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National Yang Ming Chiao Tung University

Master Thesis

平面圖的最大特徵值之研究

The Largest Eigenvalue of a Planar Graph

研究生：許峻彥(Chun-Yen Hsu)

指導教授：翁志文(Chih-wen Weng)

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平面圖的最大特徵值之研究

學生：許峻彥　　　　　　指導教授：翁志文 博士

國立陽明交通大學 應用數學系 碩士班

摘　　要

令 G 爲點數爲 n、最小度數 5 的最大平面圖。我們證明對任一點而言，與其距離不小於 2 的點所形成的子圖必含一點數 6 以上的輪圈圖。我們利用此發現以證明了 G 的最大特徵值 $\rho(G)$ 必滿足不等式 $\rho(G) \leq 2 + \sqrt{2n-11}$。

關鍵字： 最大平面圖, 最小度數, 特徵值

The Largest Eigenvalue of a Planar Graph

Student : Chun-Yen Hsu　　Advisor: Chih-wen Weng

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Abstract

Let G be a maximal planar graph of order n with a minimum degree of 5. We find a wheel of order at least 6 in the subgraph induced on the subset of vertices with distances at least 2 from every vertex in G. We leverage this property to show that the largest eigenvalue $\rho(G)$ of G satisfies $\rho(G) \leq 2 + \sqrt{2n - 11}$.

Keywords: maximal planar graph, minimum degree, eigenvalue.

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1 Introduction

Algebraic graph theory is a branch of mathematics that analyzes graphs through the algebraic properties of associated matrices. Specifically, spectral graph theory examines the relationship between graph properties and the eigenvalues and eigenvectors of matrices such as the adjacency matrix and Laplacian matrix. This approach has proven to be highly effective in the study of graphs.

In 1978, Schwenk and Wilson [12] proposed studying the eigenvalues of planar graphs. Around the same time, geographers began to consider the largest eigenvalue of planar graphs as a measure of the overall connectivity of a planar network. The first significant result was presented by Hong [5], who shows that for a planar graph of order n, the largest eigenvalue is at most $\sqrt{5n-11}$. This bound was later improved to $4+\sqrt{3n-9}$ by Dasong Cao and Andrew Vince [10], who also conjectured that the planar graph of order n with the largest spectral radius is K_2+P_{n-2}. In 2017, Michael Tait and Josh Tobin [11] proved this conjecture for sufficiently large n.

In this thesis, we focus on the maximal planar graph with minimum degree 5. It is well known that a simple planar graph of order n has at most $3n-6$ edges and does not contain $K_{3,3}$, K_5 and their subdivisions. For a planar graph with a minimum degree 5, each vertex is adjacent to at least five other vertices, significantly enhancing the graph's connectivity and complexity. This makes it difficult to describe its structure succinctly. However, planar graphs follow Euler's formula, which relates the number of vertices, edges, and faces, providing a theoretical basis for understanding the structure of the graph.

Mathematically, the largest eigenvalue can be determined by analyzing the eigenvalues of a matrix. For large or complex graphs, directly computing eigenvalues may not be feasible. In such cases, iterative methods such as the power method or Arnoldi method [13] can be used to approximate the largest eigenvalues. Additionally, by leveraging specific properties of the graph, such as the degree sequence and cycle structure, bounds or estimates on the largest eigenvalue can be established. In this paper, we use the rowsum of the adjacency matrix to determine the upper bounds on the largest eigenvalue, further exploring the effectiveness and applications of this method.

The main result of this thesis is an upper bound on the largest eigenvalues of planar graphs with a minimum degree of 5. Specifically, we demonstrate the following theorem.

Theorem 1.1. *If G is a maximal planar graph of order n with $\delta(G) = 5$, then $\rho(G) \leq 2 + \sqrt{2n - 11}$.*

The remainder of this thesis is organized as follows. Section 2 introduces the necessary notations, definitions, and preliminary concepts. Section 3 presents an upper bound for planar graphs. Section 4 focuses specifically on planar graphs with a minimum degree of 5.

2 Notation and Preliminaries

2.1 Graphs

In this subsection, we review some necessary terminologies on graph theory. A *graph* G is a triple consisting of a *vertex set* $V(G)$, an *edge set* $E(G)$, and a relation that associates with each edge e, with two vertices i and j called its *endpoints*. An edge is called a *loop* if two endpoints are the same. Two edges e and e' are called *multiple-edges* if they have the same endpoints. A *simple graph* is a graph without loops and multi-edges. When G is a simple graph, we use the notation ij to denote an edge associated with vertices i and j in G.

In this thesis, all graphs are simple of order n and have vertex set $\{1, 2, \ldots, n\}$, unless specified otherwise. We use the following descriptions interchangeably: i and j are *endpoints* of e; the edge e is *incident* to i and j; and i and j are *adjacent*. The *order* of a graph G is $|V(G)|$, and the *size* of G is $|E(G)|$. The *neighborhood* $N_G(i)$ of the vertex i in G is the set of all vertices that are adjacent to i. The *closed neighborhood* $N_G[i]$ of the vertex i in G is the set $N_G(i) \cup \{i\}$. The *degree* $\deg_G(i)$ of the vertex i is the number of edges incident to i. A subgraph H of G is a graph with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. For subset $X \subseteq V(G)$, the graph $G[X]$ with vertex set $V(G[X]) = X$ and edge set $E(G[X]) = \{ij \in E(G) \mid i, j \in X\}$ is called the *induced subgraph* of G on X.

A walk W of length k from i_0 to i_k in G is a sequence $i_0 i_1 \cdots i_k$ of vertices in G such that $i_j i_{j+1} \in E(G)$ for $j = 0, 1, \ldots, k-1$; W is *closed* if $i_0 = i_k$; and W is a *path* if i_j are all distinct. A graph G is *connected* if for all $i, j \in V(G)$ then exists a path from i to j. The vertex set $V(G)$ of G is partitioned into maximal subsets such that the induced subgraph on each is connected, called a *component* of G. For a connected graph G, an edge e is a *cut edge* if the graph $G - e$ has two components, where $G - e$ is the subgraph of G with $V(G - e) = V(G)$ and

$E(G-e) = E(G) - \{e\}$. If G is connected and $G-e$ has components $\mathcal{C}_1$ and $\mathcal{C}_2$, the cut edge e of G is *incident to* $\mathcal{C}_1$ and $\mathcal{C}_2$.

For $X \subseteq V(G)$, let $e(X) = |E(G[X])|$. Similarly, for disjoint vertex sets $X, Y \in V(G)$, let $E(X,Y)$ denote the set of edges with one endpoint in X and the other in Y, and $e(X,Y) = |E(X,Y)|$. The following five classes of graphs P_n, C_n, K_n, $K_{s,n-s}$, $\mathcal{W}_n$ of order n with according edge sets

$$
\begin{aligned}
&E(P_n) = \{i(i+1) \mid i \in \{1,2,\ldots,n-1\}\},\\
&E(C_n) = E(P_n) \cup \{1n\},\\
&E(K_n) = \{ij \mid i<j,\ i,j \in \{1,2,\ldots,n\}\},\\
&E(K_{s,n-s}) = \{ij \mid i \in \{1,2,\ldots,s\},\ j \in \{s+1,s+2,\ldots,n\}\},\\
&E(\mathcal{W}_n) = E(C_{n-1}) \cup \{in \mid i \in \{1,2,\ldots n-1\}\},
\end{aligned}
$$

are called the *path*, *cycle*, *complete graph*, *complete bipartite graph*, and *wheel*, respectively. A *Hamiltonian* graph G is a graph that has a cycle that visits every vertex of G exactly once and returns to the starting vertex. For graphs G_1 and G_2 with disjoint vertex set $V(G_1)$ and $V(G_2)$, we use $G = G_1 + G_2$ to denote the graph *joined* by G_1 and G_2, where G has vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{ij \mid i \in V(G_1), j \in V(G_2)\}$. The graph $P_2 + C_6$ is depicted in Figure 5.

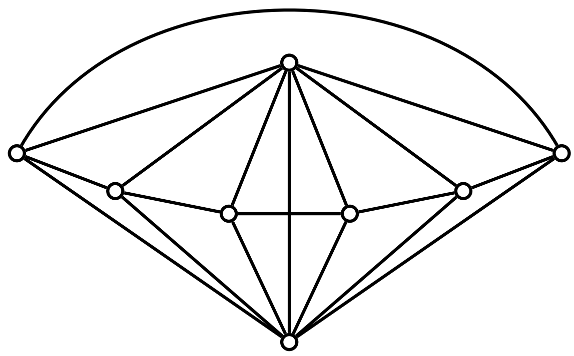

Figure 1: $P_2 + C_6$.

2.2 Graph spectrum

In this subsection, we define some matrices associated with graphs. Let A be a real symmetric matrix. It is well known that all eigenvalues of A are real.

Definition 2.1. The *adjacency matrix* of a graph G of order n is an $n \times n$ binary matrix $A = (a_{ij})$ indexed by the vertex set $V(G)$ of G such that $a_{ij} = 1$ if and only if $ij \in E(G)$.

The *spectral radius* of A, denoted by $\rho(A)$, is the maximum absolute value of its eigenvalues, and the *spectral radius* of G, denoted by $\rho(G)$, is the largest eigenvalue of its adjacency matrix. The following definition appears frequently in the study of symmetric matrices.

Definition 2.2. ([3]). Let A be an $n \times n$ real symmetric matrix and x is a nonzero column vector of size n. The *Rayleigh quotient* of A with respect to x is

$$R_A(x) = \frac{x^\top Ax}{x^\top x}.$$

Proposition 2.3. *[1, Theorem 2.4.1] Let A be an $n \times n$ real symmetric matrix. Then*

$$\min_{x \in \mathbb{R}^n \setminus \{\mathbf{0}\}} R_A(x), \qquad \max_{x \in \mathbb{R}^n \setminus \{\mathbf{0}\}} R_A(x)$$

are the least eigenvalue and largest eigenvalue of A, respectively.

For a real symmetric matrix A, we say $A \geq 0$ if every entry of A is nonnegative, and say A is irreducible if after any reordering of the indices we have

$$A \neq \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix},$$

where A_1 and A_2 are square matrices with sizes less than the size of A.

The following famous theorem is called the Perron-Frobenius Theorem.

Theorem 2.4. *[2, Theorem 6.3] Let A be an irreducible $n \times n$ symmetric matrix with $A \geq 0$. Then the following $(1) - (2)$ hold.*

1. *There is an eigenvector of A with respect to $\rho(A)$ that is entrywisely positive.*

2. *If v is an eigenvector of A that is entrywisely positive, then v is an eigenvector with respect of $\rho(A)$.*

Corollary 2.5. *Let G be a graph and $G + e$ be the graph obtained from G by adding an edge $e \notin E(G)$. Then $\rho(G) \leq \rho(G + e)$. Moreover, the strict inequality holds when G is connected.*

Proof. Let A be the adjacency matrix of G and v be the nonnegative eigenvector of A with respect to the eigenvalue $\rho(G)$ and $v^\top v = 1$. Let A' be the adjacency matrix of $G + e$. Then by Proposition 2.3, we have

$$\rho(G) = v^\top A v \leq v^\top A' v \leq \rho(G + e).$$

When G is connected, A is irreducible and v is a positive vector by theorem 2.4. Then $v^\top A v < v^\top A' v$. □

Let $A = (a_{ij})$ be an $n \times n$ real matrix. The value

$$s_i(A) = \sum_{j=1}^{n} a_{ij}$$

is called the ith *rowsum* of A. Note that $s_i(A) + s_i(B) = s_i(A + B)$ for $n \times n$ matrices.

Lemma 2.6. *[7, Lemma 2.1] If $A = (a_{ij})$ be $n \times n$ real symmetric matrix with an eigenvalue λ associated with a nonnegative eigenvector v, then*

$$\min_{1 \leq i \leq n} s_i(A) \leq \lambda \leq \max_{1 \leq i \leq n} s_i(A). \tag{2.1}$$

Moreover, if A is irreducible then strict inequality holds in (2.1) *if and only if there exist distinct i, j such that $s_i(A) \neq s_j(A)$.*

Proof. Let v be a nonnegative eigenvector of A. Since $A = (a_{ij})$ is a real symmetric matrix, $Av = vA = \lambda v$ and $a_{ij} = a_{ji}$. Suppose that $v = [i, v_2, \ldots, v_n]^\top$ and $\sum_{i=1}^{n} v_i = 1$. Then we have

$$\lambda = \lambda \sum_{i=1}^{n} v_i = \sum_{i=1}^{n} (Av)_i = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} v_j = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ji} v_j = \sum_{j=1}^{n} \sum_{i=1}^{n} a_{ji} v_j = \sum_{j=1}^{n} s_j(A) v_j.$$

Since $\sum_{i=1}^{n} v_i = 1$, λ is the convex combination of $s_i(A)$ for i in $\{1, 2, \ldots, n\}$. The proof of the first statement is complete. The second statement follows since $v > 0$ by the irreducible assumption of A. □

Lemma 2.7. *[7, Lemma 2.2] Let G be a connected graph of order n and A its adjacency matrix. If $p(x)$ is a real polynomial, then*

$$\min_{i \in V(G)} s_i(p(A)) \leq p(\rho(G)) \leq \max_{i \in V(G)} s_i(p(A)).$$

The strict inequality holds if there exists i, j such that $s_i(A) \neq s_j(A)$.

Proof. Let v be a positive eigenvector of A associated with the eigenvalue $\rho(G)$. Then v is also an eigenvector of $p(A)$ associated with the eigenvalue $p(\rho(G))$. By Lemma 2.6, the proof is complete. □

2.3 Planar graphs

A graph G is said to be *planar* if it has a drawing on a plane without edges crossing. We will fix such a drawing when studying a planar graph G. The edges of G divide the plane into maximal open connected sets, called *faces*, including a unique unbounded region called *unbounded face*, and the remaining faces are called *bounded faces*. Let H be a subgraph of G in the same drawing, and F is a face in H.

The edge in $E(H)$ (resp. in $E(G)$) with F appearing in at least one side is called a *boundary edge* of F in H (resp. in G). Let $E_H(F)$ and $E_G(F)$ denote the set of boundary edges of F in H and G, respectively. Notice that the above F is a union of some faces in G, not necessarily a face in G. Similarly, a *boundary vertex* of F is a vertex in $V(H)$ (resp. in $V(G)$) which is an endpoint of an edge in $E_H(F)$ or inside the face F, and let $V_H(F)$ (resp. $V_G(F)$) denote the set of boundary vertices in H (resp. in G). Two different faces F and F' in H are *adjacent* if $E_H(F) \cap E_H(F') \neq \emptyset$. Now we assume H is connected. Then there is a closed walk $W(F)$ that *encloses* the face F. Indeed, every edge $e \in E_G(F)$ appears once or twice according to F appearing in one side or two sides of e.

An *outerplanar graph* is a planar graph that has a drawing on the plane such that all vertices are the boundary vertices of a face. It is not necessary but for an easier realization, one might want to depict all the vertices of an outerplanar graph in the unbounded face. By abuse of the notation, we might also call the face with all vertices in it the unbounded face and the remaining faces the bounded faces of an outerplanar graph. It is well known that a planar graph of order n has at most $3n-6$ edges and does not contain $K_{3,3}$, K_5 and their subdivisions [8, Theorem 6.6.6]. Since $G + K_1$ is a planar graph for any outerplanar graph G of order n, we have $|E(G)| + n \leq 3(n+1) - 6$. Hence an outerplanar graph of order n has at most $2n-3$ edges.

2.4 Maximal planar graph

We will provide more basic properties of planar graphs in this subsection.

Definition 2.8. A planar graph G is said to be *maximal* if the addition of any edge incident to 2 nonadjacent vertices in G results in a nonplanar graph.

Note that a maximal planar graph of order $n \geq 3$ has $3n-6$ edges and every face is a triangle [8, Proposition 6.1.26]. A maximal planar graph of order 13 is depicted in Figure 2.

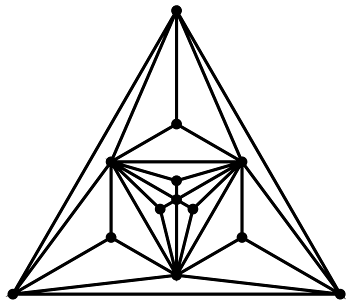

Figure 2: A maximal planar graph of order 13.

Lemma 2.9. *If G is a connected planar graph with the maximum spectral radius among planar graphs of order n, then G is a maximal planar graph.*

Proof. Suppose that G is not a maximal planar graph. Then we can add an edge to G such that G is still planar, which contradicts to Proposition 2.5. □

Lemma 2.10. *If G is a maximal planar graph of order $n > 3$, then $G[N_G(i)]$ has a Hamiltonian cycle that encloses the unbounded face of $G[N_G(i)]$ for all $i \in V(G)$. In particular, $G[N_G(i)]$ is an outerplanar graph and $G[N_G[i]]$ contains a wheel of order $\deg_G(i) + 1$.*

Proof. Since $n > 3$, we have $\sum_{i\in V(G)} \deg(i) = 6n - 12$. Hence, we have $\deg_G(i) \geq 3$ for all $i \in V(G)$. The planar graph $G - i$ has a face F that merges all triangles with the common vertex i in G. Observe that a vertex j is in the boundary of F if and only if $j \in N_G(i)$. Listing the vertices in the boundary of F in clockwise order will yield a Hamiltonian cycle in $G[N_G(i)]$. The remaining two statements are clear. □

3 The spectral radii bounds of planar graphs

We provide some basic spectral radii bounds of a planar graph of order n in this section. Recall that for a matrix A, $s_i(A)$ is the ith rowsum of A

Lemma 3.1. *If G is a maximal planar graph of order n with adjacency matrix $A = (a_{ij})$, then*

$$s_i(A^2) \leq 3n + 2s_i(A) - 9$$

for all $i \in V(G)$.

Proof. Let $\overline{N_G[i]} = V(G) \setminus (N_G[i])$. It is known that

$$s_i(A^2) = \sum_{j=1}^{n} \left(\sum_{k=1}^{n} a_{ik} a_{kj} \right) = \sum_{k \in N_G(i)} s_k(A) = s_i(A) + 2e(N_G(i)) + e\left(N_G(i), \overline{N_G[i]}\right). \tag{3.1}$$

Since the graph G is a maximal planar graph, we have

$$3n - 6 = |E(G)| = s_i(A) + e(N_G(i)) + e\left(N_G(i), \overline{N_G[i]}\right) + e\left(\overline{N_G[i]}\right). \tag{3.2}$$

Since the graph $G[N_G(i)]$ is an outerplanar graph of order $s_i(A)$, we have

$$0 \leq e(N_G(i)) \leq 2s_i(A) - 3. \tag{3.3}$$

By (3.1)–(3.3), we have

$$s_i(A^2) = 3n - 6 + e(N_G(i)) - e\left(\overline{N_G[i]}\right) \leq 3n + 2s_i(A) - 9. \tag{3.4}$$

□

Based on Lemma 2.7 and Lemma 3.1, we have an upper bound of $\rho(G)$.

Theorem 3.2. *If G is a maximal planar graph of order n, then*

$$\rho(G) \leq 1 + \sqrt{3n - 8}.$$

Proof. Let A be the adjacency matrix of G. Applying the polynomials $p(x)$ to be $x^2 - 2x$ in Lemma 2.7, we have

$$\rho(G)^2 - 2\rho(G) \leq \max_{i \in V(G)} s_i(A^2 - 2A).$$

By Lemma 3.1, we have $s_i(A^2) \leq 3n + 2s_i(A) - 9$, and then by Lemma 2.7,

$$s_i(A^2 - 2A) = s_i(A^2) - 2s_i(A) \leq 3n - 9$$

for all $i \in V(G)$ to have

$$\max_{i \in V(G)} s_i(A^2 - 2A) \leq 3n - 9,$$

which implies the inequality $\rho(G) \leq 1 + \sqrt{3n - 8}$. □

Theorem 3.3. *If G is a maximal planar graph of order n, then*

$$\rho(G) \leq 2 + \sqrt{2n - 6}.$$

Proof. Fix $i \in V(G)$. The graph $G\left[\overline{N_G[i]}\right]$ has at most $\deg_G(i) - 2$ components. Hence

$$e\left(\overline{N_G[i]}\right) \geq \left|\overline{N_G[i]}\right| - \deg_G(i) + 2 = n - 2\deg_G(i) + 1.$$

Referring to (3.4) in the proof of Lemma 3.1, we have the following better bound,

$$\begin{aligned} s_i(A^2) =& 3n - 6 + e(N_G(i)) - e\left(\overline{N_G[i]}\right) \\ \leq& 3n + 2s_i(A) - 9 - (n - 2\deg_G(i) + 1) \\ =& 2n + 4s_i(A) - 10. \end{aligned}$$

Applying the polynomial $p(x) = x^2 - 4x$ in Lemma 2.7, we have

$$\rho(G)^2 - 4\rho(G) \leq \max_{i \in V(G)} s_i(A^2 - 4A) \leq 2n - 10,$$

which implies $\rho(G) \leq 2 + \sqrt{2n-6}$. □

A little modification of the proof of Theorem 3.2 yields more results.

Theorem 3.4. *If G is a planar graph with minimum degree $\delta(G)$, then*

$$\rho(G) \leq \frac{3}{2} + \sqrt{3n - \delta(G) - \frac{27}{4}}.$$

Proof. As before from $s_i(A^2) \leq 3n + 2s_i(A) - 9$ by Lemma 3.1, we have

$$s_i(A^2 - 3A) \leq 3n - s_i(A) - 9 \leq 3n - \delta(G) - 9.$$

By Lemma 2.7, $\rho(G)^2 - 3\rho(G) \leq 3n - s_i(A) - 9 \leq 3n - \delta(G) - 9$, so

$$\rho(G) \leq \frac{3}{2} + \sqrt{3n - \delta(G) - \frac{27}{4}}.$$

□

The following is a conjecture of Boots-Royle in 1991 [9] and independently Cao-Vince in 1993 [10]. Michael Tait and Josh Tobin proved it when n is large enough in 2017 [11].

Conjecture 3.5. *For every $n \geq 9$, the planar graph of order n with maximum spectral radius is the graph $P_2 + P_{n-2}$.*

According to Conjecture 3.5, we aim to compute the spectral radius of the graph P_2+P_{n-2}. However, calculating the spectral radius of graph P_2+P_{n-2} is quite challenging. Observe that the graph P_2+P_{n-2} is closed to P_2+C_{n-2}, suggesting that they may share closed eigenvectors. Furthermore, the eigenvector of $P_2 + C_{n-2}$ can be determined using the technique of equitable quotient matrices [1, Corollary 2.5.4]. Specifically, we have the following lemma.

Lemma 3.6. *Let G be the graph $P_2 + C_{n-2}$. Then the eigenvalue $\rho(G)$ with corresponding eigenvector v is*

$$\rho(G) = \frac{3}{2} + \sqrt{2n - \frac{15}{4}}, \quad v = \left[\frac{1}{2} - \sqrt{2n - \frac{15}{4}}, \frac{1}{2} - \sqrt{2n - \frac{15}{4}}, -2, \dots, -2\right]^\top. \tag{3.5}$$

Proof. Let A be the adjacency matrix of G and Q its quotient matrix, where

$$A = \left[\begin{array}{cc|ccc} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ \hline 1 & 1 & & & \\ \vdots & \vdots & & A(C_{n-2}) & \\ 1 & 1 & & & \end{array}\right], \quad \text{and} \quad Q = \begin{bmatrix} 1 & n-2 \\ 2 & 2 \end{bmatrix}.$$

Then Q has the eigenvector $u = \left[\frac{1}{2} - \sqrt{2n - \frac{15}{4}}, -2\right]^\top$ with respect to $\rho(Q) = \frac{3}{2} + \sqrt{2n - \frac{15}{4}}$. Since Q is the equitable quotient matrix of A, A has the same spectral radius with Q and corresponding eigenvector is

$$v = \left[\frac{1}{2} - \sqrt{2n - \frac{15}{4}}, \frac{1}{2} - \sqrt{2n - \frac{15}{4}}, -2, \cdots, -2\right]^\top$$

is the eigenvector with respect to $\rho(A)$ of A. □

Lemma 3.6 provides the exact formula of the eigenvector v of $A(P_2 + C_{n-2})$. Substituting the v into the Rayleigh quotient of $A(P_2 + P_{n-2})$ allows us to estimate the lower bound of the spectral radius of $P_2 + P_{n-2}$.

Theorem 3.7. *If G is a planar graph with the maximal spectral radius among planar graphs of order n, then*

$$\rho(G) \geq \frac{3}{2} + \sqrt{2n - \frac{15}{2}} - \frac{8}{8n - 15 + \sqrt{8n-15}}.$$

Proof. Let G be the graph $P_2 + P_{n-2}$, A its adjacency matrix and $R_A(v)$ its Rayleigh quotient with respect to nonzero column vector

$$v = \left[\frac{1}{2} - \sqrt{2n - \frac{15}{4}}, \frac{1}{2} - \sqrt{2n - \frac{15}{4}}, -2, \ldots, -2\right]^\top$$

appearing in (3.5). Note that $v^\top v = 8n - 15 - \sqrt{8n-15}$. Let $3n$ be the additional edge in $P_2 + C_{n-2}$ from $P_2 + P_{n-2}$, and E_{ij} be the binary matrix with a unique 1 in the ij position. By Proposition 2.3,

$$\begin{aligned}
\rho(G) = &\max_{x \in \mathbb{R}^n \setminus \{\mathbf{0}\}} R_A(x) \geq \frac{v^\top A v}{v^\top v} = \frac{v^\top \left(A(P_2 + C_{n-2}) - E_{3n} - E_{n3}\right) v}{v^\top v} \\
= &\frac{\rho(P_2 + C_{n-2}) v^\top v - 8}{8n - 15 - \sqrt{8n-15}} \\
= &\frac{3}{2} + \sqrt{2n - \frac{15}{4}} - \frac{8}{8n - 15 - \sqrt{8n-15}}.
\end{aligned}$$

□

4 The maximal planar graphs with degree restrictions

Let G be a maximal planar graph of order n. In (3.4) of the proof of Lemma 3.1, we let $e\left(\overline{N_G[i]}\right) = 0$ to have the equality

$$s_i(A^2) = 3n - 6 + e(N_G(i)) - e\left(\overline{N_G[i]}\right) \leq 3n + 2s_i(A) - 9.$$

In this section, we give some assumptions on the degrees of vertices in G to have $e\left(\overline{N_G[i]}\right) > 0$ for all $i \in V(G)$ and have upper bounds of $\rho(G)$. We start from the range of $e(N_G(i))$ first.

Lemma 4.1. *If G is a maximal planar graph and $i \in V(G)$, then $G[N_G(i)]$ is an outerplanar graph, and* $\deg_G(i) \leq e(N_G(i)) \leq 2\deg_G(i) - 3$.

Proof. Since $G[N_G(i)]$ contains a Hamiltonian cycle C that encloses the unbounded face of $G[N_G(i)]$ by Lemma 2.10, $G[N_G(i)]$ is an outerplanar graph of order $\deg_G(i)$ with at least $\deg_G(i)$ edges, the number of edges in C. It is known that an outerplanar graph of order n has at most $2n - 3$ edges. $\square$

Let C be the Hamiltonian cycle in $G[N_G(i)]$ such that C encloses the unbounded face of $G[N_G(i)]$. Next lemma considers a very special case that $e(N_G(i)) = \deg_G(i)$. See the graph depicted in Figure 3.

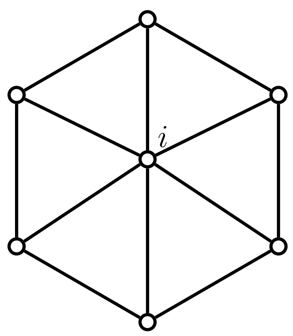

Figure 3: The graph $G(N_G[i])$ with $\deg_G(i) = 6$.

Lemma 4.2. *If G is a maximal planar graph of order n and $i \in V(G)$ such that $e(N_G(i)) = \deg_G(i)$, then*

$$3n - 6 - (\Delta(G) - 1)\deg_G(i) \le e\left(\overline{N_G[i]}\right) \le 3n - 6 - (\delta(G) - 1)\deg_G(i).$$

Proof. Since every vertex in $N_G(i)$ has degree 3 in $G[N_G[i]]$, each vertex in $N_G(i)$ is incident on at least $\delta(G) - 3$ and at most $\Delta(G) - 3$ edges in $E\left(N_G(i), \overline{N_G[i]}\right)$. Hence

$$(\delta(G) - 3)\deg_G(i) \le e\left(N_G(i), \overline{N_G[i]}\right) \le (\Delta(G) - 3)\deg_G(i).$$

Thus

$$\begin{aligned} 3n - 6 - (\Delta(G) - 1)\deg_G(i) \le & e\left(\overline{N_G[i]}\right) = |E(G)| - e(N_G[i]) - e\left(N_G(i), \overline{N_G[i]}\right) \\ \le & 3n - 6 - (\delta(G) - 1)\deg_G(i). \end{aligned}$$

□

If $e(N_G(i)) > \deg_G(i)$, then there is a *chord* for C, i.e., an edge with endpoints not consecutive in C. The next two lemmas study this case in a more general setting. Let H be an outerplanar graph with the unbounded face enclosed by a cycle C. A *side face* of H is a bounded face that has at most two adjacent faces. If F is a side face of H then $E_H(F)$ contains at most one chord for C. Hence we have the following lemma.

Lemma 4.3. *If H is an outerplanar graph with the unbounded face enclosed by a cycle C and F is a side face of H, then every vertex in $V_H(F)$ has degree 2 except at most two vertices in a chord for C.*

Lemma 4.4. *If H is an outerplanar graph with the unbounded face enclosed by a cycle C and at least one chord for C, then H has at least two side faces.*

Proof. Let $C = v_1v_2v_3 \cdots v_nv_1$ and prove the lemma by induction on n. In the base case $n = 4$, there are at most two chords, v_1v_3 and v_2v_4 for C, say $v_1v_3 \in$

$E(H)$. Then $v_1v_2v_3v_1$ and $v_1v_3v_4v_1$ are the two desired faces. In general, suppose $n \geq 4$ and v_iv_j is a chord for C, where $i < j-1$. Then we find two cycles $C_1 = v_1v_2\cdots v_iv_jv_{j+1}\cdots v_nv_1$ and $C_2 = v_iv_{i+1}\cdots v_jv_i$ with a unique common edge v_iv_j. Note that a chord for C_1 or C_2 is also a chord in C. If there is no chord for C_1 then the face enclosed by C_1 is a desired face, otherwise, by induction, there are two faces with exactly one chord for C_1 as boundary edge, and at least one of them does not contain v_iv_j in its boundary edge, which is the desire one. Similarly, we have the other desired face in the cycle C_2. The last statement is clear. □

The faces F_1 and F_2 in the outerplanar graph H depicted in Figure 4 have exactly one for C, the cycle enclosing the unbounded face.

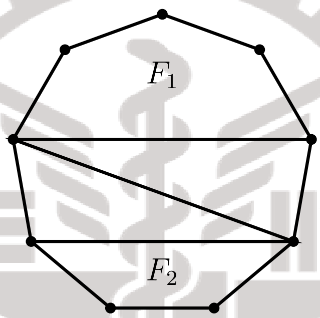

Figure 4: Two faces F_1 and F_2 in H have exactly one chord for C.

We will apply Lemma 4.4 with $H = G[N_G(i)]$. For a face F in $G[N_G(i)]$, set $V_0(F) = V_G(F) - V_{G[N_G(i)]}(F)$. Note that $V_0(F)$ collects the interior vertices in F. We use $G[F]$ to denote the subgraph of G induced on $V_G(F)$.

Example 4.5. Let G be a maximal planar graph and fix a vertex i. The subgraph $N_G(i)$ of G has a face F enclosed by $W(F) = 1234561$ as depicted in the left of Figure 5. The vertex subset $V_0(F) = \{7, 8\}$ of G are inside F. Notice that $V_0(F) \not\subseteq N_G[i]$.

Corollary 4.6. *If G is a maximal planar graph of order n with $\delta(G) \geq 4$ and $i \in V(G)$, then $\overline{N_G[i]} \neq \emptyset$.*

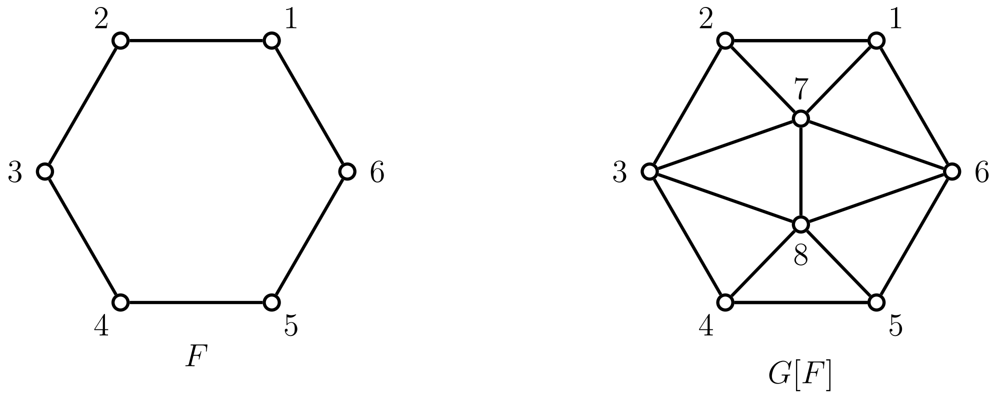

Figure 5: A face F of $G[N_G(i)]$ and the graph $G[F]$.

Proof. Applying Lemma 4.4 by $H = G[N_G(i)]$, there is at least a vertex $j \in V_{G[N_G(i)]}(F)$ of degree 2 in $G[N_G(i)]$, where F is a face that use exactly one chord for C. Since $\delta(G) \geq 4$, the vertex j has a neighbor in $V_0(F) \subseteq \overline{N_G[i]}$. $\square$

Recall that G is a maximal planar graph, C is the cycle that encloses the unbounded face ∞ in the outerplanar graph $G[N_G(i)]$ and $i \in V_G(\infty)$. Let F be a face not enclosed by C in $G[N_G(i)]$. Then $G[V(G) - V_0(F)]$ is clear to be connected. Observe that $G[V_0(F)]$ is also connected by the maximal planar graph assumption.

We will estimate the value $e(V_0(F), V(G) - V_0(F))$ in a more general setting. Let H and $G - H$ be two connected induced subgraphs of G, where $G - H$ is the subgraph of G induced on $V(G) - V(H)$. Then H and $G - H$ have a common face F. Let $\ell_G(H)$ denote the length of a closed walk in H that encloses F and $\ell'_G(H)$ denote the length of a closed walk in $G - H$ that encloses F.

Example 4.7. Let G be the graph depicted in Figure 6. Let H be the induced subgraph consisting of the vertices x, y, z, and w. Then $\ell_G(H) = 5$ with the boundary closed walk $xyzwzx$ of length 5 that encloses the face F. Similarly, $\ell'_G(H) = 5$ with the boundary closed walk $v_1v_2v_3v_4v_5v_1$ that encloses the same face F.

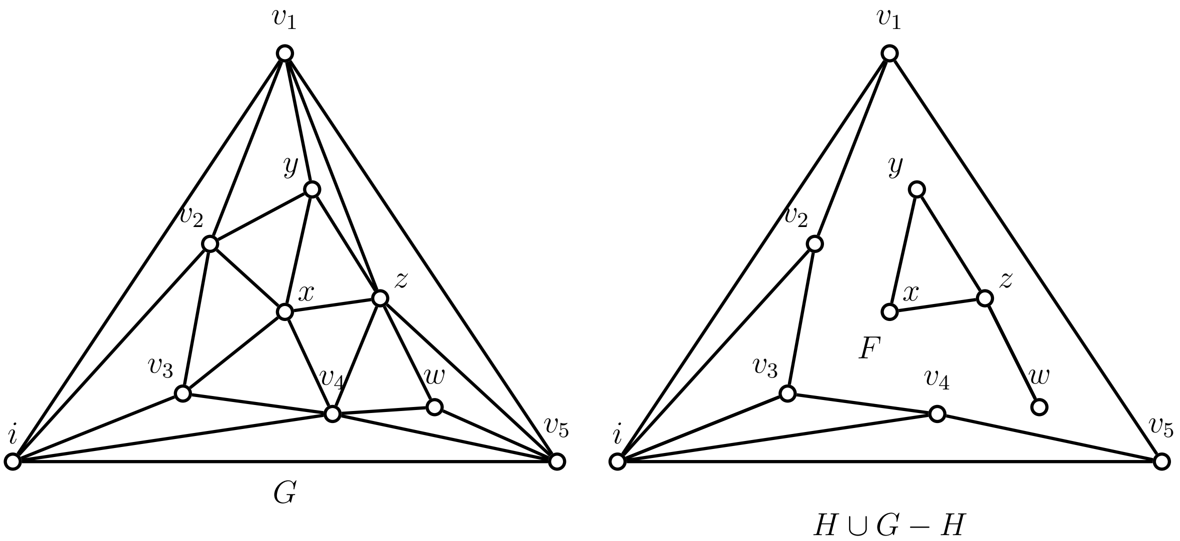

Figure 6: The graph G and $H \cup G - H$.

Lemma 4.8. *Let G be a maximal planar graph and H its induced subgraph. Then*

$$e(H, G-H) = \ell_G(H) + \ell'_G(H).$$

Proof. Let $W_1 = i_1 i_2 \ldots i_k i_1$ and $W_2 = j_1 j_2 \ldots j_\ell j_1$ be the closed walks in H and $G-H$ respectively that enclose the common face F of H and $G-H$, where $k = \ell_G(H)$ and $\ell = \ell'_G(H)$. Observe that the face F is obtained by merging the set T of $k+\ell$ triangle faces in G with each containing a unique edge in the walks W_1 and W_2, where each repeated edge in the walk is counted as two different edges. Counting the pairs (e,t) such that $e \in E(H, G-H)$, $t \in T$ and e is an edge in t in two ways, we have $e(H, G-H)2 = 2(k+\ell)$. Hence $e(H, G-H) = k+\ell = \ell_G(H) + \ell'_G(H)$. □

If H is the middle pentagon in the graph G depicted in Figure 7, one can see that $\ell_G(H) = 5$, $\ell'_G(H) = 6$ and $e(H, G-H) = \ell_G(H) + \ell'_G(H) = 11$.

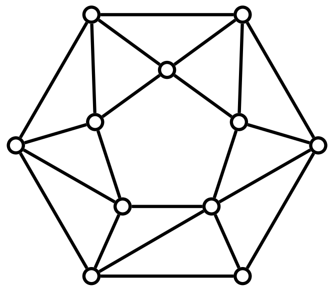

Figure 7: The graph G with $\ell_G(H) = 5$ and $\ell'_G(H) = 6$.

4.1 The case $\delta(G) = 5$

Throughout this subsection, let G be a maximal planar graph of order n and minimum degree 5. Fix $i \in V(G)$, Hamiltonian cycle C in $G[N_G(i)]$ of length at least 5 and a side face F of $G[N_G(i)]$ enclosed by a cycle $W(F)$. Recall that F being side face implies that $E_{G[N_G(i)]}(F)$ contains at most one chord for C. We will find a wheel of order at least 6 in $G\left[\overline{N_G[i]}\right]$, and use this to find a better upper bound of $\rho(G)$. An interior vertex $x \in V_0(F)$ is called *special* if there are three consecutive vertices a, b, c in the cycle $W(F)$ such that $xabx$ and $xbcx$ enclose two adjacent faces of G. In Figure 6, the vertex x is the unique special vertex in $\{x, y, z, w\}$.

Lemma 4.9. *If $x \in V_0(F)$ with $\deg_{G[V_0(F)]}(x) = 1$ then x is special.*

Proof. Let y is the unique neighbor of x in $V_0(F)$. Since $\delta(G) = 5$, there are at least 4 vertices in $W(F)$ adjacent to x. We might assume y is inside the cycle $xa_1a_2\cdots a_tx$ in clockwise direction, where $a_1, \cdots, a_t \in V(W(F))$ and $xa_j \notin E(G)$ for $2 \le j \le t-1$. Indeed all vertices in $V_0(F) - \{x\}$ are inside the cycle $1xa_1a_2\cdots a_tx$ since $G[V_0(F)]$ is connected. Every face of G is a triangle, which implies that x is adjacent to every vertex in the path from a_t to a_1 of length at least 3 along the cycle $W(F)$ in clockwise direction. □

In Figure 6, the vertex w has degree 1, but is not special in $\{x, y, z, w\}$. This is because $\delta(G) = 3 \ne 5$.

Lemma 4.10. *There is at most one special vertex in $V_0(F)$.*

Proof. If $x \in V_0(F)$ is a special vertex, then there are three consecutive vertices a, b, c in the cycle $W(F)$ such that $xabx$ and $xbcx$ enclose two faces of G, then besides the 4 neighbors i, a, c, x, there is another neighbor of b since $\delta(G) = 5$. Hence either ab or bc is the unique chord C in $E_{G[N_G(i)]}(F)$. Since no vertex in $V_0(F)$ is inside the unbounded face ∞ enclosed by C. There is at most one special vertex. □

Theorem 4.11. *$G[V_0(F)]$ contain a wheel of order at least 6.*

Proof. If there is a vertex $x \in V_0(F)$ such that $N_G[x] \subseteq V_0(F)$, then by Lemma 2.10 with $i = x$, $G[N_G[x]]$ contains a wheel of order $1 + \deg_G(x) \geq 6$. Suppose that $N_G[x] \cap N_G(i) \neq \emptyset$ for all $x \in V_0(F)$. Then $G[V_0(F)]$ is a connected outerplanar graph whose unbounded face ∞ is obtained by deleting the edges in $E(N_G(i), V_0(F))$. By Lemma 4.9-4.10, every vertex in $V_0(F) - \{x\}$ has degree at least 2. Hence $G[V_0(F)]$ contains a cycle. Let $W(V_0(F))$ and $W(F)$ be the closed walk in $G[V_0(F)]$ and the cycle in $N_G(i)$ respectively that encloses the common face ∞. Pick a special vertex $x \in V_0(F)$ if it exists, or pick any $x \in V_0(F)$. Travel from x in the closed walk $W(V_0(F))$ along a direction that ∞ appears on the left side. Let y be the first vertex of $\deg_{G[V_0(F)]}(y) \geq 3$. We call a vertex to be a repeated vertex if it appears at least twice during the travel. Then y is a repeated vertex. Pick a repeated vertex z such that no other repeated vertex between the cycle D along the travel from z back to z. An example is shown in Figure 8. Note that $\deg_{G[V(D)]}(u) = \deg_{G[V_0(F)]}(u)$ for all $u \in V(D) - \{z\}$. There is a vertex $w \in V(D)$ with degree 2 in $G[V(D)]$ by Lemma 4.3-4.4. Note that $w \neq z, x$, so w is not the special vertex. Since $\delta(G) = 5$, w has three neighbors a, b, c in $N_G(i)$. The three edges wa, wb and wc divide the side face F of $G[N_G(i)]$ into three parts, and the vertices in $V_0(F) - \{w\}$ are inside one part. Hence w is adjacent to every boundary vertex in the other two parts $V_{G[N_G(i)]}(F)$, forming at least two adjacent triangle faces, a contradiction to w not special. □

Based on Lemma 4.9 and Theorem 4.11, we can derive an upper bound on

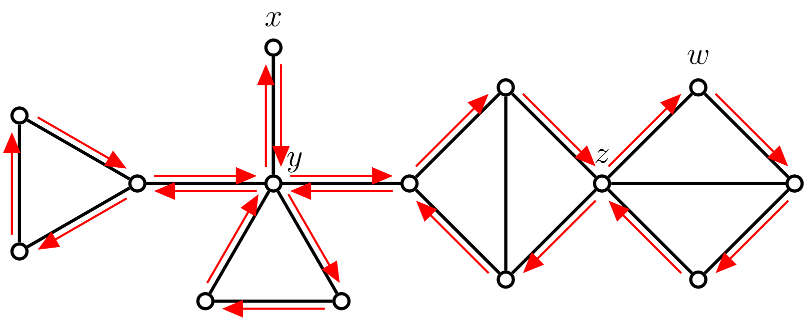

Figure 8: A closed walk in $G[V_0(F)]$.

$\deg_G(i)$ such that $N_G(i) = \deg_G(i)$.

Corollary 4.12. *If G is a maximal planar graph of order n with minimum degree 5, and $i \in V(G)$ such that $e(N_G(i)) = \deg_G(i)$, then $\deg_G(i) \leq \frac{2n}{3} - 3$.*

Proof. If $e(N_G(i)) = \deg_G(i)$, then $G\left[\overline{N_G[i]}\right]$ is connected and $\left|\overline{N_G[i]}\right| = n - \deg_G(i) - 1$. Since $G\left[\overline{N_G[i]}\right]$ contains a wheel of order at least 6 by Theorem 4.11, contributing 5 more edges than a tree,

$$\begin{aligned} e\left(\overline{N_G[i]}\right) &\geq (\left|\overline{N_G[i]}\right| - 1) + 5 \\ &= (n - \deg_G(i) - 1) + 4 \\ &= n - \deg_G(i) + 3. \end{aligned}$$

Since $e\left(\overline{N_G[i]}\right) \leq 3n - 6 - (\delta(G) - 1)\deg_G(i) \leq 3n - 6 - 4\deg_G(i)$ by Lemma 4.2, we have $3n - 6 - 4\deg_G(i) \geq n - \deg_G(i) + 3$. Therefore $\deg_G(i) \leq \frac{2n}{3} - 3$. □

Theorem 4.13. *If G is a maximal planar graph of order n with minimum degree 5, then*

$$\rho(G) \leq 1 + \sqrt{3n - 18}$$

Proof. For any vertex $i \in V(G)$, $\left|\overline{N_G[i]}\right| = n - \deg_G(i) - 1$. By Theorem 4.11, we have $G\left[\overline{N_G[i]}\right]$ contains a wheel of order at least 6 for all $i \in V(G)$. It results that

$e\left(\overline{N_G[i]}\right) \geq 10$. In (3.4) of the proof of Lemma 3.1 and $e\left(\overline{N_G[i]}\right) \geq 10$, we have

$$s_i(A^2) = 3n - 6 + e(N_G(i)) - e\left(\overline{N_G[i]}\right) \leq 3n + 2s_i(A) - 19, \qquad (4.1)$$

where A be the adjacency matrix of G. Applying the polynomials $p(x) = x^2 - 2x$ in Lemma 2.7, we have

$$\rho(G)^2 - 2\rho(G) \leq \max_{i \in V(G)} s_i(A^2 - 2A).$$

By Lemma 3.1 and (4.1), we have

$$s_i(A^2 - 2A) = s_i(A^2) - s_i(A) \leq 3n - 19$$

for all $i \in V(G)$ to have

$$\max_{i \in V(G)} s_i(A^2 - 2A) \leq 3n - 19,$$

which implies the inequality $\rho(G) \leq 1 + \sqrt{3n - 18}$. □

Lemma 4.14. *If G is a maximal planar graph of order $n \geq 39$ with minimum degree 5, and $i \in V(G)$ such that $e(N_G(i)) = \deg_G(i)$, then $e\left(\overline{N_G[i]}\right) \geq \frac{n}{3} + 7$.*

Proof. Since $e(N_G(i)) = \deg_G(i)$, then $\deg(i) \leq \frac{2n}{3} - 3$ by Corollary 4.12 and $G\left[\overline{N_G[i]}\right]$ contain a wheel of order at least 6 and $\deg_{G\left[\overline{N_G[i]}\right]}(j) \geq 2$ for all $j \in \overline{N_G[i]}$ by Theorem 4.11. Hence, $\left|\overline{N_G[i]}\right| = n - \deg_G(i) - 1 \geq \frac{n}{3} + 2$ and $e\left(\overline{N_G[i]}\right) \geq \frac{n}{3} + 2 + 5 = \frac{n}{3} + 7$.

When $n \geq 39$, it follows that $e\left(\overline{N_G[i]}\right) \geq 20$. □

Lemma 4.15. *If G is a maximal planar graph of order $n \geq 39$ with minimum degree 5, and $i \in V(G)$, then*

$$e\left(\overline{N_G[i]}\right) \geq \min\left\{\frac{n}{3} + 7\ ,\ 20\right\}$$

for all $i \in V(G)$.

Proof. For any vertex $i \in V(G)$, $\left|\overline{N_G[i]}\right| = n - \deg_G(i) - 1$.

Case 1. Suppose that $e(N_G(i)) \geq \deg_G(i) + 1$. Then there are at least two faces with exactly one chord in $G[N_G(i)]$ By Lemma 4.4. $G\left[\overline{N_G[i]}\right]$ contains two wheel of order at least 6 by Theorem 4.11. It results that $e\left(\overline{N_G[i]}\right) \geq 20$.

Case 2. Suppose that $e(N_G(i)) = \deg_G(i)$. Then $e\left(\overline{N_G[i]}\right) \geq \frac{n}{3}+7$ by Lemma 4.14.

Hence,

$$e\left(\overline{N_G[i]}\right) \geq \min\left\{\frac{n}{3} + 7\ ,\ 20\right\}$$

for all $i \in V(G)$.

□

Theorem 4.16. *If G is a maximal planar graph of order $n \geq 39$ with minimum degree 5, then*

$$\rho(G) \leq 1 + \sqrt{3n - 28}.$$

Proof. Since $n \geq 39$, we have $e\left(\overline{N_G[i]}\right) \geq 20$ by Lemma 4.15. In (3.4) of the proof of Lemma 3.1 and $e(\overline{N_G[i]}) \geq 20$, we have

$$s_i(A^2) = 3n - 6 + e(N_G(i)) - e\left(\overline{N_G[i]}\right) \leq 3n + 2s_i(A) - 29. \qquad (4.2)$$

Let A be the adjacency matrix of G. Applying the polynomials $p(x)$ to be $x^2 - 2x$ in Lemma 2.7, we have

$$\rho(G)^2 - 2\rho(G) \leq \max_{i \in V(G)} s_i(A^2 - 2A).$$

By Lemma 3.1 and (4.2), we have

$$s_i(A^2 - 2A) = s_i(A^2) - s_i(A) \leq 3n - 29$$

for all $i \in V(G)$ to have

$$\max_{i \in V(G)} s_i(A^2 - 2A) \leq 3n - 29,$$

which implies the inequality $\rho(G) \leq 1 + \sqrt{3n - 28}$. □

Theorem 4.17. *If G is a maximal planar graph of order n with $\delta(G) = 5$, then*

$$\rho(G) \leq 2 + \sqrt{2n - 11}.$$

Proof. Fix $i \in V(G)$. The graph $G\left[\overline{N_G[i]}\right]$ has at most $\deg_G(i) - 2$ components with each component $G[V_0(F)]$, where F is a face in $G[N_G(i)]$. By Theorem 4.11, then $G\left[\overline{N_G[i]}\right]$ contains a wheel of orders at least 6. contributing at least 5 more edges than a tree in the $G\left[\overline{N_G[i]}\right]$. Hence,

$$e\left(\overline{N_G[i]}\right) \geq \left|\overline{N_G[i]}\right| - (\deg_G(i) - 2) + 5 = n - 2\deg_G(i) + 6.$$

Referring to (3.4) in the proof of Lemma 3.1, we have the following better bound,

$$\begin{aligned} s_i(A^2) =& 3n - 6 + e(N_G(i)) - e\left(\overline{N_G[i]}\right) \\ \leq& 3n - 6 + (2s_i(A) - 3) - (n - 2\deg_G(i) + 6) \\ =& 2n + 4s_i(A) - 15. \end{aligned}$$

Applying the polynomial $p(x) = x^2 - 4x$ in Lemma 2.7, we have

$$\rho(G)^2 - 4\rho(G) \leq \max_{i \in V(G)} s_i(A^2 - 4A) \leq 2n - 15,$$

which implies $\rho(G) \leq 2 + \sqrt{2n - 11}$. □

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