Triangle-free distance-regular graphs

Student: Ming-hsing Lu Advisor: Chih-wen Weng

Abstract

Let $\Gamma = (X, R)$ denote a distance-regular graph with distance function ∂ and diameter $d \geq 3$. For $2 \leq i \leq d$, by a parallelogram of length i, we mean a 4-tuple $xyzu$ of vertices in X such that $\partial(x, y) = \partial(z, u) = 1$, $\partial(x, u) = i$, and $\partial(x, z) = \partial(y, z) = \partial(y, u) = i - 1$. Suppose the intersection number $a_1 = 0$, $a_2 \neq 0$ in Γ. We prove the following (i)-(ii) are equivalent. (i) Γ is Q-polynomial and contains no parallelograms of length 3; (ii) Γ has classical parameters. By applying the above result we show that if Γ has classical parameters and the intersection numbers $a_1 = 0$, $a_2 \neq 0$, then for each pair of vertices $v, w \in X$ at distance $\partial(v, w) = 2$, there exists a strongly regular subgraph Ω of Γ containing v, w. Furthermore, for each vertex $x \in \Omega$, the subgraph induced on $\Omega_2(x)$ is an a_2-regular connected graph with diameter at most 3.

1 Introduction

It is shown that a distance-regular graph with classical parameters has the Q-polynomial property [2, Theorem 8.4.1]. To describe the converse, let Γ denote a Q-polynomial distance-regular graph with diameter $d \geq 3$. Brouwer, Cohen, Neumaier proved that if Γ is a near polygon and has intersection number $a_1 \neq 0$ then Γ has classical parameters [2, Theorem 8.5.1]. Weng proves the same result by loosing the near polygon assumption, but instead assuming that the graph Γ contains no kites of length 2 and no kites of length 3 [7, Lemma 2.4]. For the complement, Weng shows Γ has classical parameters in the assumptions that Γ has diameter $d \geq 4$, intersection numbers $a_1 = 0$, $a_2 \neq 0$, and Γ contains no parallelograms of length 3 and no parallelograms of length 4 [9, Theorem 2.11]. We generalize Weng's result as following.

Theorem 1.1. *Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $d \geq 3$ and intersection numbers $a_1 = 0$, $a_2 \neq 0$. Then the following (i)-(ii) are equivalent.*

(i) Γ is Q-polynomial and Γ contains no parallelograms of length 3.

(ii) Γ has classical parameters.

By the results in [4] and [10], Theorem 1.1 has the following corollary.

Corollary 1.2. *Let Γ denote a distance-regular graph with classical parameters and intersection numbers $a_1 = 0$, $a_2 \neq 0$. Then for each pair of vertices*

$v, w \in X$ at distance $\partial(v, w) = 2$, there exists a strongly regular subgraph Ω of Γ containing v, w with intersection numbers of Ω

$$\begin{aligned} a_i(\Omega) &= a_i(\Gamma), \\ c_i(\Omega) &= c_i(\Gamma), \\ b_i(\Omega) &= a_2(\Gamma) + c_2(\Gamma) - a_i(\Gamma) - c_i(\Gamma). \end{aligned}$$

for $0 \leq i \leq 2$.

Applying Corollary 1.2, we have the following corollary.

Corollary 1.3. *Let Ω be a strongly regular graph with $a_1 = 0$, $a_2 \neq 0$. Then $\Omega_2(x)$ is an a_2-regular connected graph with diameter at most 3 for all $x \in \Omega$.*

2 Preliminaries

Let $\Gamma = (X, R)$ be a graph consisting of a finite non-empty set X of *vertices*, and a finite set R of unordered pairs of distinct vertices called *edges*. For each vertex x in a graph Γ, the number of edges incident to x is the *valency* of x. Two vertices associate with each edge are called the endpoints of the edge.

If $e = xy$ is an edge of Γ , then e is said to *join* the vertices x and y, and these vertices x and y are said to be *adjacent*. A *path* is a simple graph whose vertices can be ordered so that two vertices are adjacent if and only if they are consecutive in the list. A graph is *connected* if each pair of vertices belong to a path. The *length* of a path is the number of the edges in the path. The *distance* of two vertices x and y in Γ is the length of the shortest path from x to y, denoted by $\partial(x, y)$. The *diameter* of Γ is $\max\{\partial(x, y) \mid x, y \in X\}$

For the rest of this section, we review some definitions and basic concepts of distance-regular graphs. See Bannai and Ito[1] or Terwilliger[6] for more background information.

Throughout this thesis, $\Gamma = (X, R)$ will denote a connected, graph with vertex set X, edge set R, path-length distance function ∂, and diameter $d \geq 3$.

Γ is said to be *regular*, if all vertices in Γ have the same valency. A *k-regular* graph is a graph with valency k of each vertex of the graph. Γ is said to be a strongly regular graph $srg(v, k, \lambda, \mu)$, if Γ is k-regular with diameter 2 and has the following two properties:

(i) For any two adjacent vertices x and y, there are exactly λ vertices adjacent to x and to y.

(ii) For any two nonadjacent vertices x and y, there are exactly μ vertices adjacent to x and to y.

Note that $srg(v, k, \lambda, \mu)$ is a distance-regular graph of diameter 2 with $a_1 = \lambda$, $c_2 = \mu$, $b_0 = k$.

For a vertex $x \in X$ and $0 \leq i \leq d$, set $\Gamma_i(x) = \{y \mid \partial(x, y) = i\}$. Γ is said to be *distance-regular* whenever for all integers $0 \leq h, i, j \leq d$, and all vertices $x, y \in X$ with $\partial(x, y) = h$, the number

$$p^h_{ij} = \mid \{z \in X \mid z \in \Gamma_i(x) \cap \Gamma_j(y)\} \mid$$

is independent of x, y. The constants p^h_{ij} are known as the *intersection numbers* of Γ. For convenience, set $c_i := p^i_{1\ i-1}$ for $1 \leq i \leq d$, $a_i := p^i_{1\ i}$ for $0 \leq i \leq d$, $b_i := p^i_{1\ i+1}$ for $0 \leq i \leq d-1$, and put $b_d := 0$, $c_0 := 0$, $k := b_0$. It is immediate from the definition that $b_i \neq 0$ for $0 \leq i \leq d-1$, $c_i \neq 0$ for $1 \leq i \leq d$, and

$$k = b_0 = a_i + b_i + c_i \quad \text{for} \ \ 1 \leq i \leq d. \tag{2.1}$$

Note that $a_1 \neq 0$ implies $a_2 \neq 0$. See Figure 1.

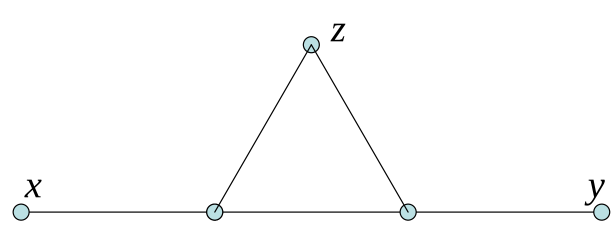

Figure 1: $\partial(x, y) = 3$. Either $\partial(x, z) = 2$ or $\partial(z, y) = 2$.

A distance-regular graph Γ is called *bipartite* whenever $a_1 = a_2 = \cdots = a_d = 0$. See Figure 2. Γ is called a *generalized odd graph* whenever $a_1 = a_2 = \cdots = a_{d-1} = 0$, $a_d \neq 0$. See Figure 3.

From now on, we fix a distance-regular graph Γ with diameter $d \geq 3$. For $0 \leq h, i, j \leq d$ let p^h_{ij} denote the intersection numbers of Γ.

Let $\mathrm{Mat}_X(\mathbb{R})$ denote the algebra of all the matrices over the real number field with the rows and columns indexed by the elements of X. The *distance matrices* of Γ are the matrices $A_0, A_1, \cdots, A_d \in \mathrm{Mat}_X(\mathbb{R})$, defined by the

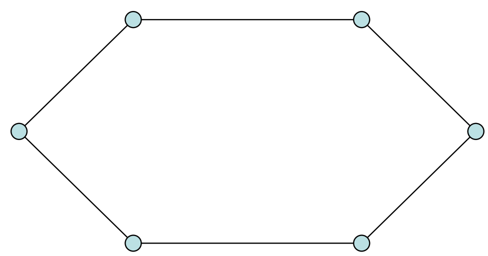

Figure 2: A bipartite distance-regular graph

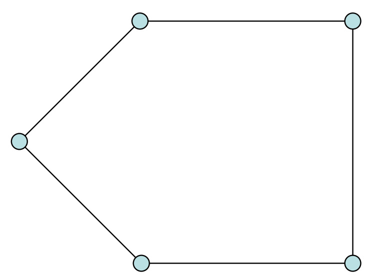

Figure 3: A generalized odd graph

rule

$$(A_i)_{xy} = \begin{cases} 1, & \text{if } \partial(x,y) = i; \\ 0, & \text{if } \partial(x,y) \neq i \end{cases} \qquad \text{for} \quad x, y \in X.$$

Then

$$A_0 = I, \tag{2.2}$$

$$A_0 + A_1 + \cdots + A_d = J \quad \text{where} \quad J = \text{all } 1's \text{ matrix}, \tag{2.3}$$

$$A_i^t = A_i \qquad \text{for} \quad 0 \leq i \leq d, \tag{2.4}$$

$$A_i A_j = \sum_{h=0}^{d} p_{ij}^h A_h \quad \text{for} \quad 0 \leq i, j \leq d, \tag{2.5}$$

$$A_i A_j = A_j A_i \qquad \text{for} \quad 0 \leq i, j \leq d. \tag{2.6}$$

Let M denote the subspace of $\mathrm{Mat}_X(\mathbb{R})$ spanned by $A_0, A_1, \ldots, A_d$. Then M is a commutative subalgebra of $\mathrm{Mat}_X(\mathbb{R})$, and is known as the *Bose-Mesner algebra* of Γ. By [1, p59, p64], M has a second basis $E_0, E_1, \cdots, E_d$ such that

$$E_0 = \mid X \mid^{-1} J, \tag{2.7}$$

$$E_i E_j = \delta_{ij} E_i \qquad \text{for } \; 0 \leq i, j \leq d, \tag{2.8}$$

$$E_0 + E_1 + \cdots + E_d = I, \tag{2.9}$$

$$E_i^t = E_i \qquad \text{for } \; 0 \leq i \leq d. \tag{2.10}$$

The $E_0, E_1, \cdots, E_d$ are known as the *primitive idempotents* of Γ, and E_0 is known as the *trivial* idempotent. Let E denote any primitive idempotent of Γ. Then we have

$$E = \mid X \mid^{-1} \sum_{i=0}^{d} \theta_i^* A_i \tag{2.11}$$

for some $\theta_0^*, \theta_1^*, \cdots, \theta_d^* \in \mathbb{R}$, called the *dual eigenvalues* associated with E.

Let $\circ$ denote entry-wise multiplication in $\mathrm{Mat}_X(\mathbb{R})$. Then

$$A_i \circ A_j = \delta_{ij} A_i \qquad \text{for } \; 0 \leq i, j \leq d,$$

so M is closed under $\circ$. Thus there exists $q_{ij}^k \in \mathbb{R}$ $\; 0 \leq i, j, k \leq d$ such that

$$E_i \circ E_j = \mid X \mid^{-1} \sum_{k=0}^{d} q_{ij}^k E_k \qquad \text{for } \; 0 \leq i, j \leq d.$$

Γ is said to be *Q-polynomial* with respect to the given ordering $E_0, E_1, \cdots, E_d$ of the primitive idempotents, if for all integers h, i, j $(0 \leq h, i, j \leq d)$,

$q_{ij}^h = 0$ (resp. $q_{ij}^h \neq 0$) whenever one of h, i, j is greater than (resp. equal to) the sum of the other two. Let E denote any primitive idempotent of Γ. Then Γ is said to be Q-polynomial with respect to E whenever there exists an ordering $E_0, E_1 = E, \cdots, E_d$ of the primitive idempotents of Γ, with respect to which Γ is Q-polynomial. If Γ is Q-polynomial with respect to E, then the associated dual eigenvalues are distinct [5, p384]. It is shown that if Γ is Q-polynomial with $a_2 = 0$, that Γ is a bipartite graph or a generalized odd graph.

Set $V = \mathbb{R}^{|X|}$ (column vectors), and view the coordinates of V as being indexed by X. Then the Bose-Mesner algebra M acts on V by left multiplication. We call V the *standard module* of Γ. For each vertex $x \in X$, set

$$\hat{x} = (0, 0, \cdots, 0, 1, 0, \cdots, 0)^t, \tag{2.12}$$

where the 1 is in coordinate x. Also, let $\langle , \rangle$ denote the dot product

$$\langle u, v \rangle = u^t v \quad \text{for} \ \ u, v \in V. \tag{2.13}$$

Then referring to the primitive idempotent E in (2.11), we compute from (2.10)-(2.13) that

$$\langle E\hat{x}, \hat{y} \rangle = \mid X \mid^{-1} \theta_i^* \quad \text{for} \ \ x, y \in X, \tag{2.14}$$

where $i = \partial(x, y)$.

The following theorem about Q-polynomial is used in this thesis.

Theorem 2.1. *[6, Theorem 3.3] Let Γ be Q-polynomial with respect to E with the distinct associated dual eigenvalues $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$. Then the following (i)-(ii) are equivalent.*

(i) For all integers $h, i, j (1 \leq h \leq d), (0 \leq i, j \leq d)$ and for all $x, y \in X$ such that $\partial(x, y) = h$,

$$\sum_{\substack{z \in X \\ \partial(x,z)=i \\ \partial(y,z)=j}} Ez - \sum_{\substack{z \in X \\ \partial(x,z)=j \\ \partial(y,z)=i}} Ez = p^h_{ij} \frac{\theta^*_i - \theta^*_j}{\theta^*_0 - \theta^*_h}(Ex - Ey). \tag{2.15}$$

(ii)

$$\theta^*_{i-2} - \theta^*_{i-1} = \sigma(\theta^*_{i-3} - \theta^*_i) \tag{2.16}$$

for appropriate $\sigma \in \mathbb{R} \setminus \{0\}$

Γ is said to have *classical parameters* (d, b, α, β) whenever the diameter of Γ is $d \geq 3$, and the intersection numbers of Γ satisfy

$$c_i = \begin{bmatrix} i \\ 1 \end{bmatrix} \left(1 + \alpha \begin{bmatrix} i-1 \\ 1 \end{bmatrix}\right) \quad \text{for} \quad 0 \leq i \leq d, \tag{2.17}$$

$$b_i = \left(\begin{bmatrix} d \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix}\right)\left(\beta - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix}\right) \quad \text{for} \quad 0 \leq i \leq d, \tag{2.18}$$

where

$$\begin{bmatrix} i \\ 1 \end{bmatrix} := 1 + b + b^2 + \cdots + b^{i-1}. \tag{2.19}$$

Γ is said to have *classical parameters* if Γ is has classical parameters (d, b, α, β) for some constants d, b, α, β. It is shown that a distance-regular graph with classical parameters has the Q-polynomial property [2, Theorem 8.4.1]. Terwilliger proves the following theorem.

Theorem 2.2. *[6, Theorem 4.2] Let Γ denote a distance-regular with diameter $d \geq 3$. Choose $b \in \mathbb{R} \setminus \{0. -1\}$, and let $\begin{bmatrix} \ \end{bmatrix}$ be as in (2.19). Then the following (i)-(ii) are equivalent.*

(i) Γ *is Q-polynomial with associated dual eigenvalues* $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$ *satisfying*

$$\theta_i^* - \theta_0^* = (\theta_1^* - \theta_0^*) \begin{bmatrix} i \\ 1 \end{bmatrix} b^{1-i}.$$

(ii) Γ *has classical parameters* (d, b, α, β) *for some real constants* α, β.

From Theorem 2.2, we have

$$\theta_i^* - \theta_{i+1}^* = b^{-i}(\theta_0^* - \theta_1^*). \tag{2.20}$$

Pick an integer $2 \leq i \leq d$. By a *parallelogram* of length i in Γ, we mean a 4-tuple $xyzw$ of vertices of X such that

$$\partial(x, y) = \partial(z, w) = 1, \quad \partial(x, w) = i,$$

$$\partial(x, z) = \partial(y, z) = \partial(y, w) = i - 1.$$

See Figure 4.

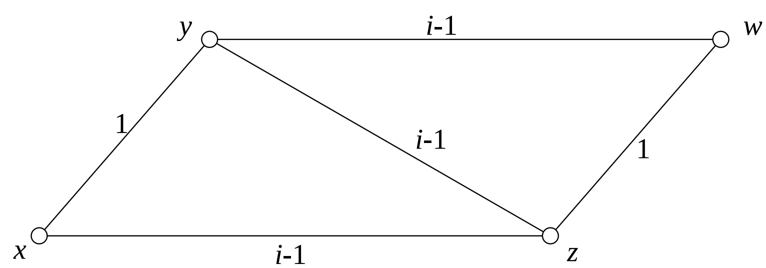

Figure 4: A parallelogram of length i.

3 The Main Theorem

Lemma 3.1. *Let* Γ *denote a* Q*-polynomial distance-regular graph with* $a_1 = 0$ *and diameter* $d \geq 3$*. Fix an integer* i *for* $2 \leq i \leq d$ *and three vertices* x,y,z *with*

$$\partial(y,x) = 1, \quad \partial(x,z) = i-1, \quad \partial(y,z) = i.$$

Then

$$s_i = s_i(x,y,z) = a_{i-1}\frac{(\theta_1^* - \theta_i^*)(\theta_{i-1}^* - \theta_1^*) + (\theta_2^* - \theta_i^*)(\theta_0^* - \theta_{i-1}^*)}{(\theta_0^* - \theta_{i-1}^*)(\theta_{i-1}^* - \theta_i^*)},$$

where

$$s_i(x,y,z) = \mid \Gamma_{i-1}(y) \cap \Gamma_{i-1}(x) \cap \Gamma_1(z) \mid . \tag{3.1}$$

Proof. Let

$$\ell_i(x,y,z) = \mid \Gamma_{i-1}(y) \cap \Gamma_i(x) \cap \Gamma_1(z) \mid .$$

Since $w \in \Gamma_{i-1}(y) \cap \Gamma_1(z)$ implies $w \in \Gamma_{i-1}(x) \cup \Gamma_i(x)$, we have

$$s_i(x,y,z) + \ell_i(x,y,z) = a_{i-1}. \tag{3.2}$$

By (2.15) we also have

$$\sum_{\substack{w \in X \\ \partial(x,w)=i-1 \\ \partial(z,w)=1}} Ew - \sum_{\substack{w \in X \\ \partial(x,w)=1 \\ \partial(z,w)=i-1}} Ew = a_{i-1}\frac{\theta_{i-1}^* - \theta_1^*}{\theta_0^* - \theta_{i-1}^*}(Ex - Ez). \tag{3.3}$$

Taking the inner product of (3.3) with $\hat{y}$ using(2.14), we obtain

$$s_i(x,y,z)\theta_{i-1}^* + \ell_i(x,y,z)\theta_i^* - a_{i-1}\theta_2^* = a_{i-1}\frac{\theta_{i-1}^* - \theta_1^*}{\theta_0^* - \theta_{i-1}^*}(\theta_1^* - \theta_i^*). \tag{3.4}$$

Solving $s_i(x,y,z)$ by using (3.2) and (3.4) we get,

$$s_i(x,y,z) = a_{i-1}\frac{(\theta_1^* - \theta_i^*)(\theta_{i-1}^* - \theta_1^*) + (\theta_2^* - \theta_i^*)(\theta_0^* - \theta_{i-1}^*)}{(\theta_0^* - \theta_{i-1}^*)(\theta_{i-1}^* - \theta_i^*)}. \tag{3.5}$$

□

From Lemma 3.1, $s_i(x,y,z)$ is a constant for any vertices x,y,z with $\partial(y,x)=1$, $\partial(x,z)=i-1$, $\partial(y,z)=i$. We use s_i for this value. Note that $s_i=0$ if and only if Γ contains no parallelogram of length i.

Lemma 3.2. *Let Γ denote a distance-regular graph with classical parameters (d,b,α,β) and $a_1=0$, $a_2\neq 0$. Then $b<-1$.*

Proof. From (2.1), (2.17), (2.18), and since $a_1=0$, $a_2\neq 0$, we have

$$-\alpha(b+1)^2 = a_2-(b+1)a_1 = a_2 > 0. \tag{3.6}$$

Hence

$$\alpha < 0. \tag{3.7}$$

By direct calculation from (2.17), we get

$$(c_2-b)(b^2+b+1) = c_3 > 0. \tag{3.8}$$

Since b is an integer and $b\neq 0,-1$[2, p.195], we have

$$b^2+b+1>0. \tag{3.9}$$

Then from (3.8), implies

$$c_2 > b. \tag{3.10}$$

By using (2.17), (3.10), we get

$$\alpha(1+b) = c_2-b-1 \geq 0. \tag{3.11}$$

Hence $b<-1$, by (3.7) and since $b\neq -1$. □

Theorem 3.3. *Let Γ denote a Q-polynomial distance-regular with diameter $d \geq 3$ and $a_1 = 0, a_2 \neq 0$. Then with referring to definition in (3.1) the following (i)-(iii) are equivalent.*

(i) $s_3 = 0$.

(ii) $s_i = 0$, *for* $3 \leq i \leq d$.

(iii) Γ *has classical parameter* (d, b, α, β).

Proof. (ii)⇒(i) Clear.

(iii)⇒(ii) From (2.20) we have,

$$\theta_i^* - \theta_{i+1}^* = b^{-i}(\theta_0^* - \theta_1^*)$$

for some $b \in \mathbb{R}\backslash\{0, -1\}$. Therefore, for $3 \leq i \leq d$,

$$(\theta_1^* - \theta_i^*) = (\theta_0^* - \theta_1^*)(b^{-1} + b^{-2} + \cdots + b^{i-1}), \tag{3.12}$$

$$(\theta_{i-1}^* - \theta_1^*) = -(\theta_0^* - \theta_1^*)(b^{-1} + b^{-2} + \cdots + b^{i-2}), \tag{3.13}$$

$$(\theta_2^* - \theta_i^*) = (\theta_0^* - \theta_1^*)(b^{-2} + b^{-3} + \cdots + b^{i-1}), \tag{3.14}$$

and

$$(\theta_0^* - \theta_{i-1}^*) = (\theta_0^* - \theta_1^*)(b^0 + b^{-1} + \cdots + b^{i-2}). \tag{3.15}$$

Evaluate (3.5) using (3.12), (3.13), (3.14), (3.15), we find $s_i = 0$ for $3 \leq i \leq d$.

(i)⇒(iii) Suppose $s_3 = 0$. Then by setting $i = 3$ in (3.5),

$$(\theta_1^* - \theta_3^*)(\theta_2^* - \theta_1^*) + (\theta_2^* - \theta_3^*)(\theta_0^* - \theta_2^*) = 0. \tag{3.16}$$

Set

$$b := \frac{\theta_1^* - \theta_0^*}{\theta_2^* - \theta_1^*}. \tag{3.17}$$

Then

$$\theta_2^* = \theta_0^* + \frac{(\theta_1^* - \theta_0^*)(b+1)}{b}. \tag{3.18}$$

Eliminating θ_2^*, θ_3^* in (3.16) using (3.18) and (2.16),we have,

$$\frac{-(\theta_1^* - \theta_0^*)^2(\sigma b^2 + \sigma b + \sigma - b)}{\sigma b^2} = 0. \tag{3.19}$$

for appropriate $\sigma \in \mathbb{R} \setminus \{0\}$. Note that $\theta_1^* \neq \theta_0^*$, hence

$$(\theta_1^* - \theta_0^*)^2(\sigma b^2 + \sigma b + \sigma - b) = 0,$$

so

$$\sigma^{-1} = \frac{b^2 + b + 1}{b}. \tag{3.20}$$

From Theorem 2.2, to prove that Γ has classical parameter, it suffices to prove that

$$\theta_i^* - \theta_0^* = (\theta_1^* - \theta_0^*) \begin{bmatrix} i \\ 1 \end{bmatrix} b^{1-i} (0 \leq i \leq d). \tag{3.21}$$

We prove (3.21) by induction on i. The case $i = 0, 1$ are trivial and case $i = 2$ is from (3.18). Now suppose $i \geq 3$. Then (2.16) implies

$$\theta_i^* = \sigma^{-1}(\theta_{i-1}^* - \theta_{i-2}^*) + \theta_{i-3}^* \tag{3.22}$$

Evaluate (3.22) using (3.20) and the induction hypothesis, we find $\theta_i^* - \theta_0^*$ is as in (3.21). Therefore Γ has classical parameter. $\square$

Theorem 3.4. *Let $\Gamma = (X, R)$ denote a distance-regular graph with intersection numbers $a_1 = 0$, $a_2 \neq 0$. Then the following (i)-(ii) are equivalent.*

(i) Γ *is* Q*-polynomial and* Γ *contains no parallelograms of length* 3*.*

(ii) Γ *has classical parameters.*

Proof. (i) $\Rightarrow$ (ii) Suppose Γ is Q-polynomial and contains no parallelogram of length 3. Then $s_3 = 0$. Hence Γ has classical parameters by Theorem 3.3.

(ii) $\Rightarrow$ (i) Suppose Γ has classical parameters. Then Γ has Q-polynomial property[8, Theorem 8.4.1]. Then (i) holds by Theorem 3.3. $\square$

By the results in [4] and [10], we have the following corollary.

Corollary 3.5. *Let* Γ *denote a distance-regular graph with classical parameters and intersection numbers* $a_1 = 0$*,* $a_2 \neq 0$*. Then for each pair of vertices* $v, w \in X$ *at distance* $\partial(v, w) = 2$*, there exists a strongly regular subgraph* Ω *of* Γ *containing* v, w*. The intersection numbers of* Ω *are*

$$\begin{aligned} a_i(\Omega) &= a_i(\Gamma), \\ c_i(\Omega) &= c_i(\Gamma), \\ b_i(\Omega) &= a_2(\Gamma) + c_2(\Gamma) - a_i(\Gamma) - c_i(\Gamma) \end{aligned}$$

for $0 \leq i \leq 2$*.*

Corollary 3.6. *Let* Ω *be a strongly regular graph with* $a_1 = 0$*,* $a_2 \neq 0$*. Then* $\Omega_2(x)$ *is an* a_2*-regular connected graph with diameter at most 3 for all* $x \in \Omega$*.*

Proof. Fix a vertex $x \in \Omega$, suppose $y \in \Omega_2(x)$, obviously, $\partial(x, y) = 2$. Hence $\mid \Omega_1(y) \cap \Omega_2(x) \mid = a_2$. This shows $\Omega_2(x)$ is a_2-regular.

Suppose that $\Omega_2(x)$ is not connected or is connected with diameter at least 4. Pick $u, v \in \Omega_2(x)$ such that there is no path in $\Omega_2(x)$ of length at most 3 connecting u, v. Observe $\partial(u, v) = 2$, since Ω has diameter 2. For each vertex $z \in \Omega_1(u) \cap \Omega_1(v)$, we must have $\partial(x, z) = 1$, otherwise $\partial(x, z) = 2$ and u, z, v is a path of length 2 in $\Omega_2(x)$. Hence we have $z \in \Omega_1(u) \cap \Omega_1(x)$ and $\Omega_1(u) \cap \Omega_1(v) \subseteq \Omega_1(u) \cap \Omega_1(x)$. Now $\Omega_1(u) \cap \Omega_1(v) = \Omega_1(u) \cap \Omega_1(x)$, since both sets have the same cardinality c_2. Similarly, we have $\Omega_1(u) \cap \Omega_1(v) = \Omega_1(v) \cap \Omega_1(x)$. Pick $w \in \Omega_1(u) \cap \Omega_2(v)$. Then $\partial(x, w) = 2$, since $w \notin \Omega_1(u) \cap \Omega_1(v) = \Omega_1(u) \cap \Omega_1(x)$. We do not have a path of length 2 in $\Omega_2(x)$ connecting w, v, otherwise we can extend this path to a path of length 3 in $\Omega_2(x)$ connecting u, v. By the same argument as above, we have $\Omega_1(w) \cap \Omega_1(v) = \Omega_1(w) \cap \Omega_1(x) = \Omega_1(v) \cap \Omega_1(x)$. Now we have

$$\Omega_1(u) \cap \Omega_1(v) = \Omega_1(v) \cap \Omega_1(x) = \Omega_1(w) \cap \Omega_1(v)$$

Pick $z \in \Omega_1(u) \cap \Omega_1(v) = \Omega_1(w) \cap \Omega_1(v)$. Then z, u, w forms a triangle, a contradiction with $a_1 = 0$. $\square$

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