}(n-1)_{u_2}, (n-1)_{u_2}n_{u_2}\}$, denoted as $j_{u_2}(j+1)_{u_2}$ with $j \neq 1, n-2, n-1$. Moreover, by Construction 4.8, there exists a Hamiltonian cycle C of $P_n \boxtimes L$ containing at least one of $(j-1)_{u_1}j_{u_1}, j_{u_1}(j+1)_{u_1}, (j+1)_{u_1}(j+2)_{u_1}$. Let $\ell_{u_1}(\ell+1)_{u_1}$ be the edge in C where ℓ is one of the $j-1, j, j+1$. As same in Theorem 4.10, we can construct a new Hamiltonian cycle $\hat{C}$ of $P_n \boxtimes T$ with edges

$$E(\hat{C}) = E(C') \cup E(C) \cup \{j_{u_2}\ell_{u_1}, (j+1)_{u_2}(\ell+1)_{u_1}\} \setminus \{j_{u_2}(j+1)_{u_2}, \ell_{u_1}(\ell+1)_{u_1}\}.$$

Now, we shall ensure that C contains at least $\frac{3}{4}n - \deg_T(v)$ edges from the set E_v for any $v \in V(T)$. This is true for any $v \neq u_1, u_2$ from the induction hypothesis. Furthermore, $\hat{C}$ contains at least $\frac{3}{4}n - \deg_L(u_1) - 1 = \frac{3}{4}n - \deg_T(u_1)$ and $\frac{3}{4}n - \deg_{T'}(u_2) - 1 = \frac{3}{4}n - \deg_T(u_2)$ edges from E_{u_1} and E_{u_2}, respectively. This completes the proof. □

Notice that we can replace the tree T in Theorem 4.10 and Theorem 4.11 by a connected graph, then the existence of the Hamiltonian cycle is still valid since every connected graphs contain a spanning tree. Therefore, Theorem 1.1 is a direct consequence of Theorem 4.4 and Theorem 4.11.

Finally, we give a counterexample to show that the condition in Theorem 4.11 cannot be relaxed to $n \geq \Delta(T)$ as in Theorem 4.10.

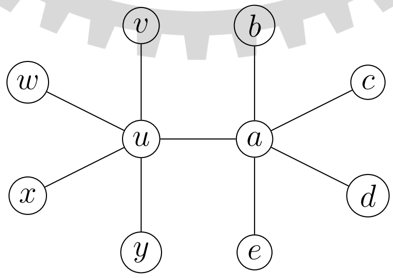

Figure 4.5: A graph with maximum degree 5

Theorem 4.12. *Let G be the graph in Figure 4.5. Then $P_5 \boxtimes G$ does not contain a Hamiltonian cycle.*

Proof. Let $H = P_5 \boxtimes G$ and suppose that H has a Hamiltonian cycle C. Observe that $N_H(1_\alpha) = \{1_u, 2_u, 2_\alpha\}$ and $N_H(5_\alpha) = \{5_u, 4_u, 4_\alpha\}$ for $\alpha = v, w, x, y$. This implies that for each $\alpha = v, w, x, y$, the vertex 1_α and 5_α must be adjacent to at least one of the $1_u, 2_u$ and $4_u, 5_u$ in C, respectively. Hence, $1_u, 2_u, 4_u, 5_u$ cannot have other neighbors in C since there are already 8 neighbors for them. By a similar statements, it follows that $1_a, 2_a$ and $4_a, 5_a$ can only be adjacent to 1_β and 5_β in C for $\beta = b, c, d, e$, respectively. Let $A = V(P_5) \times \{u, v, w, x, y\} \subseteq E(H)$ and $B = V(P_5) \times \{a, b, c, d, e\} \subseteq E(H)$. Then $3_u 3_a$ is the only remaining edge to connect A and B. However, we must have at least two edges to connect A and B for a Hamiltonian cycle, which is a contradiction. Therefore, H is not Hamiltonian. $\square$

Chapter 5

Toughness of the strong product of a path and a graph

In this section, we discuss the toughness of $P_n \boxtimes G$.

Theorem 5.1. *Let G be a graph and $n \geq 2$ be an integer. If $P_n \boxtimes G$ is $\frac{3}{2}$-tough, then G contain a $\{K_{1,1}, K_{1,2}, K_{1,3}, K_{1,4}\}$-factor.*

Proof. The proof is by contradiction. Suppose that G does not contain a $\{K_{1,1}, K_{1,2}, K_{1,3}, K_{1,4}\}$-factor. By Proposition 3.2, there exists $S \subseteq V(G)$ such that $i(G-S) > 4|S|$. Let I be the set of isolated vertices in $G-S$. Now, we consider the set

$$S' = (\{1,2\} \times S) \cup (\{2\} \times I) \subseteq V(P_n \boxtimes G).$$

Hence, $|S'| = 2|S| + |I| < \frac{3}{2}|I|$. Observe that each vertex in $\{1\} \times I$ is isolated in $(P_n \boxtimes G) - S'$. Hence, $(P_n \boxtimes G) - S'$ has at least $|I|$ components. Since $\frac{|S'|}{c(P_n \boxtimes G - S')} < \frac{\frac{3}{2}|I|}{|I|} < \frac{3}{2}$, it follows that $P_n \boxtimes G$ is not $\frac{3}{2}$-tough. □

By this Theorem, we can easily prove Corollary 1.2.

Proof of Corollary 1.2. From Theorem 5.1, if $P_n \boxtimes G$ is $\frac{3}{2}$-tough, then G must contains a $\{K_{1,1}, K_{1,2}, K_{1,3}, K_{1,4}\}$-factor. By Theorem 4.11, it follows that $P_n \boxtimes G$ is Hamiltonian. □

Proof of Corollary 1.3. (i) ⇒ (ii) is from Proposition 3.1. (ii) ⇒ (iii) is from Theorem 4.1. (iii) ⇒ (i) is from Theorem 4.10. □

Now, we consider the graphs of the form $P_n \boxtimes G$ where G is a connected graph and $n > \frac{4}{3}(\Delta(G)+2)$. We have proved that $t(P_n \boxtimes G) \geq \frac{3}{2}$ implies that $P_n \boxtimes G$ is Hamiltonian. On the other hand, Chvatal's result demonstrates that $t(P_n \boxtimes G) < 1$ implies that $P_n \boxtimes G$ is not

Hamiltonian. The only remaining case is the graphs with toughness between 1 and $\frac{3}{2}$. Observe that $P_n \boxtimes P_2$ is Hamiltonian and has toughness exactly 1. We are now interested in determining whether there exists a graph G such that $P_n \boxtimes G$ is not Hamiltonian, but its toughness is close to $\frac{3}{2}$.

Below, we discuss the toughness and Hamiltonicity of the graph $P_n \boxtimes K_{k,4k+1}$. For convenience, let X, Y be the two parts of vertex of $K_{k,4k+1}$ where $|X| = k$ and $|Y| = 4k+1$. Furthermore, for each $i = 1, 2, \ldots, n$, we define three subsets of $V(P_n \boxtimes K_{k,4k+1})$

$$V_i = \{i\} \times V(K_{k,4k+1}), \quad X_i = \{i\} \times X, \quad Y_i = \{i\} \times Y.$$

Lemma 5.2. *Let n, k be positive integers. Then the graph $P_n \boxtimes K_{k,4k+1}$ is not Hamiltonian.*

Proof. It is clear that $K_{k,4k+1}$ does not contain a $\{K_{1,1}, K_{1,2}, K_{1,3}, K_{1,4}\}$-factor. By Theorem 4.4, $P_n \boxtimes K_{k,4k+1}$ is not Hamiltonian. □

Lemma 5.3. *Let n, k be positive integers with $n \geq 3$. Then $t(P_n \boxtimes K_{k,4k+1}) \leq \frac{6k+1}{4k+2}$.*

Proof. Consider the set

$$S = X_1 \cup V_2 \subseteq V(P_n \boxtimes K_{k,4k+1}).$$

Notice that $|S| = 6k+1$ and $c(P_n \boxtimes K_{k,4k+1} - S) = |Y_1| + 1 = 4k+2$. By definition, it follows that $t(P_n \boxtimes K_{k,4k+1}) \leq \frac{6k+1}{4k+2}$. □

Now, we shall prove that $t(P_n \boxtimes K_{k,4k+1}) \geq \frac{6k+1}{4k+2}$. Notice that from our definition of toughness, we have $t(G) = \min \frac{|S|}{c(G-S)}$, where S is a vertex cut of G. We first prove the following lemma.

Lemma 5.4. *Let n, k be positive integers with $n \geq 3$ and $G = P_n \boxtimes K_{k,4k+1}$. If S is a vertex cut that attains the minimum value of $\frac{|S|}{c(G-S)}$ among all the vertex cuts, then either $V_i \cap S = \emptyset$ or $X_i \subseteq S$ for each $i = 1, 2, \ldots, n$.*

Proof. Suppose on the contrary that $S_i \neq \emptyset$ and $X_i \not\subseteq S$ where $S_i = V_i \cap S$. Let $x \in X_i \setminus S$. We have the following two cases.

Case 1: $Y_i \not\subseteq S$. Let $y \in Y_i \setminus S$. Since $xy \in E(G-S)$ and $N_G(\{x,y\}) = N_G(S_i)$, the deletion of S_i in G will not affect the number of components. That is,

$$c(G-S) = c(G-(S \setminus S_i)).$$

Furthermore, we have $|S \setminus S_i| < |S|$ since $S_i \neq \emptyset$. It follows that $\frac{|S \setminus S_i|}{c(G-(S\setminus S_i))} < \frac{|S|}{c(G-S)}$, which is a contradiction.

Case 2: $Y_i \subseteq S$. Notice that $Y_{i-1}, Y_{i+1} \subseteq N_{G-S}(x)$ (only Y_{i+1} for $i=1$ and Y_{i-1} for $i=n$). Therefore, the deletion of S in G yields at most $2k$ more components than that of the deletion of $S \setminus S_i$. That is,

$$c(G-S) \leq c(G-(S\setminus S_i)) + 2k.$$

Furthermore, since $Y_i \subseteq S$, we have that $|S| \geq |S \setminus S_i| + 4k + 1$. If $S \setminus S_i$ is not a vertex cut, then we have $c(G-S) \leq 2k+1$. This implies that $\frac{|S|}{c(G-S)} \geq \frac{4k+1}{2k+1} \geq \frac{5}{3}$, which contradicts to Lemma 5.3. If $S \setminus S_i$ is still a vertex cut, then by Lemma 5.3, it follows that

$$\frac{3}{2} > \frac{6k+1}{4k+2} \geq \frac{|S|}{c(G-S)} \geq \frac{|S \setminus S_i| + 4k + 1}{c(G-(S\setminus S_i)) + 2k}.$$

By rearranging the formula, we obtain

$$\begin{aligned} |S \setminus S_i| \cdot c(G-S) &\leq |S| \cdot c(G-(S\setminus S_i)) + 2k|S| - (4k+1)\cdot c(G-S) \\ &< |S| \cdot c(G-(S \setminus S_i)). \end{aligned}$$

This implies that $\frac{|S\setminus S_i|}{c(G-(S\setminus S_i))} < \frac{|S|}{c(G-S)}$, which is a contradiction. □

Lemma 5.5. *Let n, k be positive integers with $n \geq 3$ and $G = P_n \boxtimes K_{k,4k+1}$. Let S be a vertex cut of G that attains the minimum value of $\frac{|S|}{c(G-S)}$ among all the vertex cuts. For every $y \in Y$, we have that $1_y, n_y \notin S$. Moreover, if $i_y \in S$, then $(i-1)_y, (i+1)_y \notin S$.*

Proof. We only prove that $1_y \notin S$ for every $y \in Y$, and $n_y \notin S$ can be shown analogously. Suppose on the contrary that $1_y \in S$ for some $y \in Y$. By Lemma 5.4, we have that $X_1 \subseteq S$. Let $S' = S \setminus \{1_y\}$. Then $N_{G-S'}(1_y) \subseteq \{2_y\} \cup X_2$. If follows that either $N_{G-S'}(1_y) = \emptyset$ or

$G[N_{G-S'}(1_y)]$ is connected. This implies that $c(G-S) \leq c(G-S')$. Since $|S'| = |S|-1 < |S|$, we have that

$$\frac{|S'|}{c(G-S')} < \frac{|S|}{c(G-S)},$$

which is a contradiction.

Next, we suppose that $i_y, (i+1)_y \in S$ for some $i \in \{2, \ldots, n-1\}$ and $y \in Y$. Let $\hat{S} = S \setminus \{i_y\}$. By Lemma 5.4, we have that $X_i \subseteq S$ and $X_{i+1} \subseteq S$. Notice that $N_{G-\hat{S}}(i_y) \subseteq \{(i-1)_y\} \cup X_{i-1}$ for $i \geq 2$. By a similar argument above, we can easily show that

$$\frac{|\hat{S}|}{c(G-\hat{S})} < \frac{|S|}{c(G-S)},$$

which is a contradiction. □

Lemma 5.6. *Let k, n be integers with $n \geq \frac{40k^2+22k+3}{6k+1}$. Then $t(P_n \boxtimes K_{k,4k+1}) \geq \frac{6k+1}{4k+2}$.*

Proof. Let $G = P_n \boxtimes K_{k,4k+1}$ and S be the vertex cut of G that attains the minimum value of $\frac{|S|}{c(G-S)}$ among all the vertex cut. Let $S_i = V_i \cap S$ for each i. By Lemma 5.4, we have that either $S_i = \emptyset$ or $X_i \subseteq S_i$ for each i. Let $R = \{i \mid S_i \neq \emptyset\}$. If $R = \{1, 2, \ldots, n\}$, then the number of components of $G - S$ is at most $4k+1+m$ where $m = |S \cap (R \times Y)|$. This value is derived from the fact that $G - (R \times X)$ is a graph of $4k+1$ paths of length n and deleting one vertex of this graph can yield at most one more component. By Lemma 5.5, we must have $m \leq (4k+1)\left\lfloor \frac{n-1}{2} \right\rfloor$. It follows that

$$\frac{|S|}{c(G-S)} \geq \frac{nk+m}{4k+1+m} \geq \frac{nk+(4k+1)\left\lfloor \frac{n-1}{2} \right\rfloor}{4k+1+(4k+1)\left\lfloor \frac{n-1}{2} \right\rfloor} \geq \frac{6k+1}{4k+2}.$$

Note that the second inequality is obtained from the fact that $nk > 4k+1$, and the last inequality holds for $n \geq \frac{40k^2+22k+3}{6k+1}$.

On the other hand, suppose that $R \subsetneq \{1, 2, \ldots, n\}$. We define R_1, R_2, ..., R_s as follows. First, let each $R_i \neq \emptyset$ consists of consecutive integers in $\{0, n+1\} \cup R$ such that $\max(R_j) <$

min$(R_{j+1}) - 1$ for $j = 1, 2, \ldots, s-1$ and

$$\bigcup_{j=1}^{s} R_j = \{0, n+1\} \cup R.$$

Then we delete 0 from R_1 and $n+1$ from R_s. Let $r_j = |R_j|$ and $m_{jy} = |S \cap (R_j \times \{y\})|$ for $j = 1, 2, \ldots, s$ and $y \in Y$. Furthermore, let α_j be the number of y such that $m_{jy} \neq 0$ and $m_j = \sum_{y \in Y} m_{jy}$. By Lemma 5.5, we have that $m_1 \leq \lfloor \frac{r_1}{2} \rfloor$, $m_s \leq \lfloor \frac{r_s}{2} \rfloor$, and $m_j \leq \alpha_j \left\lfloor \frac{r_j+1}{2} \right\rfloor$ for $j = 2, 3, \ldots, s-1$. Observe that for the graph $G - (R \times X)$, deleting one vertex from $R_1 \times Y$ or $R_s \times Y$ can yield at most one more component. Similarly, for each $j = 2, 3, \ldots, s-1$ and $y \in Y$, deleting q vertices from $R_j \times \{y\}$ can yield at most $q - 1$ more component. Furthermore, if we delete at least one vertex from $R_j \times \{y\}$ for every $y \in Y$, then we obtain one more component. Therefore, the total number of components in $G - S$ is at most

$$1 + m_1 + \sum_{j=2}^{s-1} \left(m_j - \alpha_j + \mathbf{1}_{[\alpha_j = 4k+1]}\right) + m_s,$$

where **1** is the indicator function. Notice that $|S| = \sum_{j=1}^{s} r_j k + m_j$. We claim that

$$P_j := \frac{r_j k + m_j}{1 + m_j - \alpha_j + \mathbf{1}_{[\alpha_j = 4k+1]}} \geq \frac{6k+1}{4k+2}$$

for $j = 2, 3, \ldots, s-1$. If $\alpha_j = 4k+1$, then

$$P_j = \frac{r_j k + m_j}{2 + m_j - 4k - 1}.$$

It is clear that this value attains minimum when m_j attains its maximum, i.e, $m_j = (4k + 1) \left\lfloor \frac{r_j+1}{2} \right\rfloor$. It follows that

$$P_j \geq \frac{r_j k + (4k+1) \left\lfloor \frac{r_j+1}{2} \right\rfloor}{2 + (4k+1) \left\lfloor \frac{r_j-1}{2} \right\rfloor} = \begin{cases} \frac{(6k+1)r_j}{(4k+1)r_j - (8k-2)} & , r_j \text{ even} \\ \frac{(6k+1)r_j + (4k+1)}{(4k+1)r_j - (4k-3)} & , r_j \text{ odd.} \end{cases}$$

This value is larger than $\frac{6k+1}{4k+1} > \frac{6k+1}{4k+2}$.

If $\alpha_j < 4k+1$, then we must have $r_j \geq 3$ and $\alpha_j \geq 1$; otherwise,

$$c(G-S) = c(G-(S \setminus (R_j \times V(K_{k,4k+1})))),$$

which contradicts to the definition of S. Similarly, P_j has the minimum value when $m_j = \alpha_j \left\lfloor \frac{r_j+1}{2} \right\rfloor$. Hence, we have

$$P_j \geq \frac{r_j k + \alpha_j \left\lfloor \frac{r_j+1}{2} \right\rfloor}{1+\alpha_j \left\lfloor \frac{r_j-1}{2} \right\rfloor} = \begin{cases} \frac{2r_jk+\alpha_j r_j}{\alpha_j r_j - 2(\alpha_j - 1)} & , r_j \text{ even} \\ \frac{2r_jk+\alpha_j(r_j+1)}{\alpha_j(r_j+1)-2(\alpha_j-1)} & , r_j \text{ odd.} \end{cases}$$

This value is larger than $\frac{3}{2} > \frac{6k+1}{4k+2}$. By a similar argument, we can prove that $P_1 := \frac{r_1k+m_1}{1+m_1} \geq \frac{6k+1}{4k+2}$ when $R_1 \neq \emptyset$ and $P_s := \frac{r_sk+m_s}{1+m_s} \geq \frac{6k+1}{4k+2}$ when $R_s \neq \emptyset$. By above, we have that

$$\frac{|S|}{c(G-S)} \geq \frac{\sum_{j=1}^{s} r_jk + m_j}{1 + m_1 + \sum_{j=2}^{s-1} \left(m_j - \alpha_j + \mathbf{1}_{[\alpha_j = 4k+1]}\right) + m_s} \geq \frac{6k+1}{4k+2}.$$

□

Finally, we prove Theorem 1.4

Proof of Theorem 1.4. By Lemma 5.3 and Lemma 5.6, we have that $t(P_n \boxtimes K_{k,4k+1}) = \frac{6k+1}{4k+2}$ for $n \geq \frac{40k^2+22k+3}{6k+1}$. In addition, by Lemma 5.2, $P_n \boxtimes K_{k,4k+1}$ is not Hamiltonian. Notice that $P_n \boxtimes K_{k,4k+1} \in \mathcal{G}$ for n large. Since

$$\lim_{k\to\infty} \frac{6k+1}{4k+2} = \frac{3}{2},$$

there exists t-tough graph in $\mathcal{G}$ for any $t < \frac{3}{2}$. By the definition of Hamiltonian toughness, we have that $\mathrm{Ht}(\mathcal{G}) \geq \frac{3}{2}$.

On the other hand, by Corollary 1.2, every $\frac{3}{2}$-tough graph in $\mathcal{G}$ is Hamiltonian. Consequently, $\mathrm{Ht}(\mathcal{G}) \leq \frac{3}{2}$. □

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