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A Structure-Preserving Lanczos-type Algorithm with Application to Control Problems

William R. Ferng[1] Wen-Wei Lin[2] Chern-Shuh Wang[3]

Abstract

A Hamiltonian structure-preserving Lanczos-type method, named the J-Lanczos algorithm, is introduced for solving large sparse Hamiltonian eigenvalue problem which arises in both continuous-time and discrete-time optimal control applications. Shift and invert techniques are incorporated to approximate all stable eigenvalues and the associated invariant subspace. Numerical results for solving high order continuous-time Riccati equation arising from position and velocity control for a string of high speed vechicles are presented.

Keywords: optimal control, Riccati equation, Hamiltonian matrix, Lanczos method, eigenvalue problems, structure-preserving.

1 Introduction

Consider the continuous-time optimal control problem of finding a control function $u(t)$ which minimize the cost function

$$J(u) = \int_0^\infty \left[x(t)^T Kx(t) + u(t)^T Ru(t)\right] dt, \qquad (1.1)$$

where $K = K^T \geq 0$ (positive semidefinite) and $R = R^T > 0$ (positive definite), under the constraint

$$\dot{x}(t) = \frac{d}{dt}x(t) = Ax(t) + Bu(t), \qquad x(0) = x_0. \quad (1.2)$$

The standard assumptions assume the pair (A, B) is stabilizable and (C, A) is detectable, where $K = C^T C$ is a full rank factorization of K. It is well-known that, under these assumptions,

$$u(t) = -R^{-1}B^T Xx(t) \qquad (1.3)$$

is the optimal control function, where X is the unique symmetric positive definite solution to the continuous-type algebraic Riccati equation (CARE)

$$-XNX + XA + A^T X + K = 0, \qquad (1.4)$$

with $N = BR^{-1}B^T$.

Let H be the $2n \times 2n$ Hamiltonian matrix

$$H = \begin{bmatrix} A & N \\ K & -A^T \end{bmatrix}. \qquad (1.5)$$

Under the above-mentioned assumptions, H has no eigenvalue with real part zero. If $\begin{bmatrix} Y \\ Z \end{bmatrix}$, where Y, Z are $n \times n$ matrices, spans the invariant subspace of H associated with the n stable eigenvalues (eigenvalues with negative real part), then $X = -ZY^{-1}$ exists and solves the CARE (1.4).

The counterpart of the continuous-time optimal control is the discrete-time optimal control problem of finding u_k to minimize

$$q(x_k, u_k) = \frac{1}{2}\sum_{k=0}^{\infty}(x_k^T C^T C x_k + u_k^T R u_k), \qquad (1.6)$$

subject to

$$x_{k+1} = Ax_k + Bu_k. \qquad (1.7)$$

With the standard assumptions that (A, B) is stabilizable, (C, A) is detectable, and $R > 0$, it can be shown that

$$u_k = -(R + B^T XB)^{-1}B^T XAx_k \qquad (1.8)$$

is the optimal control feedback, where X is the unique positive semidefinite solution of the discrete algebraic Riccati equation (DARE)

$$A^T XA - X - A^T XB(R+B^T XB)^{-1}B^T XA + C^T C = 0. \qquad (1.9)$$

Let

$$L = \begin{bmatrix} I & N \\ 0 & A^T \end{bmatrix}, \qquad M = \begin{bmatrix} A & 0 \\ -K & I \end{bmatrix}, \qquad (1.10)$$

where $N = BR^{-1}B^T \geq 0$, $K = C^T C \geq 0$. Then $M - \lambda L$ has no eigenvalue on the unit circle. Furthermore if $\lambda \in \sigma(M, L)$, then $\bar{\lambda}^{-1} \in \sigma(M, L)$, where $\bar{\lambda}$ denotes the complex conjugate. The unique positive semidefinite solution of the discrete algebraic Riccati equations (1.6) is then given by $X = -ZY^{-1}$, where $\begin{bmatrix} Y \\ Z \end{bmatrix}$ spans the invariant subspace corresponding to the stable eigenvalues ($|\lambda| < 1$) of $M - \lambda L$. If we let

$$H = (M + L)^{-1}(M - L), \qquad (1.11)$$

[1]Department of Applied Mathematics, National Chiao Tung University, Hsin-Chu, Taiwan. Email: ferng@math.nctu.edu.tw
[2]Department of Mathematics, National Tsing Hua University, Hsin-Chu, Taiwan. Email: wwlin@am.nthu.edu.tw
[3]Department of Mathematics, National Cheng Kung University, Tainan, Taiwan.

then one can verify that H is a Hamiltonian matrix such that

$$\lambda \in \sigma(M, L) \Longrightarrow \frac{\lambda+1}{\lambda-1} \in \sigma(H) \qquad (1.12)$$

It is clear that the computational kernel of the both types of optimal control problems is the Hamiltonian eigenvalue problem. A distinct property of a Hamiltonian matrix is that if λ is an eigenvalue, then $-\bar{\lambda}$, where $\bar{\lambda}$ denotes the complex conjugate of λ, is also an eigenvalue. Effort in developing efficient and structure-preserving numerical algorithms for computing the eigenvalues and the associated invariant subspace of a Hamiltonian is therefore justified.

Many structure-preserving numerical algorithms have been proposed for computing the invariant subspace of a Hamiltonian matrix, thus for solving the algebraic Riccati equations. One type of these methods [2, 3] are based on the symplectic QR-type transformations in which the SR factorization with symplectic similarity transformations is used. The other type [7, 8, 9] exploits the squares of the Hamiltonian matrix or skew Hamiltonian matrix to compute the corresponding eigenvalues and uses them to find the stable invariant subspace. These methods are very efficient for problems of small or medium sizes, but become inadequate for very large and sparse cases.

Since there is also a wide class in control theory which leads to solve large sparse Hamiltonian eigenvalue problem [1]. Hence some Lanczos-type algorithm was proposed [5] in which a 2×2 block nonsymmetric look-ahead Lanczos algorithm is applied to reduce the Hamiltonian matrix to a block tridiagonal matrix without modifying the Hamiltonian matrix itself.

In this paper we introduce a structure-preserving Lanczos-type algorithm, named J-Lanczos algorithm, for solving large sparse Hamiltonian eigenvalue problems. In this algorithm, the Hamiltonian matrix is partially reduced to a J-tridiagonal matrix using a sequence of symplectic similarity transformations. Just like the classic Lanczos algorithm, information about the extreme eigenvalues tends to emerge long before the J-tridiagonalization process is completed. The J-Ritz pairs (eigen-pairs of J-tridiagonal submatrices) are used to approximate the extreme eigen-pairs of the Hamiltonian matrix. Since the goal of solving the algebraic Riccati equations is to find the stable invariant subspace corresponding to all stable eigenvalues lying in the open left-half complex plane, some shift and invert techniques is developed. In practice, we start with zero shift and then we use the distribution density of the computed eigenvalues to predict the next shift and the number of the J-Lanczos iterations.

We organize this paper as follows. Some related definitions are listed in section 2, the J-Lanczos method is presented in section 3, and numerical results are shown in section4. Conclusion remarks follow in section 5.

2 Preliminaries

Herein we denote the $n \times n$ identity matrix by I_n and define

$$J_n = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}.$$

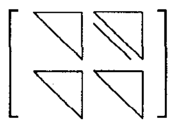
Note that $J_n^{-1} = J_n^T = -J_n$. A matrix $S \in \mathbf{R}^{2n \times 2m}$ $(n \geq m)$ is symplectic if $S^T J_n S = J_m$. A matrix $H \in \mathbf{R}^{2n \times 2n}$ is Hamiltonian if and only if $(JH)^T = JH$. If confusion is unlikely, the subscript n will be omitted. The definition of J-structure matrices is given as follows.

Definition 2.1 *Let*

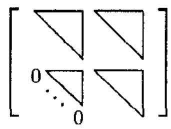
$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$$

be a $2n \times 2n$ matrix with $G_{ij} \in \mathbf{R}^{n \times n}$.

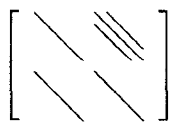
1. *G is called a J-Hessenberg matrix if G_{11}, G_{21}, and G_{22} are upper triangular and G_{12} is upper Hessenberg, that is,*

$G =$ 

2. *G is called a J-upper triangular matrix if G_{11}, G_{12}, and G_{22} are upper triangular and G_{21} is strictly upper triangular, that is,*

$G =$ 

3. *G is called a J-tridiagonal matrix if G_{11}, G_{21}, and G_{22} are diagonal and G_{12} is tridiagonal, that is,*

$G =$ 

Definition 2.2 *Let $H \in \mathbf{R}^{2n \times 2n}$ be a Hamiltonian matrix. Given $x \in \mathbf{R}^{2n}$ and a positive integer j.*

1. *The Krylov matrix of H with respect to x and j is defined by*

$$K[H, x, 2j] = \left[x, Hx, \ldots, H^{j-1}x \mid H^j x, \ldots, H^{2j-1}x\right].$$

2. *The Krylov subspace spanned by the columns of $K[H, x, 2j]$ is denoted by $K(H, x, 2j)$.*

3 J-Lanczos Algorithm

It was shown [4] that for a given Hamiltonian matrix H, if certain conditions hold, there exists a symplectic matrix S such that $T = S^{-1}HS$ is unreduced J-tridiagonal. With column partitioning, we denote

$$S = [q_1, \dots, q_n \mid q_{n+1}, \dots, q_{2n}] \qquad (3.1)$$

and

$$T = \left[\begin{array}{cccc|cccc} a_1 & & & & c_1 & b_1 & & \\ & \ddots & & & b_1 & \ddots & \ddots & \\ & & \ddots & & & \ddots & \ddots & b_{n-1} \\ & & & a_n & & & b_{n-1} & c_n \\ \hline k_1 & & & & -a_1 & & & \\ & \ddots & & & & \ddots & & \\ & & \ddots & & & & \ddots & \\ & & & k_n & & & & -a_n \end{array}\right], \qquad (3.2)$$

with $k_j \neq 0$ for $j = 1, \dots, n$ and $b_j \neq 0$, for $j = 1, \dots, n-1$. Upon comparing columns in $HS = ST$, we obtain

$$\begin{aligned} Hq_j &= a_j q_j + k_j q_{n+j} & (3.3)\\ Hq_{n+j} &= b_{j-1}q_{j-1} + c_j q_j + b_j q_{j+1} - a_j q_{n+j} & (3.4) \end{aligned}$$

for $j = 1, \dots, n$ with $b_0 \equiv 0$. If we require the following conditions hold:

$$\|q_j\|_2 = 1, \qquad q_j \perp q_{n+j}, \qquad (3.5)$$

for $j = 1, \dots, n$, then the coefficients a_j, k_j, c_j, b_j, and q_{n+j}, q_{j+1} can be uniquely determined at the j-th step by the following identities.

$$\begin{aligned} a_j &= q_j^T H q_j, & (3.6)\\ k_j &= q_j^T JHq_j, & (3.7)\\ q_{n+j} &= (Hq_j - a_j q_j)/k_j, & (3.8)\\ c_j &= -q_{n+j}^T JHq_{n+j}, & (3.9)\\ r_j &= Hq_{n+j} - b_{j-1}q_{j-1} - c_j q_j + a_j q_{n+j} & (3.10)\\ b_j &= \|r_j\|_2, & (3.11)\\ q_{j+1} &= r_j/b_j. & (3.12) \end{aligned}$$

By properly sequencing the formulae we obtain a procedure for reducing a Hamiltonian matrix to a J-tridiagonal matrix. The vectors q_j and q_{n+j} are called the J-Lanczos vectors.

The J-tridiagonalization procedure may halt (because of some $b_j = 0$) before the completion of the iteration if the initial J-Lanczos vector q_1 is contained in a proper invariant subspace. Nevertheless this is a welcome event. However, the procedure can also halt (because of some $k_j = 0$) before the J-Lanczos vector, say, q_{n+j}, can be constructed. Such termination does not guarantee an invariant subspace and is called a serious breakdown.

Let

$$S_j = [q_1, \dots, q_j \mid q_{n+1}, \dots, q_{n+j}] \qquad (3.13)$$

and

$$T_j = \left[\begin{array}{cccc|cccc} a_1 & & & & c_1 & b_1 & & \\ & \ddots & & & b_1 & \ddots & \ddots & \\ & & \ddots & & & \ddots & \ddots & b_{j-1} \\ & & & a_j & & & b_{j-1} & c_j \\ \hline k_1 & & & & -a_1 & & & \\ & \ddots & & & & \ddots & & \\ & & \ddots & & & & \ddots & \\ & & & k_j & & & & -a_j \end{array}\right]. \qquad (3.14)$$

The following theorem states the mathematical property of the procedure and shows that the matrix S_j constructed is symplectic.

Theorem 3.1 *Suppose $H \in \mathbf{R}^{2n\times 2n}$ is a Hamiltonian matrix and q_1 is a given unit vector. Let*

$$\Delta_j = \det(K_j^T J K_j), \qquad (3.15)$$

where $K_j = K[H, q_1, 2j]$ is the Krylov matrix. Then the following statements hold.

(a) $\Delta_j \neq 0$, $j = 1, \dots, m$ and rank$([K_m, M^{2m}q_1]) = 2m$ for some $1 \leq m \leq n$ if and only if the J-tridiagonalization procedure runs until $j = m$, i.e., $b_1 \cdots b_{m-1}k_1 \cdots k_m \neq 0$ and $b_m = 0$. Moreover, for $j = 1, \dots, m$, we have

$$HS_j = S_j T_j + r_j e_{2j}^T, \qquad (3.16)$$

and S_j is symplectic, range$(S_j) = K(H, q_1, 2j)$.

(b) $\Delta_j \neq 0$, $j = 1, \dots, m-1$, but $\Delta_m = 0$ and rank$([K_{m-1}, H^{2m-2}q_1]) = 2m-1$ for some $1 \leq m \leq n$ if and only if the procedure runs until $j = m - \frac{1}{2}$, i.e., $b_1 \cdots b_{m-2}k_1 \cdots k_{m-1} \neq 0$, $b_{m-1} \neq 0$, but $k_m = 0$.

Encountering a zero b_j is a welcome event in that it signals the computation of an exact invariant subspace. However, an exact zero or even a small b_j is a rarity in practice. Nevertheless, the extremal eigenvalues of T_j turn out to be surprisingly good approximations to H's extremal eigenvalues. The following theorem provides a computable criteria for convergence check.

Theorem 3.2 *Suppose that j steps of the J-tridiagonalization procedure have been performed and*

T_j has no pure imaginary eigenvalues. Let

$$\Lambda_j = U_j^{-1} T_j U_j = \left[\begin{array}{ccc|ccc} \theta_1 & & & & & \\ & \ddots & & & & \\ & & \theta_j & & & \\ \hline & & & -\theta_1 & & \\ & & & & \ddots & \\ & & & & & -\theta_j \end{array}\right] \quad (3.17)$$

be the J-diagonalization of T_j, where

$$U_j = [u_1, \ldots, u_j \mid u_{j+1}, \ldots, u_{2j}]$$

is symplectic with $\|u_i\|_2 = \|u_{j+i}\|_2 = 1$ for $i = 1, \ldots, j$. Let

$$Y_j = [y_1, \ldots, y_j \mid y_{n+1}, \ldots, y_{n+j}] = S_j U_j$$

and denote $U_j = (u_{k,l})$. Then, for $i = 1, \ldots, j$

$$\|Hy_i - \theta_i y_i\|_2 = |\beta_{j,i}|, \quad (3.18)$$

where $\beta_{j,i} = b_j u_{2j,i}$, and

$$\|Hy_{n+i} + \theta_i y_{n+i}\|_2 = |\beta_{j,j+i}|, \quad (3.19)$$

where $\beta_{j,j+i} = b_j u_{2j,j+i}$.

The vectors y_i and y_{n+i} are called the J-Ritz vectors corresponding to the J-Ritz values θ_i and $-\theta_i$, respectively. The extremal eigenpairs of H are approximated by J-Ritz pairs. We summarize the J-Lanczos method in the following algorithm.

Algorithm 3.1 (J-Lanczos) *Given a Hamiltonian matrix $H \in \mathbf{R}^{2n\times 2n}$, a unit vector q_1, and a convergence tolerance ϵ, this algorithm computes the columns of the symplectic matrix S_j and entries of the $2j \times 2j$ J-tridiagonal matrix T_j using the identities (3.6) $\sim$ (3.12), and computes the J-Ritz pair (θ_i, y_i) by symplectic QR-like algorithm [2] or conventional QR algorithm to approximate the extremal eigenpairs of H. Stop criterion is based on (3.18) and (3.19).*

Set $b_{-1} = 0$, $q_{-1} = 0$, $j = 1$.
while $j \le$ maximal iterations specified
  $a_j = q_j^T H q_j$
  $k_j = q_j^T J H q_j$
  if $k_j = 0$, serious breakdown, stop.
  $q_{n+j} = (Hq_j - a_j q_j)/k_j$
  $c_j = -q_{n+j}^T J H q_{n+j}$
  $r_j = Hq_{n+j} - b_{j-1} q_{j-1} - c_j q_j + a_j q_{n+j}$
  $b_i = \|r_j\|_2$
  If $b_j = 0$, invariant subspace found, stop.
  $q_{j+1} = r_j / b_j$
  Compute eigen decomposition $U_j^{-1} T_j U_j = \Lambda_j$
  Compute $Y_j = S_j U_j$
  for $i = 1, \ldots, j$,
    if $|\beta_{j,i}| \le \epsilon$ and $|\beta_{j,j+i}| \le \epsilon$, accept $(\theta_i, y_i), (-\theta_i, y_{n+i})$ and their conjugate pairs as the desired eigen-pairs.
  end for
  $j = j + 1$
end while

All Lanczos-type algorithms converge fast for approximating some extremal eigenvalues, not all eigenvalues. For solving Riccati equations (1.4) and (1.9), one has to compute all eigenpairs of the associated Hamiltonian matrix. Hence some shift and invert technique has to be considered and incorporated into the J-Lanczos algorithm.

An important consideration of the shift-invert approach is how to preserve the Hamiltonian structure of the transformed matrix. Such transformation can usually be written as a rational matrix function

$$f(H) = \sum_{k=-\infty}^{\infty} c_k H^k, \quad c_k \in \mathbf{C}.$$

To preserve the Hamiltonian structure, we require that $(Jf(H))^* = Jf(H)$, where $*$ denotes the complex conjugate transpose. Since H is Hamiltonian, $(JH^k)^* = (-1)^{k+1} JH^k$, hence

$$J\sum_k \bar{c}_k (-1)^{k+1} H^k = J \sum_k c_k H^k.$$

Write $c_k = \alpha_k + i\beta_k$, where $i = \sqrt{-1}$, $\alpha_k, \beta_k \in \mathbf{R}$. By comparing the coefficients in the above identity, we have

$$c_k = \begin{cases} \alpha_k & \text{if } k \text{ is odd number,} \\ i\beta_k & \text{if } k \text{ is even number.} \end{cases}$$

Since the J-Lanczos method uses real symplectic similarity transformations to partially reduce a Hamiltonian matrix to J-tridiagonal form, we would requires $c_k = 0$ for all even index k. For practical implementation, the following three types of shift-invert are considered.

1. Choose
$$f(H) = H^{-1}$$
whenever the desired eigenvalues are of the smallest modules.
2. Choose
$$\begin{aligned} f(H) &= (H \pm \delta^2 H^{-1})^{-1} \\ &= (H^2 \pm \delta^2 I)^{-1} H, \end{aligned}$$

for real shift, where $\delta \in \mathbf{R}$ and $\delta > 0$. The choice of $\pm$ sign depends on whether the desired eigenvalues are close to the real numbers $\pm\delta$ or to the imaginary numbers $\pm i\delta$.

3. Choose

$$\begin{aligned} f(H) &= (H^3 + bH + cH^{-1})^{-1} \\ &= (H^4 + bH^2 + cI)^{-1}H, \end{aligned}$$

for complex shift, where $b = 2(\beta^2 - \alpha^2)$ and $c = (\alpha^2 + \beta^2)^2$ with $\alpha, \beta \in \mathbf{R}, \alpha, \beta > 0$, and the desired eigenvalues are close to the complex numbers $\pm(\alpha \pm i\beta)$.

4 Numerical Results

The numerical experiment is conducted on applying the$J$-Lanczos algorithm to solving high order continuous-time Riccati equation (1.1) arising from position and velocity control for a string of high-speed vehicles [1]. In this application, the matrices N, K, A are

$$\begin{aligned} N &= \mathrm{diag}(1, 0, 1, 0, \ldots, 0, 1), \\ K &= \mathrm{diag}(0, 10, 0, 10, \ldots, 10, 0), \end{aligned}$$

and

$$A = \begin{bmatrix} A_{11} & A_{12} & & & & \\ & A_{22} & A_{23} & & & \\ & & \ddots & \ddots & & \\ & & & \ddots & A_{m-1,m} & 0 \\ & & & & A_{m,m} & v \\ & & & & 0 & w \end{bmatrix}$$

with

$$A_{ii} = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}, \quad A_{i,i+1} = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix},$$

$$v = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad w = -1.$$

For a string of $m = 500$ vehicles, it is necessary to solve the Riccati equation of order $n = 2m + 1 = 1001$ and consequently the associated Hamiltonian matrix is of order 2002. The J-Lanczos Algorithm incorporated with shift and invert techniques is implemented in MATLAB to solve the problem on a Sun SPARC-10 workstation with 32 MB of main memory. Table 1 shows the results with real shift δ to $(H + \delta H^{-1})^{-1}$ and Table 2 for complex shift $\pm(\alpha \pm i\beta)$ to $(H^3 + bH + cH^{-1})^{-1}$.

Some observations and remarks of the numerical experiments are in order.

Table 1: Summary of numerical results with real shift δ

J-Lanczos iterations	Total number of shifts	Total number of eigenvalues	Time per eigenvalue
j	δ	λ	sec.
20	5	58	29.4
30	5	118	21.6
40	4	112	30.6
50	3	122	29.7

Table 2: Summary of numerical results with complex shifts $\pm(\alpha \pm \beta i)$

J-Lanczos iterations	Total number of shifts	Total number of eigenvalues	Time per eigenvalue
j	$\pm(\alpha \pm \beta i)$	λ	sec.
30	60	1276	60.3
50	40	1560	42.3
80	10	902	21.9

1. In this experiment, complex shifts along a straight line on the complex plane with an argument angle $\pi/4$ are actually implemented.
2. The J-Lanczos algorithm is most efficient when 30 iterations are taken with real shift and 80 iterations with complex shift. It requires more iterations for the complex shift because the complex eigenvalues are more clustered than the real eigenvalues.
3. The relation between ν, the number of converged eigenvalues, and j, the number of J-Lanczos iterations performed, is approximately $\nu = 1.4 \times j - 25$.
4. The dominate cost of this shift-invert J-Lanczos method is the LU-factorization of the shift matrices and the associated triangular solvers. In this particular test problem, the underlined Hamiltonian matrix and all shift matrices are band matrices. Therefore fast band factorization routine and storage format are easy to implement.
5. In the implementation, $\epsilon = 10^{-10}$ is used as the tolerance for testing convergence. The obtained computed solution $\hat{X}$ to the algebraic Riccati equation has residual norm $\| - \hat{X}N\hat{X} + \hat{X}A + A^T\hat{X} + K\|_2 = 1.6 \times 10^{-5}$.

5 Conclusions

For very large and sparse Hamiltonian matrices, the general QR method [6] or structure-preserving numerical methods proposed in [2, 3, 9, 7] for computing the stable invariant subspaces become inadequate when storage and computational effort are big concern. Alternatively, we proposed the Hamiltonian structure-preserving J-Lanczos method incorporating with shift-invert techniques. Numerical results showed that this new approach can efficiently solve the problem with high accuracy.

Finally, we would like to comment that, unlike the serial oriented symplectic QR-type algorithms, parallel implementation of the shift-invert J-Lanczos algorithm is straightforward.

References

[1] M. Athans, W. S. Levine, and A. Levis, "A system for the optimal and suboptimal position and velocity control for a string of high-speed vehicles", *Proc. 5th Int. Alalogue Computation Meetings*, Lausanne, Switzerlan, Sep. 1967.

[2] A. Bunse-Gerstner and V. Mehrmann, "A symplectic QR like algorithm for the solution of real algebraic Riccati equation", *IEEE Trans. on Auto. Contr.*, 31(1986), pp.1104–1113.

[3] A. Bunse-Gerstner, R. Byers and V. Mehrmann, "A chart of numerical methods for structured eigenvalue problems", *SIAM J. Matr. Anal. Appl.*, 13(1992), pp.419–453.

[4] William R. Ferng, Wen-Wei Lin, and Chern-Shuh Wang, "The shift-inverted J-Lanczos algorithm for the numerical solutions of large sparse algebraic Riccati equations", *Technical Report NCTU-AM-94-006*, Department of Applied Mathematics, National Chiao Tung University, Hsin-Chu, Taiwan, ROC.

[5] R. W. Freund, "Lanczos-type algorithms for structured non-Hermitian Eigenvalue problems", *Proceeding of the Cornelius Lanczos International Centenary Conference*, J. David Brown, Moody T. Chu, Donald C. Ellison, and Robert J. Plemmons, eds., 1994.

[6] A. Laub, "A Schur method for solving algebraic Riccati equations", *IEEE Trans. Auto. Contr.*, 24(1979), pp.913–921.

[7] R.V. Patel, Z. Lin and P. Misra, "Computation of stable invariant subspaces of Hamiltonian matrices", *SIAM J. Matr. Anal. Appl.*, 15(1994), pp.284–298.

[8] C.F. Van Loan, "A symplectic method for approximating all the eigenvalues of a Hamiltonian matrix", *Lin. Alg. Appl.*, 61(1984), pp.233–251.

[9] H.-G. Xu, "Solving Algebraic Riccati Equation via Skew-Hamiltonian Matrices", Ph. D. thesis, Fudan Univ., Shanghai, China, 1991.

[10] Q. Ye, "A convergence analysis for nonsymmetric Lanczos algorithms", *Math. Comput.*, 56(1991), pp.677–691.