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應用數學系

碩士論文

Department of Applied Mathematics

National Yang Ming Chiao Tung University

Master Thesis

細分圖上懸掛子圖後的淨化數之探討

The pure number of subdivision graph with pendant subgraphs

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中華民國一一一年六月

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楊子賢　謹誌於

國立陽明交通大學應用數學系

中華民國一一一年六月

細分圖上懸掛子圖後的淨化數之探討

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摘要

給一個簡單無向圖 G，可以知道圖 G 的鄰接矩陣 $A(G)$，進而去計算出 $A(G)$ 的行空間的維度 $r(G)$，稱為秩。再加上 G 的最大匹配數 $m(G)$，可以定義一個函數 $f(G) := 2m(G) - r(G)$。我們稱 $f(G)$ 為圖 G 的淨化數。一個重要結果是樹型圖的淨化數為 0。本論文探討 $m(G)$、$r(G)$ 及 $f(G)$ 的交互關係，推廣前人在樹型圖及單圈圖上的研究，之後探討一類圖的淨化數。此類圖是由任意一圖進行邊細分、在所有新增細分點上懸掛樹型圖、在原始點上懸掛任意圖。如果上述樹型圖上選擇的懸掛點是匹配點，我們得到一個利用小圖的淨化數去計算大圖的淨化數的方法。我們也給了一些未來可研究的問題。

關鍵字：秩、最大匹配數、f 函數、樹型圖、單圈圖、細分圖、匹配點

The pure number of subdivision graph with pendant subgraphs

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Abstract

Let G be a simple undirected graph with the adjacency matrix $A(G)$. The rank $r(G)$ of G is the dimension of the column space of the adjacent matrix $A(G)$ of G, and the matching number $m(G)$ is the maximum number of a matching in G. The number $f(G) := 2m(G) - r(G)$ of G is called the **pure number** of G. It is well-known that every tree has pure number 0. In this thesis, we first studies the relation between the three numbers $m(G)$, $r(G)$ and $f(G)$ of G, and then extends the previous results on the pure numbers of trees and unicyclic graphs to a special class of graphs. Each graph G in this class is obtained from a graph H by edge subdivision, adding pendant-trees on new vertices, and adding pendant-subgraphs on original vertices. If further assume that the above adding pendant-trees use their matched vertices to pend, we show that $f(G)$ is the sum of the f-values of adding pendant-subgraphs, nothing to do with pendant-trees and the graph H. We give some open problems for further study.

Keywords: rank, matching number, pure number, trees, unicyclic graphs, subdivision graphs, matched vertex.

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1 Introduction

The graphs in this thesis are simple undirected. Let G be a graph with adjacency matrix $A(G)$. The rank of $A(G)$ is called the **rank** of G, denoted by $r(G)$. Let $m(G)$ denote the maximal size of a matching in G and the number $f(G) := 2m(G) - r(G)$ is called the **pure number** of G. The three numbers $m(G)$, $r(G)$ and $f(G)$ are related. It is well-known that $f(T) = 0$ for every tree T in [1]. When G is a unicyclic graph, J.-M. Guo, W. Yan and Y.-N. Yeh found that the pure number $f(G)$ take only one of the three values -1, 0 or 2, and characterized these three types of unicyclic graphs in [4]. S.-C. Gong , Y.-Z. Fan and Z.-X. Yin extend the results in [4] to find the rank of a graph with pendant-trees pending on matched vertices in [3]. Our study employs the methods in these two papers [4, 3]. There are two basic methods: One is about a rank operation of graphs by **pendant-edge deletion** as described in Section 3.2 and the other is an f-invariant operation of graphs, called the **pendant-star deletion** as described in Section 3.3. We generalize these two basic methods to a more effective f-invariant operation of graphs, called **pendant-tree deletion** in Theorem 4.5 of Section 4.3. Before doing this, we review how the two basic methods involved in the study of unicyclic graphs in Section 3.4 and the study of graphs with a pendant-tree in Section 3.5. Our main result is Theorem 4.7, which states that if G is obtained from a graph H by edge subdivision, adding pendant-trees on new vertices and adding pendant-subgraphs on original vertices and with the further assumption that the above adding pendant-trees use their matched vertices to pend, then $f(G)$ is the sum of the f-values of adding pendant-subgraphs, nothing to do with pendant trees and the graph H. Three examples of Theorem 4.7 are provided in Section 4.5 and a variant of Theorem 4.7 is given in Section 4.6. We give some problems for further study in Section 4.7.

2 Notations and Definitions

In this section, we provide the notations and definitions which are necessary in this thesis.

2.1 Graphs and subgraphs

A **simple undirected graph** (or a **graph** for short) is a pair $G = (V(G), E(G))$, where $V(G)$ is a finite set and $E(G)$ collects some 2-subsets of $V(G)$. The elements in $V(G)$ and $E(G)$ are called **vertices** and **edges** of G respectively. The number $|V(G)|$ is called the **order** of G and the number $|E(G)|$ is called the **size** of G. For an edge $e = uv \in E(G)$, the following terminologies are used interchangeably: the vertices u and v are **endpoints** of the edge; the vertices u and v are **adjacent**; the vertex u is a **neighbour** of v; the vertex u is **incident** with e; the edge e is **incident** on u. For a vertex $v \in V(G)$, define $N(v)$ to be the set of all vertices which are adjacent to v. The cardinality of $N(v)$ is called the **degree** of vertex v, denoted by $\deg(v)$. A **subgraph** of G is a graph H satisfying $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Let $S \subseteq V(G)$ be a subset of vertices of G and the **induced subgraph** $G[S]$ of G by S is the graph with vertex set S whose edge set consisting of the edges in $E(G)$ that have both endpoints in S. Given $S \subseteq V(G)$, denote by $G - S$ the induced subgraph $G[\overline{S}]$, where $\overline{S} := V(G) - S$. If H is an induced subgraph of G, then we use $G - H$ instead of $G - V(H)$. A **subdivision** of a graph G is the graph $SD(G)$ with vertex set $V(SD(G)) = V(G) \cup E(G)$ and edge set $E(SD(G)) = \{ue, ve : e = uv \in E(G)\}$. We call a vertex in $E(G) \subseteq V(SD(G))$ the **division vertex** and a vertex in $V(G) \subseteq V(SD(G))$ the **original vertex**. See Figure 1 for a graph and its subdivision.

Figure 1: A graph and its subdivision

2.2 Paths and cycles

Throughout the thesis, G is a graph. A **walk** in G is a sequence of edges $(e_1, e_2, \cdots, e_{n-1})$ for which there is a sequence of vertices $(v_1, v_2, \cdots, v_n)$ such that e_i is incident on v_i, v_{i+1} for $i = 1, 2, \ldots, n-1$. A **trail** in G is a walk in which all edges are distinct. A **path** of length k from a vertex u to a vertex v v in G is a trail whose corresponding sequence of vertices $(u = v_0, v_1, \cdots, v = v_k)$ are all distinct except possible $u = v$ in which we call the path a **cycle**. Note that a cycle has length at least 3. A **connected graph** is a graph such that for any two vertices u, v in $V(G)$, there is a path from u to v. A **connected component** of a graph G is a maximal connected induced subgraph of G. An **acyclic graph** is a graph without any cycle.

2.3 Examples of graphs

We use P_n (resp. C_n) to denote the graph itself being a path (resp. a cycle) of order n. A **tree** is a connected acyclic graph. For $n \geq 1$, a **complete graph** of order n, denoted by K_n, is a graph G with all possible $\binom{|V(G)|}{2}$ edges. For $p \geq 1$, let $\overline{K_p}$ denote the graph of order p without any edges. A **bipartite graph** G is a graph whose vertex set $V(G)$ can be divided into two disjoint subsets U and V such that every edge e has one endpoint in U and the other in V; Moreover, if $|E(G)| = |U| \cdot |V|$ holds, then G is called a **complete bipartite graph**, denoted by $G = K_{|U|,|V|}$. The complete graph $K_{1,n-1}$ is also called a **star** of order n and the vertex of degree $n-1$ in $K_{1,n-1}$ is called the

center. We use the notation S_u to denote a star with center u. We say a graph H **is** G if after suitable renaming of $V(H)$ and we have $G = H$. Graphs G and H are **disjoint** if $V(G) \cap V(H) = \emptyset$. If G and H are disjoint graphs, then we use $G \cup H$ to denote the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$, and call $G \cup H$ the **union** of G and H. If H_1 and H_2 are graphs with a unique common vertex u, then the graph, denoted by $H_1 +_u H_2$ with vertex set $V(H_1) \cup V(H_2)$ and edge set $E(H_1) \cup E(H_2)$ is called the **coalescence** of H_1 and H_2 at u. In this situation, we also say that the graph $H_1 +_u H_2$ has **pendant-**H_1 and **pendant-**H_2 pending on the vertex u. A k**-cyclic** graph is a connected graph of order n and size $n + k - 1$. A 1-cyclic graph is also called a **unicyclic** graph. The unique cycle in a 1-cyclic graph is called the **base cycle**. See Figure 2 for a 1-cyclic graph of order 12 with a pendant-tree of order 9 and a pendant-cycle of order 4 pending on the vertex u.

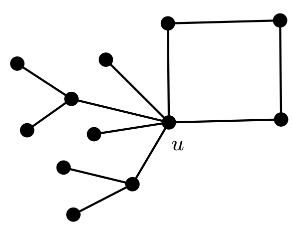

Figure 2: A graph with a pendant-tree of order 9 and a pendant-cycle of order 4 pending on u

If $H_1, \ldots, H_t$ are disjoint graphs and G is a graph such that $V(G) \cap V(H_i) = \{u_i\}$ for $1 \leq i \leq t$, then we denote the graph

$$((\cdots(G +_{x_1} H_1) +_{x_2} H_2 \cdots) +_{x_{t-1}} H_{t-1}) +_{x_t} H_t$$

by

$$G +_{x_i} \bigcup_{i=1}^{t} H_i,$$

and call the graph obtained from G by **adding pendant-H_i** on x_i.

2.4 Matching number $m(G)$, rank $r(G)$ and pure number $f(G)$

A **matching** M in a graph G is a set of edges such that any two edges have no common endpoints. The **matching number** $m(G)$ of G is the largest size of a matching of G. If $|M| = m(G)$, then M is said to be a **maximum matching** in G. If a vertex u is incident with an edge of a matching M, then u is said to be **saturated** by M; Otherwise, u is said to be **unsaturated** by M. A vertex v of G is **matched** in G if $m(G) > 1$ and v is saturated by every maximum matching of G; Otherwise, v is said to be **mismatched** in G.

Let G be a graph with vertex set $V(G)$ $=\{v_1, v_2, \cdots, v_n\}$. Then the **adjacency matrix** $A(G) = (a_{ij})$ of G is an n-by-n matrix such that

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ and } v_j \text{ are adjacent.} \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

The **column space** $\mathcal{C}(A(G))$ of G is the set of vectors consisting of the linear combinations of all column vectors of $A(G)$ over $\mathbb{R}$. The rank of G, denoted by $r(G)$, is the dimension of the **column space** $\mathcal{C}(A(G))$ of G. We define the **pure number** $f(G)$ of G as $2m(G) - r(G)$.

3 Results in the literature

We shall review some known results in this chapter. For the completeness, we also provide some of the proofs.

3.1 The union of disjoint graphs

The following lemma is immediate from the definitions of rank $r(G)$, matching number $m(G)$ and the pure number $f(G)$ of a graph G.

Lemma 3.1. *[1] Let $G = \bigcup_{i=1}^{n} G_i$, where $G_1, G_2, \ldots, G_n$ are connected components of G. Then*

$$r(G) = \sum_{i=1}^{n} r(G_i), \quad m(G) = \sum_{i=1}^{n} m(G_i), \quad f(G) = \sum_{i=1}^{n} f(G_i).$$

From the above lemma, to study $r(G)$, $m(G)$ and $f(G)$ of a graph G, it suffices to assume that G is connected.

3.2 Rank of a graph by pendant-edge deletion

Let G be a graph with a pendant-edge uv pending on vertex v. The deleting of the set $\{u, v\}$ from G is called the **pendant-edge deletion operation**. We use the notation $G - u - v$ to denote the graph $G - \{u, v\}$. The following lemma provides the effect on the ranks after the pendant-edge deletion operation of a graph.

Lemma 3.2. *([2]) Let G be a graph with a pendant-edge uv pending on the vertex v. Then*

$$r(G) = r(G - u - v) + 2.$$

Proof. For visual understanding, the adjacency matrix $A(G)$ of G is illustrated as

$$A(G) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & & y^t & \\ 0 & & & & \\ \vdots & y & & A(G-u-v) & \\ 0 & & & & \end{bmatrix}_{n \times n},$$

where y is $(|V(G)|-2)\times 1$ submatrix of $A(G)$ on $V(G-u-v)\times\{v\}$ and $n=|V(G)|$. Applying row operations by using the first row of $A(G)$ to eliminate y, and applying column operation by using the first column of $A(G)$ to eliminate y^t, this yields the following matrix

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & & & \\ \vdots & \vdots & & A(G-u-v) & \\ 0 & 0 & & & \end{bmatrix}_{n\times n},$$

which has rank $r(G-u-v)+2$. ∎

From above the Lemma 3.1 and the Lemma 3.2, we have the corollary in the following.

Corollary 3.3. *[8] If S is a pendant-star of G pending on the center of S with $|V(S)|\geq 2$, then*

$$r(G)=r(G-S)+2.$$

Proof. Let v be the center of S and let u be a neighbor of v in S. Note that $G-u-v$ equals to $G-S$ together with $|V(S)|-2$ isolated vertices from $S-\{u,v\}$. By Lemma 3.2 and Lemma 3.1, we have

$$r(G)=r(G-u-v)+2=r(\overline{K_{n-2}}\cup(G-S))+2=r(G-S)+2,$$

where $n=|V(S)|$. ∎

3.3 f-invariant operation: pendant-star deletion

We shall prove that the deleting of pendant-star of a graph is f-invariant in this section, and consequently $f(T)=0$ for every tree T.

Lemma 3.4. *[4] If S is a pendant-star of G pending on the center of S and $|V(S)| \geq 2$, then*

$$m(G) = m(G - S) + 1.$$

Proof. We show first that $m(G) \geq m(G-S)+1$. Let u be the center of pendant-star S and uv be an edge in S. Let M be a maximum matching in $G-S$ with $|M| = m(G-S)$. Since the edge $uv \notin M$, we have $m(G) \geq m(G-S)+1$. Next, we show that $m(G) \leq m(G-S)+1$. Since a matching in G with maximum size is divided into two parts: one contains a matching of $G - S$ and the other contains a matching of S. Hence, we have

$$m(G) \leq m(G - S) + m(S) = m(G - S) + 1.$$

■

By Corollary 3.3 and Lemma 3.4, we get the result of f-invariant of the deleting of pendant-star on a graph in the following.

Corollary 3.5. *[8] If S is a pendant-star of G pending on the center of S and $|V(S)| \geq 2$, then*

$$f(G - S) = f(G).$$

Proof. By Corollary 3.3 and Lemma 3.4, we have

$$\begin{aligned} f(G - S) &= 2m(G - S) - r(G - S) \\ &= 2(m(G) - 1) - (r(G) - 2) \\ &= 2m(G) - r(G) = f(G). \end{aligned}$$

■

Motivated by Corollary 3.5, an algorithm taking a tree T with a prescribed vertex u as input and a star S with center u as output by pendant-star deletion is given.

Algorithm 3.6 (Star producing from a tree T with a prescribed vertex u)**.**

Input: A tree T with a vertex u

1: If T is a star with center u, then $S_u \leftarrow T$ and go to 4; otherwise, go to 2.

2: Find a vertex v of degree 1 in T, its neighbor $w \neq u$ and a star S_w containing w and v. Go to 3.

3: Apply the pendant-S_w deletion operation and take the connected component H of $T - S_w$, where H contains u. $T \leftarrow H$ and go back to 1.

4: **return** S_u.

Output: S_u

It has been shown in [8, 6] that the above output star S_u is unique. Indeed, from our later Proposition 4.3, we have

$$V(S_u) = \{u\} \cup \{v \in N(u) : \ v \text{ is mismathced in the subtree of } T - u \text{ containing } v\}.$$

Lemma 3.7. *For any positive integer p, $f(\overline{K_p}) = 0$.*

Proof.

$$f(\overline{K_p}) = 2m(\overline{K_p}) - r(\overline{K_p}) = 2 \cdot 0 - 0 = 0.$$

■

From the Algorithm 3.6, we have the result about the pure number of a tree in the following.

Corollary 3.8. *[1] If T is a tree of order at least 2, then $f(T) = 0$.*

Proof. By Corollary 3.5, Lemma 3.7 , and Algorithm 3.6, it remains to prove that the corollary holds in the base case $T = K_{1,n-1}$. Since $r(K_{1,n-1}) = 2$ and $m(K_{1,n-1}) = 1$, we have

$$f(T) = f(K_{1,n-1}) = 2m(K_{1,n-1}) - r(K_{1,n-1}) = 2 \cdot 1 - 2 = 0.$$

■

3.4 1-cyclic graphs

We will review the result in the study of the pure number of a 1-cyclic graph in this section. Throughout this section, let G be a connected 1-cyclic graph with base cycle C. We assume that $V(C) = \{u_1, u_2, \ldots, u_t\}$ and

$$G = C +_{u_i} \bigcup_{i=1}^{t} T_i, \tag{2}$$

where T_i are disjoint trees such that $V(C) \cap V(T_i) = \{u_i\}$. See Figure 3 for an example with $t = 3$.

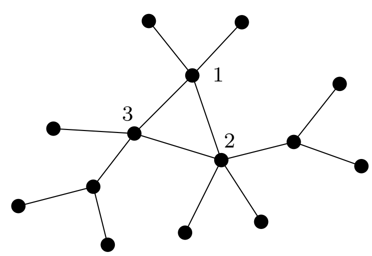

Figure 3: A 1-cyclic graph with three pendant-trees.

Let G^* be the graph obtained from G by replacing each T_i with the star S_i in the output of Algorithm 3.6 on T_i with prescribed vertex $u_i \in V(C)$. The graph G^* is called the **canonical subgraph** of G. See Figure 4 for G^* of the graph G in Figure 3.

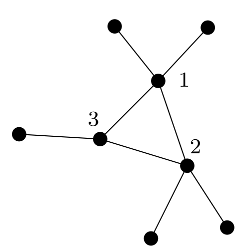

Figure 4: The canonical graph G^* of the graph G in Figure 3.

The rank of the cycle C_n can be easily found.

Lemma 3.9. *[7]*

$$r(C_n) = \begin{cases} n-2 & \text{if } n \text{ is a multiple of } 4, \\ n & \text{otherwise.} \end{cases} \tag{3}$$

The pure number of a 1-cyclic graph G is determined from G^* and the number of vertices in its base cycle C by J.-M.Guo, W.Yan and Y.-N.Yeh [4].

Theorem 3.10. *[4] Let G be a 1-cyclic graph with cycle C. Then*

(i) $f(G) = -1$ *if* $G^* = C$ *and* $|V(C)|$ *is odd,*

(ii) $f(G) = 2$ *if* $G^* = C$ *and* $|V(C)| \equiv 0 \pmod 4$*,*

(iii) $f(G) = 0$ *if* ($G^* = C$ *and* $|V(C)| \equiv 2 \pmod 4$) *or* $G^* \neq C$*.*

Proof. By Corollary 3.5 and Corollary 3.8, G^* is obtained from G by repeated deleting of pendant-star, which is f-invariant, we have $f(G) = f(G^*)$. Assume $G^* \neq C$ in the case (iii), then G^* has a pendant-star S. By Corollary 3.5, we have

$$f(G^*) = f(G^* - S) = f(T) = 0,$$

where a tree T is obtained from G^* by deleting the star S. Another cases, when $G^* = C$, we determine $f(G^*)$ according to the assumptions in the cases (i)-(iii) about $t = |V(C)|$ and Lemma 3.9, we have

(i) $f(G^*) = f(C) = 2m(C) - r(C) = 2(t-1)/2 - t = -1$,

(ii) $f(G^*) = f(C) = 2m(C) - r(C) = 2t/2 - (t-2) = 2$,

(iii) $f(G^*) = f(C) = 2m(C) - r(C) = 2t/2 - t = 0$.

■

3.5 Graph with pendant-tree pending on a matched vertex

S.-C. Gong , Y.-Z. Fan and Z.-X. Yin study the rank of a graph with a pendant-tree. We review their results in this section.

Lemma 3.11. *[3] If G has a pendant-star S pending on the center u of S, then u is matched in G.*

Proof. Suppose the u is mismatched in G and M is a maximum matching of G such that the u is unsaturated by M. Let $v \neq u$ be a vertex in the pendant-star S of G. Then v is also unsaturated by M. Hence, $M \cup \{uv\}$ is a matching of size $|M| + 1$, a contradiction to the maximal of $|M|$. ∎

The following corollary is immediate from Lemma 3.11.

Corollary 3.12. *[3] Let T be a tree of order at least 2. Then there is a matched vertex in a tree T.*

Proof. By Lemma 3.11, this is clear since a tree has a pendant-star. ∎

Theorem 3.13. *[3] Let G be a graph with a pendant-tree T pending on a matched vertex u of T and $|V(T)| \geq 2$. Then*

$$r(G) = r(T) + r(G - T).$$

Proof. We use the induction on the matching number $m(T)$ of T. If $m(T) = 1$, then $T = K_{1,p}$ with center u, where $r(K_{1,p}) = 2$ by Lemma 3.2. Let $v \neq u$ be another vertex in $K_{1,p}$. By Lemma 3.1 and Lemma 3.2,

$$r(G) = r(G - u - v) + 2 = r(\overline{K_{p-1}} \cup G_1) + 2 = r(G_1) + 2 = r(G - K_{1,p}) + r(K_{1,p}),$$

where G_1 is the graph $G - K_{1,p}$. Assume $m(T) \geq 2$. Since T is not a star, there is a pendent-$K_{1,q}$ pending on the center $w \neq u$ of $K_{1,q}$ such that $T - K_{1,q}$ is still a tree. Since $m(K_{1,q}) = 1 < m(T)$, and by induction hypothesis with applying $G = T$ in the statement, we have

$$r(T) = r(K_{1,q}) + r(T - K_{1,q}) = 2 + r(T - K_{1,q}). \qquad (4)$$

Similarly, since $m(K_{1,q}), m(T - K_{1,q}) < m(T)$, by induction hypothesis and (4), we have

$$\begin{aligned} r(G) =& r(K_{1,q}) + r(G - K_{1,q}) \\ =& 2 + r(T - K_{1,q}) + r(G - K_{1,q} - (T - K_{1,q})) \\ =& r(T) + r(G - T). \end{aligned}$$

■

4 Our results

We provide our results in this Chapter.

4.1 Relationship between $f(G)$, $r(G)$ and $m(G)$

The parameters $f(G)$, $r(G)$ and $m(G)$ of a graph G have close relationship in doing the graph deletion.

Proposition 4.1. *Let G be a graph with an induced subgraph H. Then any two statements of the following (i)–(iii) imply the third.*

(i) $f(G) = f(H) + f(G - H)$,

(ii) $r(G) = r(H) + r(G - H)$,

(iii) $m(G) = m(H) + m(G - H)$.

Proof. We will apply Lemma 3.1 to find f, r and m values of disjoint graphs H and $G-H$ in the following:

- $(i),(ii) \Rightarrow (iii)$: From $f(G) = f(H) + f(G-H)$ and $r(G) = r(H) + r(G-H)$, we have

$$
\begin{aligned}
m(G) &= \frac{1}{2}(f(G) + r(G)) \\
&= \frac{1}{2}(f(H) + f(G-H) + r(H) + r(G-H)) \\
&= \frac{1}{2}(f(H) + r(H) + f(G-H) + r(G-H)) \\
&= m(H) + m(G-H).
\end{aligned}
$$

- $(i),(iii) \Rightarrow (ii)$: From $f(G) = f(H) + f(G-H)$ and $m(G) = m(H) + m(G-H)$, we have

$$
\begin{aligned}
r(G) &= 2m(G) - f(G) \\
&= 2(m(H) + m(G-H)) - (f(H) + f(G-H)) \\
&= (2m(H) - f(H)) + (2m(G-H) - f(G-H)) \\
&= r(H) + r(G-H).
\end{aligned}
$$

- $(ii),(iii) \Rightarrow (i)$: From $r(G) = r(H) + r(G-H)$ and $m(G) = m(H) + m(G-H)$, we have

$$
\begin{aligned}
f(G) &= 2m(G) - r(G) \\
&= 2(m(H) + m(G-H)) - (r(H) + r(G-H)) \\
&= (2m(H) - r(H)) + (2m(G-H) - r(G-H)) \\
&= f(H) + f(G-H).
\end{aligned}
$$

■

4.2 A characterization of the output star S of a tree T

In this section, we will introduce the relation between output star S_u in Algorithm 3.6 of a tree T and a prescribed vertex u of T.

Lemma 4.2. *Let u be a vertex in G. Then u is mismatched in G if and only if v is matched in $G - u$ for every $v \in N(u)$.*

Proof. We will prove two things in the following:

($\Rightarrow$) Suppose that u is mismatched in G. Then $m(G) = m(G-u)$. Let M be a matching of $G - u$ with $|M| = m(G - u)$. If v is unsaturated by M, then $M \cup \{uv\}$ is a matching of G with size $m(G - u) + 1 = m(G) + 1$, a contradiction.

($\Leftarrow$) Suppose that for every $v \in N(u)$, v is matched in $G - u$. Let M be a matching of $G - u$ with size $m(G - u)$. Then every $v \in N(u)$ is saturated by M. Let N be the maximum matching of G. Then,

$$m(G) = |N| \leq |N \cap E(\{u\})| + |N \cap E(G - u)| \leq |M| = m(G - u).$$

Hence, M is a maximum matching of G. We conclude that u is not saturated by M. Hence, u is mismatched in G.

■

Proposition 4.3. *Let T be a tree and $u \in V(T)$. Let S_u be the star in the output of the star producing algorithm in Algorithm 3.6. Then the following (i)-(iii) are equivalent.*

(i) *u is mismatched in T;*

(ii) *$V(S_u) = \{u\}$;*

(iii) *$V(S_v) \neq \{v\}$ for every $v \in N(u)$, where S_v is the star in the output of the star producing algorithm when the input is the component T_v containing v in $T - u$.*

Moreover,

$V(S_u) = \{u\} \cup \{v \in N(u)$: *v is mismathced in the component T_v containing v of $T - u\}$.*

Proof. Let $d(T_u)$ be the maximum length of a path from u to a vertex in T. We prove the equivalent of (i)-(iii) and S_u is unique by induction on $d(T_u)$. If $d(T_u) = 0$, then T is the graph with single vertex u, so u is mismatched in T, $V(S_u) = \{u\}$ and $N(u) = \emptyset$. If $d(T_u) = 1$, then u is matched in T, $S_u = T \neq \{u\}$, and $V(S_v) = \{v\}$ for $v \in N(u)$. Suppose that $d(T_u) \geq 2$. Before doing the next step, notice that $d(T_v) < d(T_u)$ for $v \in N(u)$, and by induction, S_v is unique. By the construction of the algorithm,

$$V(S_u) = \{u\} \cup \{v \in N(u) : \; S_v = \{v\}\}.$$

Hence, the equivalent of (i)-(ii) implies

$$V(S_u) = \{u\} \cup \{v \in N(u) : \; v \text{ is mismatched in the component } T_v \text{ containing } v \text{ of } T - u\}.$$

Now, we use the induction hypothesis to prove the equivalent of (i)-(iii).

((i)⇒(iii)) Suppose u is mismatched in T. Then by Lemma 4.2, v is matched in T_v for every $v \in N(u)$. Since $d(T_v) < d(T_u)$ and by induction hypothesis, $V(S_v) \neq \{v\}$ for every $v \in N(u)$.

((iii)⇒(i), (ii)) Suppose $V(S_v) \neq \{v\}$ for every $v \in N(u)$. By induction hypothesis, v is matched in T_v. Hence, u is mismatched in T by Lemma 4.2. This proves (i). By the construction of the algorithm and the unique of S_v, we have $V(S_u) = \{u\}$. This prove (ii).

((ii)⇒(iii)) Suppose $V(S_u) = \{u\}$. For $v \in N(u)$, by the construction of the algorithm and the unique of S_v, we have $V(S_v) \neq \{v\}$.

■

Proposition 4.4. *Let $G = C +_{u_i} \bigcup_{i=1}^{t} T_i$ be a 1-cyclic graph with base cycle C as described in (2), and G^* be the canonical subgraph of G as described in Section 3.4. Then $G^* = C$ if and only if $V(S_{u_i}) = \{u_i\}$ for every $u_i \in V(C)$.*

Proof. This result is immediately from the definition of G^* and Proposition 4.3. ■

4.3 f-invariant operation: pendant-tree deletion

In this section, we generalize the f-invariant operation of pendant-star deletion to the operation of pendant-tree deletion pending on a matched vertex of the tree.

Theorem 4.5. *Let $G = F +_u T$ be a graph with a pendant-tree T pending on a matched vertex u of T. Then*

$$f(G) = f(G - T).$$

Proof. We follow Algorithm 3.6 to choose a star S_w containing an edge wv from T, where v has degree 1 and apply pendant-S_w deletion to have $f(G) = f(G - S_w) = f(F + uH)$ by Corollary 3.5 and Lemma 3.7, where H is the connected component of $T - S_w$ containing u. Repeatedly doing this the process of Algorithm 3.6. Since u is a matched vertex in T, we have $f(G) = f(F + uS_u)$ in the end of Algorithm 3.6, where $|V(S_u)| \geq 2$. Applying pendant-S_u deletion from $F + uS_u$, we have

$$f(G) = f(F + uS_u) = f(F - u) = f(G - T).$$

∎

We will provide another proof of Theorem 4.5 in the end of this section. We need two more lemmas.

Lemma 4.6. *Let G be a graph with a pendant-graph H pending on a vertex u of H. If u is a matched vertex in H, then*

$$m(G) = m(H) + m(G - H).$$

Proof. We claim $m(G) \geq m(H) + m(G - H)$ and $m(G) \leq m(H) + m(G - H)$.

(i) Let M_1 be a maximum matching of H and M_2 be a maximum matching of $G-H$. Let $M = M_1 \cup M_2$. Since H and $G-H$ are disjoint graphs, $M_1 \cap M_2 = \emptyset$. Hence, the M is a matching of G and $|M| = m(H)+m(G-H)$. Hence, $m(G) \geq m(H)+m(G-H)$.

(ii) Since u is a matched vertex in H, we have $m(H-u) = m(H) - 1$. Let N be a maximum matching of G. If u is unsaturated by N or u is saturated by $N \cap E(H)$, then

$$m(G) = |N| = |N \cap E(H)| + |N \cap E(G-H)| \leq m(H) + m(G-H);$$

Otherwise, u is saturated by $N \cap E(G-(H-u))$, we have

$$\begin{aligned} m(G) =& |N| = |N \cap E(H-u)| + |N \cap E(G-(H-u))| \\ \leq& (m(H)-1) + (m(G-H)+1) = m(H) + m(G-H). \end{aligned}$$

■

Now, we are ready to give another proof of Theorem 4.5 in the following:

Second proof of Theorem 4.5. Since the u is matched in T_u, and by Theorem 3.13, we have $r(G) = r(T_u) + r(G - T_u)$. Since T_u is a tree, we have $f(T_u) = 0$ by Corollary 3.8. By Lemma 4.6, we have $m(G) = m(T_u) + m(G - T_u)$. Now by Proposition 4.1, we have

$$f(G) = f(T_u) + f(G - T_u) = f(G - T_u).$$

■

4.4 Pendant-trees to a subdivision graph

We study the pure number of a graph G obtained from a subdivision graph $SD(H)$ of a graph H by adding pendant-graphs pending on the original vertices and adding pendant-trees pending on division vertices which are matched vertices of the corresponding pendant-trees.

Theorem 4.7. *Let H be a graph and H_x be disjoint graphs for $x \in V(H) \cup E(H)$ such that $(V(H) \cup E(H)) \cap V(H_x) = \{x\}$ for $x \in V(H) \cup E(H)$. If H_x is a tree and the vertex*

x is matched for $x \in E(H)$, then graph $G = SD(H) +_x \bigcup_{x \in V(H) \cup E(H)} H_x$ obtained from the subdivision graph $SD(H)$ of H by adding pendant-H_x pending on x has f-value

$$f(G) = \sum_{x \in V(H)} f(H_x).$$

Proof. Applying Theorem 4.5 in G with H_x for every $x \in E(H)$ repeatedly, we have

$$f(G) = f(\bigcup_{x \in V(H)} H_x).$$

By Lemma 3.1, we have

$$f(\bigcup_{x \in V(H)} H_x) = \sum_{x \in V(H)} f(H_x).$$

Hence,

$$f(G) = \sum_{x \in V(H)} f(H_x).$$

■

We apply Theorem 4.7 to the following special type of graphs.

Corollary 4.8. *Let H be a graph and H_x be disjoint graphs for $x \in V(H) \cup E(H)$ such that $(V(H) \cup E(H)) \cap V(H_x) = \{x\}$ for $x \in V(H) \cup E(H)$. Suppose that H_x is a 1-cyclic graph for $x \in V(H)$. If H_x is a tree and x is matched in H_x for $x \in E(H)$, then graph*

$$f(SD(H) +_x \bigcup_{x \in V(H) \cup E(H)} H_x) = 2a - b,$$

where a is the number of H_x for $x \in V(H)$ whose canonical subgraph H_x^* is a cycle of length multiple of 4 and b is the number of H_x for $x \in V(H)$ whose canonical subgraph H_x^* is a cycle of length odd.

Proof. By Theorem 4.7 and Theorem 3.10, we have $f(G) = \sum_{x \in V(H)} f(H_x) = 2a - b$. ■

4.5 An application of Theorem 4.7

We provide three applications of Theorem 4.7 in this section.

Example 4.9. Applying $H = K_3$, $H_x = C_4$ for $x \in V(H)$ and $H_x = K_2$ for $x \in E(H)$ to Corollary 4.8, one can check that $SD(H) = C_6$ and the result graph is depicted in the left of Figure 5. Deleting $H_x = K_2$ for $x \in E(H)$ from the graph in in Figure 5, three disjoint C_4 are remaining as depicted the graph in the right of Figure 5. Applying Corollary 4.8, both graphs G in Figure 5 have the same f-value $f(G) = 2 \cdot 3 - 0 = 6$.

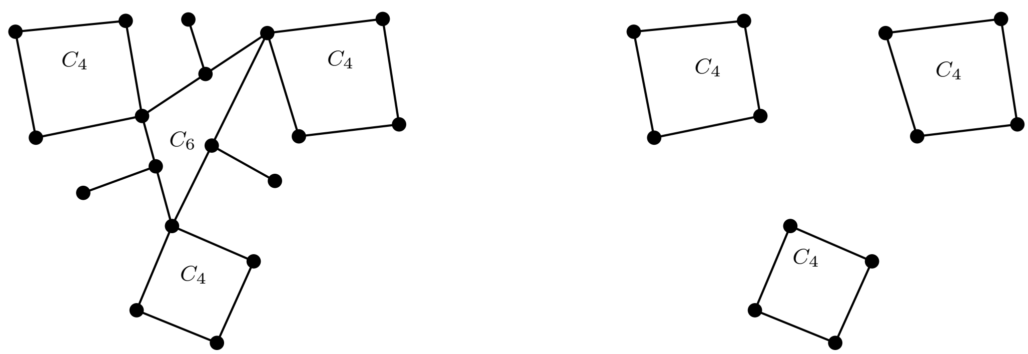

Figure 5: Two graphs with the same f-value 6.

Example 4.10. Applying $H = C_4$, $H_x = C_3$ for $x \in V(H)$ and $H_x = K_{1,3}$ for $x \in E(H)$ to Corollary 4.8, and the result graph is depicted in the left of Figure 6. Deleting $H_x = K_{1,3}$ for $x \in E(H)$ from the graph in in the left of Figure 6, four disjoint C_3 are remaining in the right. Applying Corollary 4.8, Lemma 3.9 and Corollary 3.8, both graphs in Figure 6 have the same f-value -4.

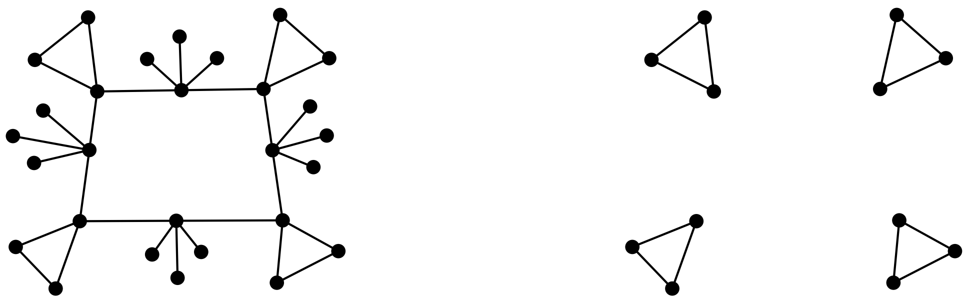

Figure 6: Two graphs with the same f-value -4.

Example 4.11. The graph G in the left of Figure 7 is obtained from the subdivision $SD(e)$ of an edge $e = uv$ by adding pendent-T_w on the division vertex e and adding two pendant-1-cyclic graphs G_1 and G_2 in the right of Figure 7 with cycles C_1 and C_2 on pending vertices u and v respectively. Applying Corollary 3.5 on the pendant-T_w and Lemma 3.1, we have $f(G) = f(G_1) + f(G_2)$. Since the G_1 and G_2 has some pendant-trees on some mismatched roots, by Proposition 4.4 and Proposition 4.3 , and Theorem 3.10, we have

$$f(G) = f(G_1) + f(G_2) = f(G_1^*) + f(G_2^*) = f(C_1) + f(C_2) = -2.$$

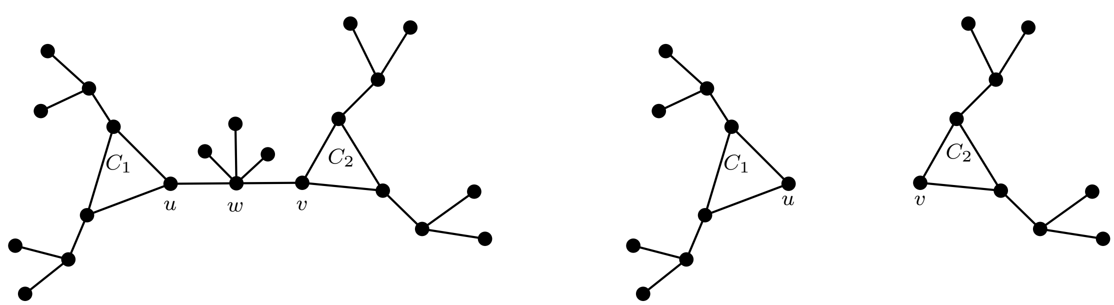

Figure 7: Two graphs with the same f-value -2.

4.6 Variant of Theorem 4.7

A simple modification of Theorem 4.7 yields the following variant version with the same proof.

Theorem 4.12. *Let H be a graph and H_x be disjoint graphs for $x \in V(H) \cup E(H)$ such that $(V(H) \cup E(H)) \cap V(H_x) = \{x\}$ for $x \in V(H) \cup E(H)$. If H_x is a tree and the vertex x is matched for $x \in V(H)$, then*

$$f(SD(H) +_x \bigcup_{x \in V(H) \cup E(H)} H_x) = \sum_{x \in E(H)} f(H_x).$$

Proof. Applying Theorem 4.5 with $G = SD(H) +_x \bigcup_{x \in V(H) \cup E(H)} H_x$ and $T = H_x$ for every $x \in V(H)$ repeatedly, we have

$$f(SD(H) +_x \bigcup_{x \in V(H) \cup E(H)} H_x) = f(\bigcup_{x \in E(H)} H_x).$$

By Lemma 3.1, we have

$$f(\bigcup_{x \in E(H)} H_x) = \sum_{x \in E(H)} f(H_x).$$

Hence,

$$f(SD(H) +_x \bigcup_{x \in V(H) \cup E(H)} H_x) = \sum_{x \in E(H)} f(H_x).$$

■

Example 4.13. Applying $H = C_4$, $H_x = C_3$ for $x \in E(H)$ and $H_x = K_{1,3}$ for $x \in V(H)$ to Theorem 4.12, and the result graph is depicted in the left of Figure 8. Deleting $H_x = K_{1,3}$ for $x \in V(H)$ from the graph in Figure 8, 4 disjoint C_3's are remaining in the right of Figure 8. Both graphs in Figure 8 have the same f-value -4.

Figure 8: Two graphs with the same f-value -4

4.7 Problems for further study

We list a few problems for further study in this section. Lemma 3.2 is a crucial tool in the development of this thesis. The following is a more general problem.

Problem 4.14. Let G and H be two graphs such that $V(G) \cap V(H) = \{x\}$. Express $\text{rank}(G +_x H)$ in terms of $\text{rank}(G)$, $\text{rank}(H)$, $\text{rank}(G - x)$ and $\text{rank}(H - x)$.

Theorem 4.5 gives the equality $f(F +_u T) = f(F - u)$, where $F +_u T$ is the coalescence of graph F and tree T, and u is matched in T. The following problem is the complement part.

Problem 4.15. Let F be a graph and T be a tree such that $V(F) \cap V(T) = \{x\}$ and x is mismatched in T. Express $f(F +_x T)$ in terms of $f(F)$, $f(T)$, $f(F - x)$ and $f(T - x)$.

It worths mentioning that to work on Problem 4.15, the results in [3] need to be studied first. A **cycle-disjoint graph** is a graph whose cycles are pairwise disjoint. In particular, a 1-cyclic graph is a cycle-disjoint graph. The following problem generalizes Theorem 3.10.

Problem 4.16. Determine $f(G)$ for a cyclic graph G.

If G is cycle-disjoint, then contracting every cycle of G yields a tree $T(G)$. Conversely, if some vertices of a tree are replaced by corresponding cycles and each neighbor of a replaced vertex in the tree is connected to a unique vertex in the corresponding cycle, then the resulting graph is a cycle-disjoint graph.

The first step to study Problem 4.16 is to solve the following conjecture.

Conjecture 4.17. If G is a cycle-disjoint graph such that every cycle in G contracting to a matched vertex of degree at least 2 in $T(G)$, then $f(G) = 0$.

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