Reachability and Observability of Periodic Descriptor Systems

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Abstract

In this paper, the necessary and sufficient conditions are derived for complete reachability and observability of periodic time-varying descriptor systems.

Key words. periodic systems, descriptor systems, reachability and observability.

1 Introduction

In the second-half of the last century, the development of systems and control theory, together with the achievements of digital control and signal processing, has set the stage for renewed interest in the study of periodic systems, both in continuous and discrete time; see, e.g., [24, 45, 36, 10, 43, 12] and the survey papers [3, 4]. This has been amplified by specific application demands in the aerospace realm [17, 25, 16], computer control of industrial processes [5] and communication systems [35, 10, 34, 44]. The number of contributions on linear time-varying discrete-time periodic systems has been increasing in recent times; see, e.g., [13, 15, 18, 37, 39, 40] and the references therein. This increasing interest in periodic systems has also been motivated by the large variety of processes that can be modelled through linear discrete-time periodic systems (e.g., multirate sampled-data systems, chemical processes, periodically time-varying filters and networks, and seasonal phenomena [2, 3, 7, 14, 23, 26, 42]).

We consider here a periodic time-varying descriptor system of the form

$$E_k x_{k+1} = A_k x_k + B_k u_k, \quad y_k = C_k x_k, \quad k \in \mathbb{Z}, \tag{1.1}$$

where the matrices $E_k,\ A_k \in \mathbb{R}^{n\times n}$, $B_k \in \mathbb{R}^{n\times m}$, $C_k \in \mathbb{R}^{p\times n}$ are periodic with period $K \geq 1$, i.e., $E_k = E_{k+K}$, $A_k = A_{k+K}$, $B_k = B_{k+K}$, $C_k = C_{k+K}$, and the matrices E_k are allowed to be singular for all k. Recently, this class of periodic descriptor systems (1.1) is discussed and studied extensively in the problem of solvability and conditionability [29], the computation of H_∞-norm and system zeros [28, 41], and the compensating and regularization problems for periodic descriptor systems [8, 19].

It is well known that the dynamics of the discrete-time periodic descriptor system (1.1) depend critically on the regularity and the eigenstructure of the periodic matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$ which satisfy the homogeneous systems of (1.1):

$$E_k x_{k+1} = A_k x_k, \quad k \in \mathbb{Z}. \tag{1.2}$$

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The set of matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$ is said to be regular when $\det[C((\alpha_k, \beta_k)_{k=0}^{K-1})] \not\equiv 0$, where

$$C((\alpha_k, \beta_k)_{k=0}^{K-1}) \equiv \begin{bmatrix} \alpha_0 E_0 & 0 & \cdots & 0 & -\beta_0 A_0 \\ -\beta_1 A_1 & \alpha_1 E_1 & & & 0 \\ & \ddots & \ddots & & \vdots \\ & & \ddots & \ddots & 0 \\ 0 & & 0 & -\beta_{K-1} A_{K-1} & \alpha_{K-1} E_{K-1} \end{bmatrix}, \tag{1.3}$$

in which α_k, β_k are complex variables for $k = 0, \ldots, K-1$. Note that we are considering the regularity of the *set* of matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$, rather than the regularity of the individual matrix pairs (E_k, A_k).

Definition 1.1. *[21] Let $\{(E_k, A_k)\}_{k=0}^{K-1}$ be $n \times n$ a regular set of matrix pairs. If there exist complex numbers $\alpha_0, \cdots, \alpha_{K-1}$, $\beta_0, \cdots, \beta_{K-1}$ which satisfy*

$$\det[C((\alpha_k, \beta_k)_{k=0}^{K-1})] = 0 \ , \quad \left(\prod_{k=0}^{K-1} \alpha_k, \prod_{k=0}^{K-1} \beta_k \right) \equiv (\pi_\alpha, \pi_\beta) \neq (0, 0) \tag{1.4}$$

then (π_α, π_β) is an eigenvalue pair of $\{(E_k, A_k)\}_{k=0}^{K-1}$.

Note that if (π_α, π_β) is an eigenvalue pair of $\{(E_k, A_k)\}_{k=0}^{K-1}$, then (π_α, π_β) and $(\tau\pi_\alpha, \tau\pi_\beta)$ represent the same eigenvalue for any nonzero τ. If $\pi_\beta \neq 0$, then $\lambda = \pi_\alpha/\pi_\beta$ is a finite eigenvalue; otherwise $(\pi_\alpha, 0)$ represents an infinite eigenvalue. The spectrum, or the set of all eigenvalue pairs, of $\{(E_k, A_k)\}_{k=0}^{K-1}$ is denoted by $\sigma\left(\{(E_k, A_k)\}_{k=0}^{K-1}\right)$. We shall assume throughout the paper that the set of periodic matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$ is regular, and also use the notation $\sigma(M)$ to denote the spectrum of a square matrix M.

It is easily seen that the determinant of $C((\alpha_k, \beta_k)_{k=0}^{K-1})$ is a homogeneous polynomial in π_α and π_β of degree n of the form

$$\sum_{k=0}^{n} c_k \pi_\alpha^k \pi_\beta^{n-k}, \tag{1.5}$$

where $c_0, \cdots, c_n$ are complex numbers uniquely determined by $\{(E_k, A_k)\}_{k=0}^{K-1}$. For the regular set of matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$, at least one of the c_k's is nonzero, and hence we see from Definition 1.1 that there are exact n eigenvalue pairs (counting multiplicity) for $\{(E_k, A_k)\}_{k=0}^{K-1}$.

It was shown in [29] that the solvability of (1.2) is equivalent to the condition that the pencil

$$\alpha\mathcal{E} - \beta\mathcal{A} := \begin{bmatrix} \alpha E_0 & 0 & \cdots & 0 & -\beta A_0 \\ -\beta A_1 & \alpha E_1 & & & 0 \\ & \ddots & \ddots & & \vdots \\ & & \ddots & \ddots & 0 \\ 0 & & 0 & -\beta A_{K-1} & \alpha E_{K-1} \end{bmatrix} \tag{1.6}$$

is regular i.e. $\det(\alpha\mathcal{E} - \beta\mathcal{A}) \not\equiv 0$. From (1.5) it is easy to check that

$$\sigma\left(\{(E_k, A_k)\}_{k=0}^{K-1}\right) = \left\{(\alpha^K, \beta^K) \mid \det(\alpha\mathcal{E} - \beta\mathcal{A}) = 0\right\}. \tag{1.7}$$

Hence, from (1.7), the solvability of (1.2) is equivalent to the regularity of $\{(E_k, A_k)\}_{k=0}^{K-1}$.

For discrete-time descriptor systems, the concepts of reachability and observability Gramians, causal and noncausal Hankel singular values, and balanced realization are well-established [1, 33]. Moreover, numerical methods are proposed in [30, 9] to solve the projected generalized Lyapunov equations for continuous-time descriptor systems. However, to our best knowledge, similar results have not been developed for periodic descriptor systems.

2 Preliminaries

For period $K = 1$ and a regular matrix pair (E, A), it is well known that the discrete-time descriptor system (E, A, B, C) is asymptotically stable if and only if all finite eigenvalues of (E, A) lie inside the unit circle [11, 31, 32]. Similarly, the asymptotic stability of the periodic descriptor system (1.1) can be characterized in terms of the spectrum of the periodic matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$.

Definition 2.1. *Let $\{(E_k, A_k)\}_{k=0}^{K-1}$ be $n \times n$ a regular set of matrix pairs. The periodic descriptor system (1.1) is asymptotically stable if and only if all finite eigenvalues of the periodic matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$ lie inside unit circle. The periodic matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$ are called pd-stable if the set of periodic matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$ is regular and all their finite eigenvalues lie inside the unit circle.*

In a similar fashion to the Kronecker canonical form for a regular matrix pair, we can transform a regular set of periodic matrix pairs into a periodic Kronecker canonical form [19] (see also [27] for the history of the canonical form).

Lemma 2.1. *Suppose that the set of periodic matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$ in systems (1.1) is regular. Then for $k = 0, \ldots, K-1$, there exist nonsingular matrices X_k and Y_k such that*

$$X_k E_k Y_{k+1} = \begin{bmatrix} I & 0 \\ 0 & E_k^b \end{bmatrix}, \quad X_k A_k Y_k = \begin{bmatrix} A_k^f & 0 \\ 0 & I \end{bmatrix}, \tag{2.1}$$

where $Y_K \equiv Y_0$, $A_{k+K-1}^f A_{k+K-2}^f \cdots A_k^f \equiv J_k$ is an $n_1 \times n_1$ Jordan matrix corresponding to the finite eigenvalues, $E_k^b E_{k+1}^b \cdots E_{k+K-1}^b \equiv N_k$ is an $n_2 \times n_2$ nilpotent Jordan matrix corresponding to the infinite eigenvalues, and $n = n_1 + n_2$.

Remark. If ν_k is the nilpotency of the N_k for $k = 0, 1, \ldots, K-1$, then these K values are defined as the indices [19] of a regular set of periodic matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$. Hence we define the index of the periodic descriptor system (1.1) as $\nu \equiv \max\{\nu_0, \nu_1, \cdots, \nu_{K-1}\}$. We say that the periodic descriptor system (1.1) is of index at most 1 if $\nu \leq 1$, i.e., E_k are all nonsingular or $N_k = 0$ for all k.

For each $k \in \mathbb{Z}$, we let

$$x_k = Y_k \begin{bmatrix} x_k^f \\ x_k^b \end{bmatrix} \begin{matrix} \}n_1 \\ \}n_2 \end{matrix}, \quad X_k B_k = \begin{bmatrix} B_k^f \\ B_k^b \end{bmatrix} \begin{matrix} \}n_1 \\ \}n_2 \end{matrix}, \quad C_k Y_k = \begin{array}{c} \begin{bmatrix} C_k^f & C_k^b \end{bmatrix} \\ \begin{matrix} n_1 & n_2 \end{matrix} \end{array}, \tag{2.2}$$

and by using Lemma 2.1 we can decompose the original system (1.1) into forward and backward periodic subsystems, respectively:

$$x_{k+1}^f = A_k^f x_k^f + B_k^f u_k, \quad y_k^f = C_k^f x_k^f, \tag{2.3}$$

$$E_k^b x_{k+1}^b = x_k^b + B_k^b u_k, \quad y_k^b = C_k^b x_k^b, \tag{2.4}$$

with $y_k = y_k^f + y_k^b, \quad k \in \mathbb{Z}$.

Notice that the state transition matrix of the forward subsystem (2.3) equals $\Phi_f(i, j) = A_{i-1}^f A_{i-1}^f \cdots A_j^f$ when $i > j$ with $\Phi_f(i, i) := I_{n_1}$. The state transition matrix of the backward subsystem (2.4) is $\Phi_b(i, j) = E_i^b E_{i+1}^b \cdots E_{j-1}^b$ when $i < j$ with $\Phi_b(i, i) := I_{n_2}$. The state transition matrix over one period $\Phi_f(\tau + K, \tau) \in \mathbb{R}^{n_1 \times n_1}$ is called the monodromy matrix of the forward subsystem (2.3) at time τ. It is well known that its eigenvalues, called the characteristic multipliers, are independent of τ [38, 22].

For $k = 0, 1, \ldots, K-1$, the $n \times n$ matrices

$$P_r(k) = Y_k \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} Y_k^{-1}, \quad P_l(k) = X_k^{-1} \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} X_k, \tag{2.5}$$

are respectively the spectral projections onto the kth right and left deflating subspaces of the periodic matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$ corresponding to the finite eigenvalues. Moreover, the fundamental matrices $\Psi_{i,j}$ $(i, j \in \mathbb{Z})$ of the periodic descriptor system (1.1) are defined by

$$\Psi_{i,j} = \begin{cases} Y_i \begin{bmatrix} \Phi_f(i, j+1) & 0 \\ 0 & 0 \end{bmatrix} X_j, & \text{if } i > j, \\ Y_i \begin{bmatrix} 0 & 0 \\ 0 & -\Phi_b(i,j) \end{bmatrix} X_j, & \text{if } i \leq j. \end{cases} \tag{2.6}$$

These matrices play an essential role for the periodic discrete-time descriptor system (1.1). For the discrete-time descriptor system with period $K = 1$, these fundamental matrices coincide with the coefficient matrices of the Laurent expansion of the generalized resolvent $(\lambda E - A)^{-1}$ at infinity [20, 33].

Some of the results in this paper can be developed through the application of the results of Stykel [32], after "lifting" the periodic system into a descriptor system of higher dimension [6]. We shall not following this path in this paper.

3 Complete reachability and observability

In this Section we shall give a characterization of complete reachability and observability for the periodic discrete-time descriptor systems (1.1). Proofs can be found in [9] and are omitted.

Definition 3.1. (i) *The periodic descriptor system (1.1) is reachable at time t if for any state $\bar{x} \in \mathbb{R}^n$, there exist two integers s, ℓ with $s < t < \ell$ and a set of control inputs $\{u_i\}_{i=s}^{\ell}$ which carry $x_s = 0$ into $x_t = \bar{x}$. The periodic descriptor system (1.1) is called completely reachable if it is reachable at all time t.*

(ii) *The forward subsystem (2.3) is reachable at time t if for any state $\bar{\xi}_1 \in \mathbb{R}^{n_1}$, there exists an integer s with $s < t$ and a set of control inputs $\{u_i\}_{i=s}^{t-1}$ which carry $x_s^f = 0$ into $x_t^f = \bar{\xi}_1$. The periodic subsystem (2.3) is called completely reachable if it is reachable at all time t.*

(iii) *The backward subsystem (2.4) is reachable at time t if for any state $\bar{\xi}_2 \in \mathbb{R}^{n_2}$, there exists an integer ℓ with $\ell > t$ and a set of control inputs $\{u_i\}_{i=t}^{\ell}$ such that $x_t^b = \bar{\xi}_2$. The periodic subsystem (2.4) is completely reachable if it is reachable at all time t.*

Remark. It is easily seen from Definition 3.1 that the periodic discrete-time descriptor system (1.1) is completely reachable if and only if both its forward and backward subsystems are completely reachable.

Theorem 3.1 (Forward Reachability). *The following statements are equivalent.*

(a) *The forward subsystem (2.3) is completely reachable.*

(b) *For $t = 0, 1, 2, \cdots, K-1$, the matrices*

$$\mathcal{R}^f(t) \equiv \left[B_{t-1}^f,\ A_{t-1}^f B_{t-2}^f,\ \cdots,\ \Phi_f(t, t - n_1 K + 1) B_{t-n_1 K}^f \right]$$

have full row rank .

(c) *For $t = 0, 1, 2, \cdots, K-1$, and*

$$\mathcal{B}_t^f \equiv \left[B_{t-1}^f,\ A_{t-1}^f B_{t-2}^f,\ A_{t-1}^f A_{t-2}^f B_{t-3}^f,\ \cdots,\ \Phi_f(t, t-K+1) B_{t-K}^f \right],$$

the matrices

$$\left[\mathcal{B}_t^f,\ \Phi_f(t, t-K) \mathcal{B}_t^f,\ (\Phi_f(t, t-K))^2 \mathcal{B}_t^f,\ \cdots,\ (\Phi_f(t, t-K))^{n_1 - 1} \mathcal{B}_t^f \right]$$

have full row rank.

(d) *For* $\prod_{i=0}^{K-1} \alpha_i \in \sigma(\Phi_f(K,0))$*, the matrix*

$$U^f(\alpha_0,\cdots,\alpha_{K-1}) \equiv \left[\begin{array}{ccccc|cccc} \alpha_0 I & 0 & \cdots & 0 & -A_0^f & B_0^f & & & \\ -A_1^f & \alpha_1 I & \ddots & & 0 & & B_1^f & & \\ 0 & -A_2^f & \ddots & \ddots & \vdots & & & \ddots & \\ \vdots & \ddots & \ddots & \ddots & 0 & & & & \ddots \\ 0 & \cdots & 0 & -A_{K-1}^f & \alpha_{K-1} I & & & & B_{K-1}^f \end{array}\right]$$

has full row rank.

(e) *For* $t = 0,1,2,\cdots,K-1$,

$$y^T\Phi_f(t+K,t) = \lambda y^T \ \ \textit{and} \ \ y^T\Phi_f(t,j)B_{j-1}^f = 0 \ \ \textit{for} \ \ j = t-K+1,\cdots,t-1,t$$

imply $y = 0$.

Proof. (a) $\Rightarrow$ (e): Suppose the statement (a) is true. For any $t \in \{0,1,\cdots,K-1\}$, assume that

$$y^T\Phi_f(t+K,t) = \lambda y^T \ \text{ and } \ y^T\Phi_f(t,j)B_{j-1}^f = 0 \ \text{ for } \ j = t-K+1,\cdots,t-1,t. \tag{3.1}$$

Since the forward subsystem (2.3) is reachable at time t, there exist an integer s with $s < t$ and control inputs u_i, $s \le i \le t-1$, which carry $x_s^f = 0$ into $x_t^f = y$. Thus, we have

$$y = x_t^f = \sum_{i=s}^{t-1} \Phi_f(t,i+1)B_i^f u_i.$$

Moreover, from the assumptions (3.1), it follows that

$$y^T y = y^T \sum_{i=s}^{t-1} \Phi_f(t,i+1)B_i^f u_i = 0.$$

Therefore, $y = 0$ and hence the condition (e) holds.

(e) $\Rightarrow$ (d): Assume that the condition (e) holds, and let vectors $y_0, y_1, \cdots, y_{K-1} \in \mathbb{R}^{n_1}$ satisfy

$$(y_0^T, y_1^T, \cdots, y_{K-1}^T)U^f(\alpha_0,\alpha_1,\cdots,\alpha_{K-1}) = 0,$$

or

$$\begin{cases} \alpha_0 y_0^T = y_1^T A_1^f \\ \alpha_1 y_1^T = y_2^T A_2^f \\ \cdots\cdots\cdots\cdots\cdots \\ \alpha_{K-2} y_{K-2}^T = y_{K-1}^T A_{K-1}^f \\ \alpha_{K-1} y_{K-1}^T = y_0^T A_0^f \end{cases} \quad \text{and} \quad \begin{cases} y_0^T B_0^f = 0 \\ y_1^T B_1^f = 0 \\ \cdots\cdots\cdots \\ y_{K-2}^T B_{K-2}^f = 0 \\ y_{K-1}^T B_{K-1}^f = 0. \end{cases} \tag{3.2}$$

Notice that for the vector y_{K-1}, it can be easily checked from (3.2), for $j = -K+1,\cdots,-2,-1,0$, that

$$\begin{aligned} y_{K-1}^T\Phi_f(K,0) &= y_{K-1}^T A_{K-1}^f A_{K-2}^f \cdots A_0^f \\ &= (\alpha_{K-2}\alpha_{K-3}\cdots\alpha_0\alpha_{K-1})y_{K-1}^T, \end{aligned}$$

and

$$y_{K-1}^T\Phi_f(0,j)B_{j-1}^f = 0 \ .$$

By condition (e), if the product $\alpha_{K-2}\alpha_{K-3}\cdots\alpha_0\alpha_{K-1} \in \sigma(\{(E_k,A_k)\}_{k=0}^{K-1})$, we then have $y_{K-1} = 0$. Similarly, it can be shown that $y_{K-2} = y_{K-3} = \cdots = y_0 = 0$. Therefore, the matrix $U^f(\alpha_0,\cdots,\alpha_{K-1})$ has full row rank and condition (d) is proved.

(d) $\Rightarrow$ (c): Suppose that (d) holds. It suffices to prove condition (c) for time instant $t = 0$. Since condition (d) holds, it follows that

$$U^f(1,\cdots,1,\lambda) = \left[\begin{array}{ccccc|ccccc} I & 0 & \cdots & 0 & -A_0^f & B_0^f & & & & \\ -A_1^f & I & \ddots & & 0 & & B_1^f & & & \\ 0 & -A_2^f & \ddots & \ddots & \vdots & & & \ddots & & \\ \vdots & \ddots & \ddots & \ddots & 0 & & & & \ddots & \\ 0 & \cdots & 0 & -A_{K-1}^f & \lambda I & & & & & B_{K-1}^f \end{array}\right] \tag{3.3}$$

has full row rank for all $\lambda \in \sigma(\Phi_f(K,0))$. By elementary row operations, the matrix $U^f(1,\cdots,1,\lambda)$ can be transformed to

$$\tilde{U}^f \equiv \left[\begin{array}{ccccc|ccccc} I & * & \cdots & \cdots & * & * & & & & \\ 0 & I & \ddots & & \vdots & \vdots & \ddots & & & \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & & \ddots & & \\ \vdots & & \ddots & I & * & * & \cdots & \cdots & * & \\ 0 & \cdots & \cdots & 0 & \lambda I - \Phi_f(K,0) & \Phi_f(K,1)B_0^f & \cdots & \cdots & A_{K-1}^f B_{K-2}^f & B_{K-1}^f \end{array}\right],$$

which is of full row rank for all $\lambda \in \sigma(\Phi_f(K,0))$. Equivalently, the last row blocks of $\tilde{U}^f$ has full row rank for all $\lambda \in \sigma(\Phi_f(K,0))$, i.e.,

$$\operatorname{rank}\left[\begin{array}{c|c} \lambda I - \Phi_f(K,0) & \mathcal{B}_0^f \end{array}\right] = n_1 \quad \text{for } \lambda \in \sigma(\Phi_f(K,0)),$$

where the matrix $\mathcal{B}_0^f$ is defined in the (c). This proves condition (c) for $t = 0$. Similar arguments apply for $1 \le t \le K-1$.

(c) $\Rightarrow$ (b): This follows from the periodicity of the matrices A_k^f and B_k^f, i.e., $A_{k+K}^f = A_k^f$ and $B_{k+K}^f = B_k^f$ for all k.

(b) $\Rightarrow$ (a): Assume that condition (b) holds. For any time $t \in \mathbb{Z}$ (mod K) and any given state $\bar{\xi}_1 \in \mathbb{R}^{n_1}$, there exist an integer $s \equiv t - n_1 K$ and a set of control inputs u_i, $s \le i \le t-1$, which satisfy

$$\sum_{i=s}^{t-1} \Phi_f(t, i+1) B_i^f u_i = \bar{\xi}_1,$$

since $\mathcal{R}^f(t)$ has full row rank. With these control inputs, the given state $\bar{\xi}_1$ can be reached at time t from the initial state $x_s^f = 0$, and hence the complete reachability of the forward subsystem (2.3) is proved. $\square$

Theorem 3.2 (Backward Reachability). *The following statements are equivalent.*

(a) *The backward subsystem (2.4) is completely reachable.*

(b) *For $t = 0, 1, 2, \cdots, K-1$, the matrices*

$$\mathcal{R}^b(t) \equiv \left[B_t^b,\, E_t^b B_{t+1}^b,\, \cdots,\, \Phi_b(t, t+\nu K - 1) B_{t+\nu K-1}^b\right]$$

have full row rank.

(c) *For $t = 0, 1, 2, \cdots, K-1$, and*

$$\mathcal{B}_t^b \equiv \left[B_t^b,\, E_t^b B_{t+1}^b,\, \cdots,\, E_t^b E_{t+1}^b \cdots E_{t+K-2}^b B_{t+K-1}^b\right],$$

the matrices $\left[N_t,\, \mathcal{B}_t^b\right]$ have full row rank.

(d) *The pair* $(\mathcal{E}_b, \mathcal{B}_b)$ *is reachable, where*

$$\mathcal{E}_b \equiv \begin{bmatrix} 0 & E_0^b & & & \\ 0 & 0 & E_1^b & & \\ \vdots & \vdots & \ddots & \ddots & \\ 0 & 0 & & \ddots & E_{K-2}^b \\ E_{K-1}^b & 0 & \cdots & \cdots & 0 \end{bmatrix} \quad \text{and } \mathcal{B}_b \equiv \begin{bmatrix} B_0^b & & & & \\ & B_1^b & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & B_{K-1}^b \end{bmatrix}. \quad (3.4)$$

Proof. (a) $\Leftrightarrow$ (b): For any time $t \in \mathbb{Z}$, it can be easily seen that

$$x_t^b = -\sum_{i=t}^{t+\nu K-1} \Phi_b(t,i) B_i^b u_i = -\mathcal{R}^b(t) \begin{bmatrix} u_t \\ u_{t+1} \\ \vdots \\ u_{t+\nu K-1} \end{bmatrix}.$$

Therefore, any given state $\bar{\xi}_2 \in \mathbb{R}^{n_2}$ can be reached at time t, i.e., $x_t^b = \bar{\xi}_2$, through a set of control inputs $\{u_i\}_{i=t}^{t+\nu K-1}$ if and only if the matrix $\mathcal{R}^b(t)$ is of full row rank.

(b) $\Leftrightarrow$ (c): Since $N_t = \Phi_b(t, t+K) = E_t^b E_{t+1}^b \cdots E_{t+K-1}^b$ and $N_t^\nu = 0$ for $t = 0, 1, \ldots, K-1$, it follows that

$$\mathcal{R}^b(t) = \left[\mathcal{B}_t^b,\, N_t \mathcal{B}_t^b,\, \cdots,\, N_t^{\nu-1} \mathcal{B}_t^b\right].$$

Thus, $\operatorname{rank}(\mathcal{R}^b(t)) = n_2$ if and only if $\operatorname{rank}\left[\lambda I - N_t,\, \mathcal{B}_t^b\right] = n_2$ for any $\lambda \in \sigma(N_t)$. Since the matrix N_t is nilpotent, $\sigma(N_t) = \{0\}$, and hence $\operatorname{rank}(\mathcal{R}^b(t)) = n_2$ if and only if $\operatorname{rank}\left[-N_t,\, \mathcal{B}_t^b\right] = n_2$. Equivalently, $\operatorname{rank}\left[N_t,\, \mathcal{B}_t^b\right] = n_2$.

(b) $\Leftrightarrow$ (d): Notice that the matrix $\mathcal{E}_b$ is nilpotent with the property that $\mathcal{E}_b^\nu = 0$ and $\mathcal{E}_b^{\nu-1} \neq 0$. It is well known that the pair $(\mathcal{E}_b, \mathcal{B}_b)$ is reachable if and only if $\mathbb{B}_b \equiv \left[\mathcal{B}_b,\, \mathcal{E}_b\mathcal{B}_b,\, \cdots,\, \mathcal{E}_b^{\nu-1}\mathcal{B}_b\right]$ has full row rank. Furthermore, it can be checked that the row blocks of the matrix $\mathbb{B}_b$ are just $\mathcal{R}^b(t)$ with different t. Therefore, statements (b) and (d) are equivalent. $\square$

Definition 3.2. (i) *The periodic descriptor system (1.1) is observable at time t if there exist two integers s, ℓ with $s < t < \ell$ such that any state at time t can be determined from knowledge of $\{y_i\}_{i=s}^{\ell}$ and $\{u_i\}_{i=s}^{\ell}$. The periodic descriptor system (1.1) is called completely observable if it is observable at all time t.*

(ii) *The forward subsystem (2.3) is observable at time t if there exists an integer ℓ with $\ell > t$ such that any state at time t can be determined from knowledge of $\{y_i\}_{i=t}^{\ell}$ and $\{u_i\}_{i=t}^{\ell}$. The periodic subsystem (2.3) is called completely observable if it is observable at all time t.*

(iii) *The backward subsystem (2.4) is observable at time t if there exists an integer s with $s < t$ such that any state at time t can be determined from knowledge of $\{y_i\}_{i=s}^{t}$ and $\{u_i\}_{i=s}^{t}$. The periodic subsystem (2.4) is completely observable if it is observable at all time t.*

Remark. It is easily seen from Definition 3.2 that the periodic discrete-time descriptor system (1.1) is completely observable if and only if both its forward and backward subsystems are completely observable.

We shall state the following Theorems without proofs, which are similar to those of Theorems 3.1 and 3.2.

Theorem 3.3 (Forward Observability). *The following statements are equivalent.*

(a) *The forward subsystem (2.3) is completely observable.*

(b) *For $t = 0, 1, 2, \cdots, K-1$, the matrices*

$$\mathcal{O}^f(t) \equiv \begin{bmatrix} C_t^f \\ C_{t+1}^f A_t^f \\ C_{t+2}^f A_{t+1}^f A_t^f \\ \vdots \\ C_{t+n_1K-1}^f \Phi_f(t + n_1K - 1, t) \end{bmatrix}$$

have full column rank.

(c) *For* $t = 0, 1, 2, \cdots, K-1$, *and*

$$\mathcal{C}_t^f \equiv \left[(C_t^f)^T,\, (A_t^f)^T (C_{t+1}^f)^T,\, \cdots,\, \Phi_f(t+K-1,t)^T (C_{t+K-1}^f)^T \right]^T ,$$

the matrices

$$\begin{bmatrix} \mathcal{C}_t^f \\ \mathcal{C}_t^f \Phi_f(t+K,t) \\ \mathcal{C}_t^f (\Phi_f(t+K,t))^2 \\ \vdots \\ \mathcal{C}_t^f (\Phi_f(t+K,t))^{n_1-1} \end{bmatrix}$$

have full row rank.

(d) *For* $\prod_{i=0}^{K-1} \alpha_i \in \sigma(\Phi_f(K,0))$, *the matrix*

$$V^f(\alpha_0, \cdots, \alpha_{K-1}) \equiv \left[\begin{array}{ccccc} \alpha_0 I & 0 & \cdots & 0 & -A_{K-1}^f \\ -A_0^f & \alpha_1 I & \ddots & & 0 \\ 0 & -A_1^f & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -A_{K-2}^f & \alpha_{K-1} I \\ \hline C_0^f & & & & \\ & C_1^f & & & \\ & & \ddots & & \\ & & & C_{K-2}^f & \\ & & & & C_{K-1}^f \end{array}\right]$$

has full column rank.

(e) *For* $t = 0, 1, 2, \cdots, K-1$,

$$\Phi_f(t+K,t)x = \lambda x \ \ \text{and} \ \ C_i^f \Phi_f(i,t)x = 0 \ \ \text{for} \ \ i = t, t+1, \cdots, t+K-1$$

imply $x = 0$.

Theorem 3.4 (Backward Observability). *The following statements are equivalent.*

(a) *The backward subsystem (2.4) is completely observable.*

(b) *For* $t = 0, 1, 2, \cdots, K-1$, *the matrices*

$$\mathcal{O}^b(t) \equiv \begin{bmatrix} C_t^b \\ C_{t-1}^b E_{t-1}^b \\ C_{t-2}^b E_{t-2}^b E_{t-1}^b \\ \vdots \\ C_{t-\nu K+1}^b \Phi_b(t-\nu K+1,t) \end{bmatrix}$$

have full column rank .

(c) *For* $t = 0, 1, 2, \cdots, K-1$, *and*

$$\mathcal{C}_t^b \equiv \left[(C_t^b)^T,\, (E_{t-1}^b)^T (C_{t-1}^b)^T,\, \cdots,\, \Phi_b(t-K+1,t)^T (C_{t-K+1}^b)^T \right]^T ,$$

the matrices

$$\begin{bmatrix} \mathcal{C}_t^b \\ \mathcal{C}_t^b N_t \\ \mathcal{C}_t^b N_t^2 \\ \vdots \\ \mathcal{C}_t^b N_t^{\nu-1} \end{bmatrix}$$

have full column rank

(d) *The pair* $(\mathcal{E}_b, \mathcal{C}_b)$ *is observable, where* $\mathcal{E}_b$ *is defined in (3.4) and* $\mathcal{C}_b \equiv \mathrm{diag}(C_0^b, C_1^b, \cdots, C_{K-1}^b)$.

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