Periodic Invariant Subspaces in Control

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Abstract

In this paper we present several different characterizations of invariant subspaces of periodic eigenvalue problems. We analyze their equivalence and discuss their use in control theory.

1 Main results

Consider the (homogeneous) linear time varying system :

$$E_k x_{k+1} = A_k x_k + B_k u_k, \qquad k \in \mathbf{N} \tag{1.1}$$

where $\mathbf{N}$ is the set of natural numbers, x_k is an n-dimensional vector of descriptor variables, u_k is an m-dimensional vector of input variables, the matrices E_k and A_k are $n \times n$, and B_k is $n \times m$. This system is said to be periodic with period K if $E_k = E_{k+K}$, $A_k = A_{k+K}$ and $B_k = B_{k+K}$, for all $k \in N$, and K is the smallest positive integer for which this holds. If we allow both the E_k and A_k matrices to be singular, then x_k may still uniquely be defined in the context of a two point boundary value problem, as e.g. in optimal control of periodic systems. It is shown in [4] that a necessary and sufficient condition for this is that the pencil

$$\lambda \mathcal{E} - \mathcal{A} := \begin{bmatrix} \lambda E_1 & & & -A_1 \\ -A_2 & \lambda E_2 & & \\ & \ddots & \ddots & \\ & & -A_K & \lambda E_K \end{bmatrix} \tag{1.2}$$

is regular (i.e. $\det(\lambda \mathcal{E} - \mathcal{A}) \not\equiv 0$). We call such periodic systems *regular*.

We can always define a periodic similarity transformation of a regular periodic system (1.1) by multiplying it from the left by the invertible transformation S_k and substituting x_k by $x_k \doteq T_{k-1}\hat{x}_k$, where S_k and T_k are again K-periodic. Defining the *periodic*

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similarity transformation

$$
\begin{bmatrix} S_1 & & & \\ & S_2 & & \\ & & \ddots & \\ & & & S_K \end{bmatrix}
\begin{bmatrix} \lambda E_1 & & & -A_1 \\ -A_2 & \lambda E_2 & & \\ & \ddots & \ddots & \\ & & -A_K & \lambda E_K \end{bmatrix}
\begin{bmatrix} T_1 & & & \\ & T_2 & & \\ & & \ddots & \\ & & & T_K \end{bmatrix} \tag{1.3}
$$

$$
\doteq \begin{bmatrix} \lambda \hat{E}_1 & & & -\hat{A}_1 \\ -\hat{A}_2 & \lambda \hat{E}_2 & & \\ & \ddots & \ddots & \\ & & -\hat{A}_K & \lambda \hat{E}_K \end{bmatrix} \tag{1.4}
$$

one then obtains a new system

$$
\hat{E}_k \hat{x}_{k+1} = \hat{A}_k \hat{x}_k + \hat{B}_k u_k, \qquad k \in \mathbf{N}, \tag{1.5}
$$

where $\hat{B}_k \doteq S_k B_k$.

One important case of such a periodic similarity transformation is the so-called "Floquet transform" of a periodic system :

$$
\begin{bmatrix} \lambda \hat{E}_1 & & & -\hat{A}_1 \\ -\hat{A}_2 & \lambda \hat{E}_2 & & \\ & \ddots & \ddots & \\ & & -\hat{A}_K & \lambda \hat{E}_K \end{bmatrix}
=
\begin{bmatrix} \lambda I & & & -M \\ -M & \lambda I & & \\ & \ddots & \ddots & \\ & & -M & \lambda I \end{bmatrix}. \tag{1.6}
$$

This reduces the homogeneous problem to a time-invariant one, but the transformation may not always exist [6, 5]. Another important case is the Periodic Schur Form [1], where the transformation matrices S_k and T_k are constrained to be unitary, and the resulting matrices $\hat{E}_k$ and $\hat{A}_k$ can all be chosen upper triangular. This second form always exists and can be computed in a numerically stable manner [1].

Both forms are closely linked to the concept of *periodic invariant subspaces*, which play a fundamental role in the solution of important control problem of periodic systems [3], [5], [2] :

- solution of periodic Riccati equations and their use in optimal and robust control,
- solution of periodic Lyapunov equations and their use in stability analysis,
- solution of periodic Sylvester equations and their use in decoupling,
- inverse eigenvalue problems and their use in pole placement.

In the paper, we describe several characterizations of invariant subspaces of a regular periodic system. A first characterization uses the classical concept of deflating subspace, directly applied to the pencil (1.2) (we assume $K = 3$ for the sake of simplicity) :

$$\begin{bmatrix} \lambda E_1 & & -A_1 \\ -A_2 & \lambda E_2 & \\ & -A_3 & \lambda E_3 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} (\lambda I - M), \tag{1.7}$$

where $X \doteq \begin{bmatrix} X_1^T & X_2^T & X_3^T \end{bmatrix}^T$ and $Y \doteq \begin{bmatrix} Y_1^T & Y_2^T & Y_3 \end{bmatrix}^T$ are assumed to have full rank d. We can also rewrite this as follows :

$$\begin{bmatrix} \lambda E_1 & & -A_1 \\ -A_2 & \lambda E_2 & \\ & -A_3 & \lambda E_3 \end{bmatrix} \begin{bmatrix} X_1 & & \\ & X_2 & \\ & & X_3 \end{bmatrix} = \begin{bmatrix} Y_1 & & \\ & Y_2 & \\ & & Y_3 \end{bmatrix} \begin{bmatrix} \lambda I & & -M \\ -M & \lambda I & \\ & -M & \lambda I \end{bmatrix} \tag{1.8}$$

so that every d-dimensional eigenspace of the type (1.7) induces a $d \cdot K$-dimensional eigenspace of the type (1.8), *provided* all individual matrices X_k and Y_k have full column rank. Conditions on M for which this holds are analyzed in the paper.

A slightly more general form involving again full rank matrices X_k and Y_k is the following :

$$\begin{bmatrix} \lambda E_1 & & -A_1 \\ -A_2 & \lambda E_2 & \\ & -A_3 & \lambda E_3 \end{bmatrix} \begin{bmatrix} X_1 & & \\ & X_2 & \\ & & X_3 \end{bmatrix} = \begin{bmatrix} Y_1 & & \\ & Y_2 & \\ & & Y_3 \end{bmatrix} \begin{bmatrix} \lambda \bar{E}_1 & & -\bar{A}_1 \\ -\bar{A}_2 & \lambda \bar{E}_2 & \\ & -\bar{A}_3 & \lambda \bar{E}_3 \end{bmatrix} \tag{1.9}$$

or, equivalently :

$$\left\{ \begin{array}{l} E_1 X_1 = Y_1 \bar{E}_1 \\ E_2 X_2 = Y_2 \bar{E}_2 \\ E_3 X_3 = Y_3 \bar{E}_3 \end{array} \right. , \quad \text{and} \quad \left\{ \begin{array}{l} A_1 X_3 = Y_1 \bar{A}_1 \\ A_2 X_1 = Y_2 \bar{A}_2 \\ A_3 X_2 = Y_3 \bar{A}_3 \end{array} \right. . \tag{1.10}$$

This last form (1.9) is very close to the periodic Schur form [1], which says that there always exist unitary matrices Q_k and Z_k, $k = 1, \ldots, K$ such that :

$$\begin{bmatrix} Z_1 & & \\ & Z_2 & \\ & & Z_3 \end{bmatrix} \begin{bmatrix} \lambda E_1 & & -A_1 \\ -A_2 & \lambda E_2 & \\ & -A_3 & \lambda E_3 \end{bmatrix} \begin{bmatrix} Q_1 & & \\ & Q_2 & \\ & & Q_3 \end{bmatrix} = \begin{bmatrix} \lambda \hat{E}_1 & & -\hat{A}_1 \\ -\hat{A}_2 & \lambda \hat{E}_2 & \\ & -\hat{A}_3 & \lambda \hat{E}_3 \end{bmatrix} \tag{1.11}$$

where the matrices $\hat{E}_k$ and $\hat{A}_k$ are upper-triangular.

Using the regularity of (1.2) we show that there exist "swapping" factors $\tilde{A}_k$ and $\tilde{E}_k$ such that

$$\bar{E}_k \tilde{A}_k = \bar{A}_k \tilde{E}_k,$$

which allow us to rewrite (1.10) in a form which does not involve Y_k anymore :

$$\left\{ \begin{array}{l} E_1 X_1 \tilde{A}_1 = A_1 X_3 \tilde{E}_1 \\ E_2 X_2 \tilde{A}_2 = A_2 X_1 \tilde{E}_2 \\ E_3 X_3 \tilde{A}_3 = A_3 X_2 \tilde{E}_3 \end{array} \right. . \tag{1.12}$$

Finally, we point out that (1.10) can be expressed geometrically as the conditions

$$\left\{ \begin{array}{l} \dim (E_1 \mathcal{X}_1 + A_1 \mathcal{X}_3) = d = \dim \mathcal{X}_1 \\ \dim (E_2 \mathcal{X}_2 + A_2 \mathcal{X}_1) = d = \dim \mathcal{X}_2 \\ \dim (E_3 \mathcal{X}_3 + A_3 \mathcal{X}_2) = d = \dim \mathcal{X}_3, \end{array} \right. \tag{1.13}$$

where $\mathcal{X}_k$ is the column space of X_k. We thus obtain equivalences between deflating subspace defined by (1.7), the general form (1.9), the algebraic form (1.12), the geometric form (1.13), and the periodic Schur decomposition (1.11).

In the paper we show that (1.12), (1.13), and the block triangular periodic Schur decomposition (1.11) are all equivalent for arbitrary E_k, A_k, as long as (1.2) is a regular pencil. On the other hand, (1.7) and (1.9) only apply to the invariant subspaces with finite nonzero reduced spectrum. For this reason, we propose an algorithm that *deflates* the 0 and ∞ eigenvalues, yielding a smaller dimensional periodic pencil without these "troublesome" eigenvalues. We then introduce the concept of regular periodic matrix pairs $((A_k, E_k))_{k=1}^{K}$ and define their "periodic eigenvalues" and "periodic eigenvectors", in contrast to those of the pencil (1.2). Finally, we give a new "constructive" proof of the periodic Schur theorem.

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References

[1] A. Bojanczyk, G. Golub, and P. Van Dooren, "The periodic Schur decomposition. Algorithms and applications", in *Proceedings of the SPIE Conference*, San Diego, Vol. 1770, July 1992, pp. 31-42.

[2] W. R. Ferng, W.-W. Lin, and C.-S. Wang, "On computing the stable deflating subspaces of cyclic symplectic matrix pairs", submitted for publication, 1998.

[3] J. Sreedhar, P. Van Dooren, "A Schur approach for solving some periodic matrix equations," in *Systems and Networks : Mathematical Theory and Applications*, pp. 339-362, Eds. U. Helmke, R. Mennicken, J. Saurer, *Mathematical Research*, Vol 77, Akademie Verlag, Berlin 1994.

[4] J. Sreedhar, P. Van Dooren, "Periodic descriptor systems : solvability and conditionability", to appear in the *IEEE Trans. Aut. Contr.*, 1997.

[5] J. Sreedhar, P. Van Dooren, "Forward/backward decomposition of periodic descriptor systems and two point boundary value problems", in *Proceedings of the European Control Conf.*, paper FR-A-L7, 1997.

[6] P. Van Dooren and J. Sreedhar, "When is a periodic discrete-time system equivalent to a time-invariant one ?", *Linear Algebra & Applications*, Vol. 212/213, pp. 131-152, 1994.