1 Introduction

Spectral characterization is an important subject in algebraic graph theory. Some properties of a graph can be recognized from the spectrum of it. For example, a graph is bipartite if and only if its eigenvalues are located symmetrically to the origin [1, Theorem 8.8.2]. Another well-known result is that a graph is regular if and only if its largest eigenvalue equal to its average valency [2, Lemma 3.2.1]. See Lemma 2.2 below. Spectral characterization of strongly regular graphs can also be done [3, Theorem 8.6.37]. Here we are interested in the question: Is a graph with the same spectrum of a distance-regular graph distance-regular? That is, for a graph, is the distance-regularity determined by its spectrum? The answer is no and there exist many counterexamples. See Remark 4.4. But furthermore, by imposing some restrictions on the conditions of the distance-regular graphs we consider, such as distance-regular graphs with some special given intersection parameters, the answer can be yes. In [8], by assuming the odd cycles in a given distance-regular graph have length greater than 2 times its diameter, Huang and Liu proved a graph with the same spectrum of it is a distance-regular graph too. In [5][6], the result is generalized that for two graphs with the same spectrum, if the girth g satisfies $g \geq 2d-1$ in the known distance-regular one, then the other is distance-regular too where d is the diameter of G. In [4], by assuming $c_{d-1} = 1$ in one graph, the other cospectral graph is shown to be distance-regular and have the same intersection parameters. In this thesis, we present a uniform way to this line of study. As a consequence, we can reprove the above mentioned results [3, Theorem 8.6.37],[4],[5],[6]. See Theorem 4.1, Corollary 4.2. Furthermore, we show that if two cospectral graphs have the same average number of vertices in the t-subconstituent with respect to a vertex for each t, then one is distance-regular implies the other is distance-regualr. See Theorem 4.3 for details.

2 Preliminary

Let $G = (X, R)$ be a finite undirected, connected graph, without loops or multiple edges, with vertex set X, edge set R, path length distance function δ and diameter $d := \max\{\delta(x, y)|x, y \in X\}$. Sometimes we write diam(G) to denote the diameter of G. By a subgraph of G, we mean a graph (Δ, Ξ), where Δ is a non-empty subset of X and $\Xi = \{\{xy\}|x, y \in \Delta, \{xy\} \in R\}$. We refer to (Δ, Ξ) as the subgraph induced on Δ and, by abuse of notation, we refer to this subgraph as Δ. For any $x \in X$ and any integer i, set

$$G_i(x) := \{y|\ y \in X, \delta(x, y) = i\}.$$

The *valency* $k(x)$ of a vertex is the cardinality of $G_1(x)$. The graph G is said to be *regular* with valency k whenever each vertex in X has valency k. For any $x \in X$, for any integer i, and for any $y \in G_i(x)$, set

$$\begin{aligned} B(x, y) &:= G_1(y) \cap G_{i+1}(x), \\ A(x, y) &:= G_1(y) \cap G_i(x), \\ C(x, y) &:= G_1(y) \cap G_{i-1}(x). \end{aligned}$$

For all $x, y \in X$ with $\delta(x, y) = i$, the numbers

$$c_i := |C(x, y)|, \quad a_i := |A(x, y)|, \quad b_i := |B(x, y)|.$$

are said to be *well-defined* if they are independent of x and y. G is said to be *t-distance-regular* whenever for all integers i $(0 \le i \le t)$, a_{i-1}, b_{i-1}, c_i are all well-defined. A d-distance-regular graph is also called a *distance-regular graph*. The constants c_i, a_i and b_i $(0 \le i \le d)$ are known as the *intersection numbers* or *intersection parameters* of G. Note that the valency $k = b_0,\ c_0 = 0,\ c_1 = 1,\ b_d = 0$ and

$$k = c_i + a_i + b_i \quad (0 \le i \le d).$$

Let $k_i(x)$ denotes the cardinality of $G_i(x)$. For a distance-regular graph G, we know that $k_i(x)$ is a constant for any $x \in G$ for all i. We denote this constant by k_i. A graph is said to be *strongly regular with parameters* (n, k, a, c) if it is regular with valency k, every pair of adjacent vertices has a common neighbors, and every pair of distinct nonadjacent vertices has c

common neighbors, where $c > 0$. We see that a strongly regular graph is a distance-regular graph with diameter 2 with intersection parameters

$$\begin{array}{lll} c_0 = 0, & a_0 = 0, & b_0 = k, \\ c_1 = 1, & a_1 = a, & b_1 = k - a - 1, \\ c_2 = c, & a_2 = k - c, & b_2 = 0. \end{array}$$

For the *adjacency matrix* A of a graph G, we mean a symmetric $(0,1)-$matrix determined by G with rows and columns indexed by the vertices of G, and with entries

$$A_{xy} = \begin{cases} 1, & \text{if } x, y \text{ is adjacent,} \\ 0, & \text{otherwise.} \end{cases}$$

Since the adjacency matrix A of a graph G is a real symmetric matrix, we have that the eigenvalues of A are all real numbers. We represent the distinct eigenvalues of A with their corresponding multiplicities by an array as follows:

$$\begin{pmatrix} \theta_0 & \theta_1 & \cdots & \theta_d \\ m_0 & m_1 & \cdots & m_d \end{pmatrix}$$

where $\theta_0 > \theta_1 > \cdots > \theta_d$. Note $m_0 + m_1 + \cdots + m_d = v$ where v is the number of vertices in G. This array is said to be the *spectrum* of G. Two graphs are said to be *cospectral* if they have the same spectrum.

The following Lemma follows immediately from linear algebra.

Lemma 2.1. *$Tr(A^n) = \sum_{i=0}^{d} m_i \theta_i^n$ for any $n \in \mathbb{N}$.*

Lemma 2.2. *Let $G = (X, R)$ be a graph with v vertices and have average valency $\overline{k} = \frac{1}{v} \sum_{x \in X} k(x)$. Let A be the adjacency matrix of G with eigenvalues $\theta_0 \geq \theta_1 \geq \cdots \geq \theta_v$. We have $\theta_0 \geq \overline{k}$, with equality if and only if G is regular.*

Proof. Let $\beta = \{u_1, u_2, \cdots, u_v\}$ be an orthonormal basis of $\mathbb{R}^v$ which are all eigenvectors of A, and let θ_i be the corresponding eigenvalue of u_i. Consider the all-1 vector $\mathbf{1}$ in $\mathbb{R}^v$. We can express $\mathbf{1}$ in terms of a linear combination of β,that is, $\mathbf{1} = \sum_{i=1}^{v} a_i u_i$. We have that $v = \mathbf{1}^t \mathbf{1} = \sum_{i=1}^{v} a_i^2$, and $\sum_{x \in X} k(x) =$

$v\overline{k} = \mathbf{1}^t A\mathbf{1} = \sum_{i=1}^{v} a_i^2\theta_i \leq \sum_{i=1}^{v} a_i^2\theta_0 = v\theta_0$, so $\overline{k} \leq \theta_0$. The equality holds if and only if $a_i(\theta_0 - \theta_i) = 0$ for all i $(o \leq i \leq v)$, that is, $A\mathbf{1} = \theta_0\mathbf{1}$, i.e.,$G$ is regular with valency θ_0. □

Theorem 2.3. *Let $G = (X, R)$ be a graph with v vertices and has spectrum*

$$\begin{pmatrix} \theta_0 & \theta_1 & \cdots & \theta_d \\ m_0 & m_1 & \cdots & m_d \end{pmatrix}$$

where $\theta_0 > \theta_1 > \cdots > \theta_d$. Then the following (i)-(ii) are equivalent.

(i) $\sum_{i=0}^{d} m_i\theta_i^2 = v\theta_0$

(ii) G is regular with valency θ_0.

Proof. Observe $(A^2)_{xx} = k(x)$ for all $x \in X$. Hence we have that $\sum_{i=1}^{d} m_i\theta_i^2 = Tr(A^2) = \sum_{x\in X} k(x) = v\overline{k}$.Then simply applying Lemma 2.2, we have that (i)-(ii) are equivalent.

□

We quote a Theorem from [1, Lemma 8.12.1].

Theorem 2.4. *If G is a graph with diameter d, then $A(G)$ has at least $d+1$ distinct eigenvalues.*

3 t-distance-regular graphs

Let $G = (X, E)$ and $G' = (X', E')$ be two connected graphs with the same spectrum

$$\begin{pmatrix} \theta_0 & \theta_1 & \cdots & \theta_d \\ m_0 & m_1 & \cdots & m_d \end{pmatrix}.$$

Let $t \leq d$ be a positive integer. Suppose G is t-distance-regular. That is in G the parameters a_i, b_i, and c_{i+1}, $(0 \leq i \leq t-1)$ are well-defined. Hence k_i $(0 \leq i \leq t)$ is well defined. Suppose G' is $(t-1)$-distance-regular , the parameters a'_i, b'_i, and c'_{i+1}, $(0 \leq i \leq t-2)$ are well-defined. Furthermore assume these parameters are the same as the corresponding intersection parameters of G. Hence $k'_i = k_i$ $(0 \leq i \leq t-1)$. Let A, A' denote the adjacency matrices of G, G' respectively.

Lemma 3.1. $\sum_{x \in X'} \sum_{y \in G'_{t-1}(x)} a'_{t-1}(x, y) = v k_{t-1} a_{t-1}$, *where* $v = |X| = \sum_{i=0}^{d} m_i$.

Proof. The number of closed walks of length $2t-1$ through x in G' is $(A'^{2t-1})_{xx}$. These closed walks divide into 2 parts. One contains an edge in the induced subgraph $G'_{t-1}(x)$ and the other does not. The number of the first part is

$$\sum_{y \in G'_{t-1}(x)} a'_{t-1}(x, y)(c^2_{t-1} c^2_{t-2} \ldots c^2_2).$$

Let K denote the number of remaining closed walks. Hence

$$K + \sum_{y \in G'_{t-1}(x)} a'_{t-1}(x, y)(c^2_{t-1} c^2_{t-2} \ldots c^2_2) = (A'^{2t-1})_{xx}. \tag{3.1}$$

Note K can be expressed in terms of the known (and well-defined) intersection parameters. Then we know that

$$\begin{aligned} Tr(A'^{2t-1}) &= \sum_{x \in X'} (A'^{2t-1})_{xx} \\ &= vK + \sum_{x \in X'} \sum_{y \in G'_{t-1}(x)} a'_{t-1}(x, y)(c^2_{t-1} c^2_{t-2} \cdots c^2_2). \end{aligned} \tag{3.2}$$

Similarly,

$$\begin{aligned} Tr(A^{2t-1}) &= vK + \sum_{x\in X}\sum_{y\in G_{t-1}(x)} a_{t-1}(c_{t-1}^2 c_{t-2}^2 \dots c_2^2) \\ &= vK + vk_{t-1}a_{t-1}(c_{t-1}^2 c_{t-2}^2 \dots c_2^2). \end{aligned} \tag{3.3}$$

Since A and A' have the same spectrum, we know that

$$Tr({A'}^{2t-1}) = Tr(A^{2t-1}).$$

Thus by (3.2)-(3.3)

$$\sum_{x\in X'}\sum_{y\in G'_{t-1}(x)} a'_{t-1}(x,y) = vk_{t-1}a_{t-1}.$$

□

Corollary 3.2. *Suppose either $a'_{t+1}(x,y) \geq a_{t-1}$ or $a'_{t-1}(x,y) \leq a_{t-1}$ for any $x \in X', y \in G'_{t-1}(x)$. Then $a'_{t-1} = a_{t-1}$.*

Proof. It's trivial by Lemma 3.1. □

Lemma 3.3. $\sum_{x\in X'}\sum_{z\in G'_t(x)} c'_t(x,z) = vk_{t-1}b_{t-1} = vk_tc_t.$

Proof. For each $x \in X'$, by counting the number of edges between $G'_{t-1}(x)$ and $G'_t(x)$ in two ways and Lemma 3.1,

$$\begin{aligned} \sum_{x\in X'}\sum_{z\in G'_t(x)} c'_t(x,z) &= \sum_{x\in X}\sum_{y\in G'_{t-1}(x)} b'_{t-1}(x,y) \\ &= \sum_{x\in X'}\sum_{y\in G'_{t-1}(x)} (k_1 - c_{t-1} - a'_{t-1}(x,y)) \\ &= vk_{t-1}(k_1 - c_{t-1}) - \sum_{x\in X}\sum_{y\in G'_{t-1}(x)} a'_{t-1}(x,y) \\ &= vk_{t-1}(k_1 - c_{t-1}) - vk_{t-1}a_{t-1} \\ &= vk_{t-1}b_{t-1} \\ &= vk_tc_t. \end{aligned}$$

□

Corollary 3.4. *Let $\overline{k'_t}$ denotes $\frac{1}{v}$ times the cardinality of the set $\{(x,z)|\ x,z \in G',\ d(x,z)=t\}$. Suppose either $\overline{k'_t} \geq k_t$ and $c'_t(x,z) \geq c_t$, or $\overline{k'_t} \leq k_t$ and $c'_t(x,z) \leq c_t$ for any $x \in X, z \in G'_t(x)$. Then $\overline{k'_t} = k_t$ and $c'_t = c_t$.*

Proof. It's trivial by Lemma 3.3. □

Lemma 3.5.

$$\begin{aligned} Tr(A'^{2t}) = vC \quad &+ \sum_{x\in X'}\sum_{y\in G'_{t-1}(x)} a'_{t-1}(x,y)^2(c^2_{t-1}c^2_{t-2}\cdots c^2_2) \\ &+ \sum_{x\in X'}\sum_{z\in G'_t(x)} c'_t(x,z)^2(c^2_{t-1}c^2_{t-2}\cdots c^2_2) \\ &+ \ vk_{t-1}a_{t-1}(c^2_{t-1}c^2_{t-2}\cdots c^2_2)(a_{t-2}+\cdots+a_1) \end{aligned}$$

for some constant C determined by a_i, b_i, c_{i+1} $(0 \leq i \leq t-2)$.

Proof. For any vertex x of G', we count the number of closed walks $x = x_0, x_1, \cdots, x_{2t} = x$. There are 4 cases.

Case 1 : $x_{t-1}, x_t, x_{t+1} \in G'_{t-1}(x)$. The number of closed walks in this case can be expressed as

$$\sum_{x_t\in G'_{t-1}(x)} a'_{t-1}(x,x_t)^2(c^2_{t-1}c^2_{t-2}\dots c^2_2).$$

Case 2 : $x_t \in G'_t(x)$. The number of closed walks in this case can be expressed as

$$\sum_{x_t\in G'_t(x)} c'_t(x,x_t)^2(c^2_{t-1}c^2_{t-2}\dots c^2_2).$$

Case 3 : $x_t \in G'_{t-1}(x), |\{x_{t-1}, x_{t+1}\} \cap G'_{t-1}(x)| = 1$. The number of closed walks in this case can be expressed as

$$\sum_{x_t\in G'_{t-1}(x)} a'_{t-1}(x,x_t)(c^2_{t-1}c^2_{t-2}\dots c^2_2)(a_{t-2}+\dots+a_1).$$

By simply apply Lemma 3.1, we know that this term equals

$$vk_{t-1}a_{t-1}(c^2_{t-1}c^2_{t-2}\dots c^2_2)(a_{t-2}+\dots+a_1).$$

Case 4 : The remaining cases. The number of closed walks in this case can be expressed as a known constant C.

As before, we know the number of the closed walks of length $2t$ is $Tr(A'^{2t})$. Hence

$$\begin{aligned} Tr(A'^{2t}) = vC &+ \sum_{x\in X'}\sum_{y\in G'_{t-1}(x)} a'_{t-1}(x,y)^2(c_{t-1}^2c_{t-2}^2\cdots c_2^2) \\ &+ \sum_{x\in X'}\sum_{z\in G'_{t}(x)} c'_{t}(x,z)^2(c_{t-1}^2c_{t-2}^2\cdots c_2^2) \\ &+ vk_{t-1}a_{t-1}(c_{t-1}^2c_{t-2}^2\cdots c_2^2)(a_{t-2}+\cdots+a_1). \end{aligned}$$

□

Corollary 3.6. *Let C be as in Lemma 3.5. Then*

$$\begin{aligned} Tr(A^{2t}) = vC &+ vk_{t-1}a_{t-1}^2(c_{t-1}^2c_{t-2}^2\ldots c_2^2) \\ &+ vk_tc_t^2(c_{t-1}^2c_{t-2}^2\ldots c_2^2) \\ &+ vk_{t-1}a_{t-1}(c_{t-1}^2c_{t-2}^2\ldots c_2^2)(a_{t-2}+\ldots+a_1). \end{aligned}$$

Proof. We express $Tr(A^{2t})$ by the way in Lemma 3.5.

$$\begin{aligned} Tr(A^{2t}) = vC &+ \sum_{x\in X}\sum_{y\in G_{t-1}(x)} a_{t-1}(x,y)^2(c_{t-1}^2c_{t-2}^2\cdots c_2^2) \\ &+ \sum_{x\in X}\sum_{z\in G_{t}(x)} c_{t}(x,z)^2(c_{t-1}^2c_{t-2}^2\cdots c_2^2) \\ &+ \sum_{x\in X}\sum_{y\in G_{t-1}(x)} a_{t-1}(x,y)(c_{t-1}^2c_{t-2}^2\cdots c_2^2)(a_{t-2}+\cdots+a_1). \end{aligned}$$

Since the intersection parameters a_{t-1}, c_t are well-defined and known in G, we simply substitute the parameters in and get the result. □

Lemma 3.7.

$$\begin{aligned} Tr(A'^{2t}) \geq vC &+ vk_{t-1}a_{t-1}^2(c_{t-1}^2c_{t-2}^2\ldots c_2^2) \\ &+ vk_tc_t(c_{t-1}^2c_{t-2}^2\ldots c_2^2) \\ &+ vk_{t-1}a_{t-1}(c_{t-1}^2c_{t-2}^2\ldots c_2^2)(a_{t-2}+\ldots+a_1). \quad (3.4) \end{aligned}$$

Furthermore, the following (i)-(ii) are equivalent.

(i) Equality holds in (3.4).

(ii) $a'_{t-1}(x,y) = a_{t-1}, c'_t(x,z) = 1$ *for any* $x \in X', y \in G'_{t-1}(x), z \in G'_t(x)$.

Proof. Applying Cauchy's inequality and $c'_t(x,z)^2 \geq c'_t(x,z)$ on the expression of $Tr(A'^{2t})$ in Lemma 3.5, it follows that

$$\begin{aligned}
Tr(A'^{2t}) \geq vC &+ \frac{1}{vk_{t-1}}(\sum_{x\in X'}\sum_{y\in G'_{t-1}(x)} a'_{t-1}(x,y)(c_{t-1}c_{t-2}\ldots c_2))^2 \\
&+ \sum_{x\in X'}\sum_{z\in G'_t(x)} c'_t(x,z)(c^2_{t-1}c^2_{t-2}\ldots c^2_2) \\
&+ vk_{t-1}a_{t-1}(c^2_{t-1}c^2_{t-2}\ldots c^2_2)(a_{t-2}+\ldots+a_1) \\
= vC &+ \frac{1}{vk_{t-1}}(vk_{t-1}a_{t-1}(c_{t-1}c_{t-2}\ldots c_2))^2 \\
&+ vk_{t-1}b_{t-1}(c^2_{t-1}c^2_{t-2}\ldots c^2_2) \\
&+ vk_{t-1}a_{t-1}(c^2_{t-1}c^2_{t-2}\ldots c^2_2)(a_{t-2}+\ldots+a_1) \\
= vC &+ vk_{t-1}a^2_{t-1}(c^2_{t-1}c^2_{t-2}\ldots c^2_2) \\
&+ vk_{t-1}b_{t-1}(c^2_{t-1}c^2_{t-2}\ldots c^2_2) \\
&+ vk_{t-1}a_{t-1}(c^2_{t-1}c^2_{t-2}\ldots c^2_2)(a_{t-2}+\ldots+a_1).
\end{aligned}$$

The above equality holds if and only if $a'_{t-1}(x,y) = a_{t-1}$ and $c'_t(x,z) = 1$ for any x, y, z with $\delta(x,y) = t-1$ and $\delta(x,z) = t$.
The equivalence of (i)-(ii) is clear. □

Lemma 3.8.

$$\begin{aligned}
Tr(A'^{2t}) \geq vC &+ vk_{t-1}a^2_{t-1}(c^2_{t-1}\cdots c^2_2) \\
&+ \frac{(vk_tc_tc_{t-1}\cdots c_2)^2}{v\overline{k'_t}} \\
&+ vk_{t-1}a_{t-1}(c^2_{t-1}c^2_{t-2}\ldots c^2_2)(a_{t-2}+\ldots+a_1). \qquad (3.5)
\end{aligned}$$

Furthermore, the following (i)-(ii) are equivalent.

(i) Equality holds in (3.5), $\overline{k'_t} = k_t$.

(ii) $a'_{t-1}(x,y) = a_{t-1},\ c'_t(x,z) = c_t$ for any $x \in X, y \in G'_{t-1}(x), z \in G'_t(x)$.

Proof. Applying Cauchy's inequality on the expression of $Tr(A'^{2t})$ in Lemma 3.5, it follows

$$\begin{aligned} Tr(A'^{2t}) \geq vC &+ \frac{1}{vk_{t-1}}(\sum_{x\in X'}\sum_{y\in G'_{t-1}(x)} a'_{t-1}(x,y)c_{t-1}c_{t-2}\dots c_2)^2 \\ &+ \frac{1}{v\overline{k'_t}}(\sum_{x\in X'}\sum_{z\in G'_t(x)} c'_t(x,z)c_{t-1}c_{t-2}\dots c_2)^2 \\ &+ vk_{t-1}a_{t-1}(c_{t-1}^2c_{t-2}^2\dots c_2^2)(a_{t-2}+\dots+a_1). \\ = vC &+ \frac{1}{vk_{t-1}}(vk_{t-1}a_{t-1}c_{t-1}c_{t-2}\dots c_2)^2 \\ &+ \frac{(vk_tc_tc_{t-1}\cdots c_2)^2}{v\overline{k'_t}} \\ &+ vk_{t-1}a_{t-1}(c_{t-1}^2c_{t-2}^2\dots c_2^2)(a_{t-2}+\dots+a_1). \end{aligned}$$

(i)⇒(ii) is clear. (ii)⇒(i) is from the observation that the last term in the above equation is $Tr(A^{2t})$ which is equal to $Tr(A'^{2t})$. □

Lemma 3.9. *Suppose* $c_t = 1$. *Then* a'_{t-1}, b'_{t-1}, c'_t *are well-defined, and are the same as the corresponding ones in* G *.*

Proof. Comparing to Collary 3.6 and using $c_t = 1$, we find the equality in Lemma 3.7 holds. Hence a'_{t-1}, c'_t are well-defined. Note $b'_{t-1} = b_0 - c_{t-1} - a'_{t-1}$. □

4 Applications

Theorem 4.1. [4, Theorem 1] Let $G = (X, E)$ and $G' = (X', E')$ be two connected graphs with the same spectrum

$$\begin{pmatrix} \theta_0 & \theta_1 & \cdots & \theta_d \\ m_0 & m_1 & \cdots & m_d \end{pmatrix}.$$

Suppose that G is distance-regular with intersection parameters a_i, b_i, c_i for $0 \leq i \leq d$. Suppose $c_j = 1$ for $1 \leq j \leq d-1$. Then G' is a distance-regular graph with the same intersection parameters of G.

Proof. We first show $a'_i = a_i, b'_i = b_i, c'_{i+1} = c_{i+1} = 1 \quad (0 \leq i \leq d-2)$ by induction on i. $a'_0 = 0 = a_0$, $c'_1 = 1 = c_1$ are clear. $b'_0 = b_0$ is from Theorem 2.3. Hence we have the case $i = 0$. Suppose this is true for $i \leq t-2$. The case $i = t-1$ is true from Lemma 3.9. So we have $a'_i = a_i, b'_i = b_i, c'_{i+1} = c_{i+1} = 1 \quad (0 \leq i \leq d-2)$. For the remaining parameters, we know $k'_i = k_i$ is well-defined for each $0 \leq i \leq d-1$. Note the diameter of G' is at most d by Lemma 2.4. Hence $k'_d = v - k_0 - k_1 \cdots - k_{d-1}$ is well-defined. Then the equality in Lemma 3.8 (iii) holds for $t = d$, so by Lemma 3.8 (ii) we have $a'_{d-1} = a_{d-1}, c'_d = c_d$. Note $a'_d = b_0 - c_d = a_d$. □

Corollary 4.2. *Let G be a strongly regular graph. Suppose that G' is a graph with the same spectrum of G. Then G' is a strongly regular graph with the same intersection parameters of G.*

Proof. This is immediate from Theorem 4.1 since $c_1 = 1$. □

Theorem 4.3. *Let G be a distance-regular graph. Suppose G' is a graph with the same spectrum of G. Furthermore, with refering to Corollary 3.4, suppose $\overline{k'_t} = k_t$. Then G' is a distance-regular graph with the same intersection parameters.*

Proof. We show $a'_i = a_i, b'_i = b_i,\ c'_{i+1} = c_{i+1} \quad (0 \leq i \leq d-1)$ by induction on i. $a'_0 = a_0$, $c'_1 = 1 = c_1$ are clear. $b'_0 = b_0$ is from Theorem 2.3. Hence we have the case $i = 0$. Suppose this is true for $i \leq t-2$. Since Lemma 3.8 (iii) holds, we have Lemma 3.8 (ii). Then $a'_{t-1} = a_{t-1}$ and $c'_t = c_t$. Note $b'_{t-1} = b_0 - c_{t-1} - a_{t-1}$. □

Remark 4.4. [6, Example 2.] The Gosset graph Γ is the unique distance-regular graph on 56 vertices with intersection array $\{27, 10, 1; 1, 10, 27\}$. Notice that in Γ, $k_0 = 1,\ k_1 = 27,\ k_2 = 27,\ k_3 = 1$. We have a graph Γ' with diameter 2 which is obtained by taking some special kind of switching on Γ such that in Γ', $k'_0 = 1, k'_1 = 27, k'_2 = 28$ where Γ and Γ' are cospectral.

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