Calderón's Problem for Some Systems of PDE

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Abstract

The note is mainly for personal record, if you want to read it, please be careful. This note was given by Prof. Imanuvilov in IAS inverse problems conference.

Calderón's Problem Elliptic Systems

Let $\Omega \subset \mathbb{R}^2$ be an open bounded domain and $u(x) = (u_1(x), u_2(x), \cdots, u_N(x)) : \Omega \to \mathbb{R}^N$. Consider the following elliptic operator

$$\begin{cases} L(x,D)u = \Delta u + 2A\partial_z u + 2B\partial_{\bar{z}} u + Qu = 0 & \text{in } \Omega, \\ u|_{\Gamma_0 = 0} \text{ and } u|_{\widetilde{\Gamma}} = f & \text{on } \partial\Omega, \end{cases}$$

where $A(x), B(x), Q(x)$ are $N \times N$ matrices and $\widetilde{\Gamma} \subset \partial\Omega$, $\Gamma_0 = \partial\Omega \backslash \widetilde{\Gamma}$ are subset of $\partial\Omega$. Define the Dirichlet-to-Neumann map

$$\Lambda_{A,B,Q} f = \frac{\partial u}{\partial \nu}|_{\widetilde{\Gamma}}.$$

The problem is how to reconstruct A, B and Q ? This result was partially proved by Iamnuvilov and Yamamoto in $\Omega \subset \mathbb{R}^n$ for $n = 2$; Eskin proved the case when $n = 3$.

Let $\mathbb{Q}$ be a non-singular regular matrix in $\overline{\Omega}$ with $\mathbb{Q}|_{\widetilde{\Gamma}} = I$ and $\frac{\partial \mathbb{Q}}{\partial \nu}|_{\widetilde{\Gamma}} = 0$. Now, we consider the conjugated differential operator

$$\mathbb{Q}^{-1} L(x,D) \mathbb{Q}.$$

Set $w = \mathbb{Q}^{-1} u$, then

$$\begin{cases} \mathbb{Q}^{-1} L(x,D)\mathbb{Q} w = 0 & \text{in } \Omega, \\ w|_{\Gamma_0} = 0 \text{ and } w|_{\widetilde{\Gamma}} = f & \text{on } \partial\Omega. \end{cases}$$

Moreover, $\frac{\partial w}{\partial \nu}|_{\widetilde{\Gamma}} = \frac{\partial u}{\partial \nu}|_{\widetilde{\Gamma}}$. We state the main result in the following.

Theorem 0.1. *Let $A_j(x), B_j(x) \in C^{5+\alpha}(\overline{\Omega})$, $Q(x) \in C^{4+\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$ and $j = 1, 2$. Let*

$$L_j(x,D) := \Delta + 2A_j\partial_z + 2B_j\partial_{\bar{z}} + Q_j$$

be second order elliptic operators for $j = 1, 2$. Assume $L_j^*(x,D)$ to be the adjoint operators for L_j and zero is not an eigenvalue of L_j^*. If $\Lambda_{A_1,B_1,Q_1} = \Lambda_{A_2,B_2,Q_2}$, then we have

$$A_1 = A_2 \text{ and } B_1 = B_2 \text{ on } \widetilde{\Gamma}.$$

Moreover, there exists a non-singular matrix $\mathbb{Q}(x) \in C^{5+\alpha}(\overline{\Omega})$ *such that*

$$
\begin{aligned}
&A_2 = 2\mathbb{Q}^{-1}\partial_{\bar z}\mathbb{Q} + \mathbb{Q}^{-1}A_1\mathbb{Q} \ \textit{in } \Omega,\\
&B_2 = 2\mathbb{Q}^{-1}\partial_{z}\mathbb{Q} + \mathbb{Q}^{-1}B_1\mathbb{Q} \ \textit{in } \Omega,\\
&Q_2 = \mathbb{Q}^{-1}Q_1\mathbb{Q} + \mathbb{Q}^{-1}\Delta\mathbb{Q} + 2\mathbb{Q}^{-1}A_1\partial_z\mathbb{Q} + 2\mathbb{Q}^{-1}B_1\partial_{\bar z}\mathbb{Q} \ \textit{in } \Omega.
\end{aligned}
$$

Proof. (Sketch) **Step1**. Find $\mathbb{Q}$ such that $\det\mathbb{Q} \neq 0$, $\mathbb{Q}|_{\widetilde{\Gamma}} = I$ and $\dfrac{\partial\mathbb{Q}}{\partial\nu}|_{\widetilde{\Gamma}} = 0$ such that $A_2 = 2\mathbb{Q}^{-1}\partial_{\bar z}\mathbb{Q} + \mathbb{Q}^{-1}A_1\mathbb{Q}$ or

$$2\partial_{\bar z}\mathbb{Q} + A_1\mathbb{Q} - \mathbb{Q}A_2 = 0 \text{ in } \Omega.$$

Step 2. Let $\widetilde{L_1}(x,D) = \mathbb{Q}^{-1}L_1(x,D)\mathbb{Q} = \Delta + 2A_2\partial_z + 2\widehat{B_1}\partial_{\bar z} + \widehat{Q_1}$ with $\widehat{B_1} = \overline{B_1}$ and $\widehat{Q_1} = Q_1$. Recall that u_1, u_2 satisfy

$$
\begin{cases}
L_1(x,D)u_1 = 0 & \text{in } \Omega,\\
u_1|_{\Gamma_0} = 0 \text{ and } u_1|_{\widetilde{\Gamma}} = f_\tau & \text{on } \partial\Omega,\\
L_2(x,D)u_2 = 0 & \text{in } \Omega,\\
u_2|_{\Gamma_0} = 0 \text{ and } u_2|_{\widetilde{\Gamma}} = f_\tau & \text{on } \partial\Omega,
\end{cases}
$$

and $\dfrac{\partial u_1}{\partial\nu} = \dfrac{\partial u_2}{\partial\nu} = 0$ on $\widetilde{\Gamma}$. Now, let $u = u_1 - u_2$, then

$$
\begin{cases}
L_2(x,D)u = 2(A_1 - A_2)\partial_z u_1 + 2(B_1 - B_2)\partial_{\bar z}u_1 + (Q_1 - Q_2)u_1 = 0 & \text{in } \Omega,\\
u = 0 \text{ and } \dfrac{\partial u}{\partial\nu}|_{\widetilde{\Gamma}} = 0 & \text{on } \partial\Omega.
\end{cases}
$$

Then

$$\int_\Omega \{2(A_1 - A_2)\frac{\partial u_1}{\partial z} + 2(B_1 - B_2)\frac{\partial u_1}{\partial \bar z} + (Q_1 - Q_2)u_1\} \cdot v dx = \varphi_1\tau + O(\tau),$$

where

$$u_1 = u_0 e^{\tau\Phi} + \widehat{u_0}e^{\tau\Phi} + \text{l.o.t}, \ v = v_0 e^{-\tau\Phi} + \widehat{v_0}e^{-\tau\Phi} + \text{l.o.t}.$$

Note that $((2\partial_z - B_2^*)v_0, (2\partial_{\bar z} - A_2^*)\widehat{v_0}) = 0$ with $v_0 + \widehat{v_0} = 0$ on Γ_0 and

$$\int_{\partial\Omega} (v_1 + iv_2)\Phi'(P_1 a, P_2 b) + (v_1 - iv_2)\overline{\Phi'}(c_1\overline{a}, c_2\overline{b})d\sigma = 0,$$

where $a(z) = (a_1(z), \cdots, a_N(z))$ and $b(z) = (b_1(z), \cdots, b_N(z))$ with $\text{Im}a|_{\Gamma_0} = \text{Im}b|_{\Gamma_0} = 0$. $\exists\Theta(z) \in W^{\frac{1}{2},2}(\Omega)$ and $\widetilde{\Theta}(\bar z) \in W^{\frac{1}{2},2}(\Omega)$ such that

$$\widetilde{\Theta}|_{\widetilde{\Gamma}} = C_2^*C_1, \ \Theta|_{\widetilde{\Gamma}} = P_2^*P_1 \text{ and } \Theta = \widetilde{\Theta} \text{ on } \Gamma_0.$$

Now we choose $\mathbb{Q} = P_1\Theta^{-1}P_2^*$, then $\mathbb{Q}||_\Gamma = I$ and $\dfrac{\partial\mathbb{Q}}{\partial\nu}|_{\widetilde{\Gamma}} = 0$. Note that

$$2\partial_{\bar z}\mathbb{Q} + A_1\mathbb{Q} - \mathbb{Q}A_2 = 0 \text{ in } \Omega\backslash X,$$

where $X = \{\det\Theta = 0\}$ ($\det\mathbb{Q} \neq 0$ in $\Omega\backslash X$). Note that $\widetilde{\mathbb{Q}} = \widehat{P}\widehat{\Theta}^{-1}\widehat{P_2}^* = \mathbb{Q}$.

Finally, we generate the same DtN map in the following. Let $G(x)$ be the solution of

$$2\partial_{\bar z}G - A_1^*G + GA_2^* = 0 \text{ in } \Omega\backslash\widetilde{X},$$

where $\widetilde{X}$ denotes some singular set as before (here we consider $G = (\mathbb{Q}^*)^{-1}$). Let $G = M_1(x)Y^{-1}(z)M_2^*(x)$, then $G|_{\widetilde{\Gamma}} = I$ and $\dfrac{\partial G}{\partial\nu}|_{\widetilde{\Gamma}} = 0$. □