Dirichlet-Neumann Relation

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Example 1. Look at heat equation $u_t = u_{xx}$

$\begin{cases} G_t = Gxx \\ G(x,0) = \delta(x) \end{cases}$, then $G(x,t) = \dfrac{1}{4\pi t} e^{-\frac{x^2}{4t}}$, where G is a Green's function for this initial value problem.

Now, consider the initial boundary value problem: Let $h(x,t)$ be the source of the heat equation, and $u^0 = u(0,t))$ is Dirichlet boundary condition and $u_x^0 = u_x(0,t)$ is Neumann boundary condition. Then the heat equation $u_t = u_{xx} + h(x,t)$ has the following representation:

$$\begin{aligned} u(x,t) &= \int_0^t \int_0^\infty h(y,s)G(x-y,t-s)dyds + \int_0^\infty u(y,0)G(x-y,t)dy \\ &+ \int_0^t G(-y,t-s)u_x(0,s)ds - \int_0^t G_x(-y,t-s)u(0,s)ds. \end{aligned}$$

Goal: PDE $\Longrightarrow$Dirichlet-Neumann Relation !

Example 2. $\begin{cases} u_t = u_{xx} & x,t>0 \\ u(x,0) = 0 & x>0 \\ u(0,t) = u^0(t) & t>0 \\ u_x(0,t) = u_x^0(t) & t>0 \end{cases}$ and take the Laplace transform in t and x separately, and we define

$$U(x,s) = \int_0^\infty e^{-st}u(x,t)dt \ , \ \hat{U}(\xi,s) = \int_0^\infty e^{-\xi x}U(x,s)dx$$

and use the well-known results in L-transform, we have

$$L(u_s) = -u|_{s=0} + sL(u) \text{ and } L(u_{xx}) = -U_x|_{x=0} - \xi U|_{x=0} + \xi^2 L(U),$$

where L is the Laplace transform and we have used the notation U.

By using above consequences, the heat equation becomes

$$s\hat{U} = \xi\hat{U} - U_x^0 - \xi U^0.$$

Therefore,

$$\hat{U} = \frac{1}{(\xi-\sqrt{s})(\xi+\sqrt{s})}\left(U_x^0 + \xi U^0\right),$$

we use the inverse Laplace transform in ξ variable, but we need to avoid the singularities, that is, we want to integrate over the part which U is analytic, so we can do the following thing:

$$U(x,s) = \frac{1}{2\pi i}\int_{-i\infty+L}^{i\infty+L} \hat{U}(\xi,s)e^{\xi x}d\xi.$$

By the residue theorem, we have

$$U(x,s) = e^{\sqrt{s}x}\left\{\frac{U_x^0+\sqrt{s}U^0}{\sqrt{s}+\sqrt{s}}\right\} + e^{-\sqrt{s}x}\left\{\frac{U_x^0-\sqrt{s}U^0}{-\sqrt{s}-\sqrt{s}}\right\},$$

note that $e^{\sqrt{s}x}\to\infty$ as , it is an unstable mode. Thus, we want

$$\frac{U_x^0+\sqrt{s}U^0}{\sqrt{s}+\sqrt{s}} = 0,$$

it is the Dirichlet-Neumann Relation. So we have $U_x^0 = -\sqrt{s}U^0 = -\frac{s}{\sqrt{s}}U^0$,that is,

$$\frac{1}{s}U_x^0 = -\frac{1}{\sqrt{s}}U^0.$$

To invert Laplace transform in t, we have

$$u_x^0 = \left[\left(L^{-1}(\frac{1}{\sqrt{\bullet}}) * u^0\right)(t)\right]_t.$$

Note that the trick is that we we the term $\frac{1}{s}$ to invert the L-transform in t. Now, we need to compute $L^{-1}(\frac{1}{\sqrt{s}})$:

$$\begin{aligned}
L^{-1}\left(\frac{1}{\sqrt{s}}\right) &= \frac{1}{2\pi i}\left[\int_\infty^0 \frac{e^{i\tau t}}{\left(\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right)\sqrt{\tau}}(-i)d\tau + \int_0^\infty \frac{e^{-i\tau t}}{\left(\frac{1}{\sqrt{2}}-i\frac{1}{\sqrt{2}}\right)\sqrt{\tau}}(i)d\tau\right] \\
&= \frac{1}{2\pi}\int_0^\infty \frac{1}{\sqrt{\tau}}\left\{\frac{e^{i\tau t}}{\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}} + \frac{e^{-i\tau t}}{\frac{1}{\sqrt{2}}-i\frac{1}{\sqrt{2}}}\right\}d\tau \\
&= \frac{1}{2\pi}\int_0^\infty \frac{1}{\sqrt{\tau}}\left\{\sqrt{2}\cos\tau t + \sqrt{2}\sin\tau t\right\}d\tau \\
&= \frac{1}{\sqrt{t}}\frac{1}{2\pi}\int_0^\infty \sqrt{\frac{2}{\gamma}}(\cos\gamma+\sin\gamma)d\gamma \\
&= \frac{1}{\sqrt{t}}D_0,
\end{aligned}$$

where we set $\gamma = \tau t$ and $D_0 = \frac{1}{2\pi}\int_0^\infty \sqrt{\frac{2}{\gamma}}(\cos\gamma+\sin\gamma)d\gamma$. Therefore,

$$u_x^0 = \partial_t\left\{\left(\frac{D_0}{\sqrt{t}} * u^0\right)(t)\right\},$$

we call this the Dirichlet-Neumann Relation.

Remark 3. If we consider $\begin{cases} u_t = u_{xx} + h(x,t) \\ u(x,0) \text{ is given} \\ u(0,t) = 0 \end{cases}$, we extend $u(x,t)$ as an odd function in x to reduce the problem to pace initial value problem, which can be solved explicitly by $G(x,t) = \frac{1}{4\pi t}e^{-\frac{x^2}{4t}}$, the Green's function of the initial value problem. Then we still have the Dirichlet-Neumann Relation. On the other hand, if we have the Neumann boundary condition on this problem, that is, we have $u_x(0,t) = 0$ but we don't know what is $u(x,0)$, then we just need to extend the function as an even function in x.

Now, we consider another example to observe the Dirichlet-Neumann relation in PDEs !

Example 4. Let us consider the equation

$$\begin{cases} u_t + u_x + v_y = \Delta u = u_{xx} + u_{yy} \\ v_t - v_x + u_y = \Delta v = v_{xx} + v_{yy} \end{cases}$$

where the $u_t + u_x + v_y$ and $v_t - v_x + u_y$ are wave operators. We suppose

$$\begin{pmatrix} U \\ V \end{pmatrix}(x,\eta,s) = \int_{-\infty}^{\infty} e^{iy\eta} d\eta \int_0^{\infty} e^{-st} \begin{pmatrix} u \\ v \end{pmatrix}(x,y,t)dt,$$

Fourier transform in y and Laplace transform in t and

$$\begin{pmatrix} \hat{U} \\ \hat{V} \end{pmatrix}(\xi,\eta,s) = \int_0^{\infty} e^{-\xi x} \begin{pmatrix} U \\ V \end{pmatrix}(x,\eta,s)dx$$

Laplace in x.

$$\begin{pmatrix} s+\xi-\xi^2+\eta^2 & i\eta \\ i\eta & s-\xi-\xi^2+\eta^2 \end{pmatrix}\begin{pmatrix} \hat{U} \\ \hat{V} \end{pmatrix} = \begin{pmatrix} (1-\xi)U^0 - U_x^0 \\ (-1-\xi)V^0 - V_x^0 \end{pmatrix},$$

where $U_x^0 = U_x(0,\eta,s)$, $U^0 = U(0,\eta,s)$...etc. Then rewrite

$$\begin{pmatrix} \hat{U} \\ \hat{V} \end{pmatrix} = \begin{pmatrix} s+\xi-\xi^2+\eta^2 & i\eta \\ i\eta & s-\xi-\xi^2+\eta^2 \end{pmatrix}^{-1}\begin{pmatrix} (1-\xi)U^0 - U_x^0 \\ (-1-\xi)V^0 - V_x^0 \end{pmatrix},$$

where

$$\begin{aligned} \begin{pmatrix} s+\xi-\xi^2+\eta^2 & i\eta \\ i\eta & s-\xi-\xi^2+\eta^2 \end{pmatrix}^{-1} &= \frac{1}{p(\xi,\eta,s)} \\ &\quad \times \begin{pmatrix} s-\xi-\xi^2+\eta^2 & -i\eta \\ -i\eta & s+\xi-\xi^2+\eta^2 \end{pmatrix} \\ &\quad \times \begin{pmatrix} (1-\xi)U^0 - U_x^0 \\ (-1-\xi)V^0 - V_x^0 \end{pmatrix}, \end{aligned}$$

where $p(\xi,\eta,s) = \det\begin{pmatrix} s+\xi-\xi^2+\eta^2 & i\eta \\ i\eta & s-\xi-\xi^2+\eta^2 \end{pmatrix} = \xi^4 - (1+2s+2\eta^2)\xi^2 + s^2 + (1+2s)\eta^2 + \eta^4$.The roots of p are

$$\lambda_1 = \sqrt{\frac{1}{2} + s + \frac{1}{2}\sqrt{1+4s} + \eta^2} \text{ and } \lambda_2 = \sqrt{\frac{1}{2} + s - \frac{1}{2}\sqrt{1+4s} + \eta^2}$$

and $\lambda_3 = -\lambda_2$, $\lambda_4 = -\lambda_1$. Note that $\lambda_4 < \lambda_3 < 0 < \lambda_2 < \lambda_1$. Stability requires that

$$Res_{\lambda_1} = Res_{\lambda_2} = 0.$$

Then we want to find the Dirichlet-Neumann relation in U^0, V^0, U_x^0, V_x^0. By using $Res_{\lambda_1} = Res_{\lambda_2} = 0$, we have

$$U_x^0 = \frac{1}{2}(1-\lambda_2-\lambda_1)U^0 + \frac{i(1+\lambda_2-\lambda_1)(1-2\lambda_2+\lambda_2^2-\lambda_1^2)}{4\eta}V^0$$

and

$$\begin{aligned} V_x^0 &= [\frac{i(\lambda_2-\lambda_1)}{8(1+4s)\eta^3}(-\lambda_2-\lambda_2^2-\lambda_1(1+\lambda_1)(1+4s+\lambda_1^2-3\lambda_1^3+\lambda_2^2(-1+2\lambda_1)) \\ &\quad \times(-1+\lambda_1^2-2s(2+\lambda_1+\lambda_2))+\lambda_2^2(1+\lambda_1)-2\eta^2-2\lambda_1\eta^2+\lambda_2(\lambda_1+\lambda_1^2-\eta^2)]U^0 \\ &\quad +\frac{1}{2}(-1-\lambda_2-\lambda_1)V^0. \end{aligned}$$

Invert Laplace in s first: (λ_1 and λ_2)

Trick: Use $\int_{-i\infty}^{i\infty} \alpha e^{st} ds = -\frac{1}{t}\int_{-i\infty}^{i\infty} \frac{\partial \alpha}{\partial s} e^{st} ds$, then we have

$$\frac{\partial \lambda_1}{\partial s} = \frac{\frac{1}{2}+\sqrt{\frac{1}{4}+s}}{2\sqrt{\frac{1}{4}+s}\sqrt{\left(\frac{1}{2}+\sqrt{\frac{1}{4}+s}\right)^2+\eta^2}}.$$

For $\frac{\partial \lambda_1}{\partial s}$, which is analytic around $Res = 0$ but $\frac{\partial \lambda_2}{\partial s}$ is not uniformly (in η) analytic around $Res = 0$. Since $\left(\frac{1}{2}+\sqrt{\frac{1}{4}+s}\right)^2+\eta^2 < 0$ is a trouble for λ_2, then we look for s such that $\left(\frac{1}{2}+\sqrt{\frac{1}{4}+s}\right)^2+\eta^2 = -\tau^2$ for some τ real. By easy calculation, we have

$$s = -(\eta^2+\tau^2) \pm i\sqrt{\eta^2+\tau^2}.$$

For $\tau = 0$, $s = -\eta^2 \pm i\eta$, there are two curves have problems, then we need to avoid the path $C = \{s = -\eta^2 \pm i\eta\}$ of the integration.

$$\begin{aligned} \int_{-\infty}^{\infty} e^{i\eta y} dy \int_{\mathbb{R}^2-C} e^{st}\frac{\partial \lambda_2}{\partial s} ds &= \int\int e^{i\eta y} e^{st} \frac{\frac{1}{2}-\sqrt{\frac{1}{4}+s}}{2\sqrt{\frac{1}{4}+s}\sqrt{\left(\frac{1}{2}-\sqrt{\frac{1}{4}+s}\right)^2+\eta^2}} ds d\eta \\ &= \int\int e^{i\eta y} e^{[-i\sqrt{\eta^2+\tau^2}-(\eta^2+\tau^2)]t} d\tau d\eta \\ &= \int\int e^{i\eta y + i\tau x} e^{[-i\sqrt{\eta^2+\tau^2}-(\eta^2+\tau^2)]\tau} d\tau dy|_{x=0} \\ &= \text{wave solution (Kirchhoff \& Hadamard).} \end{aligned}$$

Dirichlet-Neumann Relation: Hadamard-Kirchhoff convolve with initial values.

Example 5. Let $u \in \mathbb{R}^2$ satisfy

$$u_t + \begin{pmatrix} 1+\Lambda & 0 \\ 0 & -1+\Lambda \end{pmatrix} u_x + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} u_y = \Delta u = u_{xx} + u_{yy}.$$

Assume $u(0,y,t) = u^0(y,t)$, $u_x(0,y,t) = u^0(y,t)$ and $u(x,y,0) = 0$ (throwing initial data to the source). Take Fourier in y, and note $F_y u(x,\eta_2,t) = v(x,\eta_2,t)$. Then we have

$$v_t + \begin{pmatrix} 1+\Lambda & 0 \\ 0 & -1+\Lambda \end{pmatrix} v_x + \begin{pmatrix} 0 & -i\eta^2 \\ -i\eta^2 & 0 \end{pmatrix} v = v_{xx} - (\eta_2)^2 v.$$

In addition, we take Laplace transform in t and write $(L_t v)(x,\eta_2,s) = V(x,\eta_2,s) = \int_0^\infty e^{-st} v(x,\eta_2,t)dt$, then we have

$$sV + \begin{pmatrix} 1+\Lambda & 0 \\ 0 & -1+\Lambda \end{pmatrix} V_x + \begin{pmatrix} 0 & -i\eta^2 \\ -i\eta^2 & 0 \end{pmatrix} V = V_{xx} - (\eta_2)^2 V.$$

Finally, take the Laplace transform in x and write $L_x(V) = \hat{V}(\xi,\eta_2,s)$, we can obtain

$$s\hat{V} + \begin{pmatrix} 1+\Lambda & 0 \\ 0 & -1+\Lambda \end{pmatrix} (\xi\hat{V} - V^0) + \begin{pmatrix} 0 & -i\eta^2 \\ -i\eta^2 & 0 \end{pmatrix} \hat{V} = \xi^2\hat{V} - \xi V^0 - V_x^0 - (\eta_2)^2\hat{V}.$$

Now, we want to invert the Laplace transform in ξ first, then we have

$$\begin{pmatrix} s+(1+\Lambda)\xi - \xi^2 + +\eta_2^2 & -i\eta_2 \\ -i\eta_2 & s+(-1+\Lambda)\xi - \xi^2 + \eta_2^2 \end{pmatrix} \hat{V} = -V_x^0 + \begin{pmatrix} -1-\Lambda-\xi & 0 \\ 0 & 1-\Lambda-\xi \end{pmatrix} V^0$$

and call $\det A(\xi,\eta_2,s) = p(\xi,\eta_2,s)$, regarded as the polynomial in ξ and $A(\xi,\eta_2,s) = \begin{pmatrix} s+(1+\Lambda)\xi - \xi^2 + +\eta_2^2 & -i\eta_2 \\ -i\eta_2 & s+(-1+\Lambda)\xi - \xi^2 + \eta_2^2 \end{pmatrix}$. So we calculate that

$$\begin{aligned} p(\xi,\eta_2,s) &= \xi^4 - 2\Lambda\xi^3 + (-1-2s-2\eta_2^2+\Lambda^2)\xi^2 + (2s\Lambda + 2\eta_2^2)\xi \\ &\quad + s^2 + \eta_2^2 + 2s\eta_2^2 + \eta_2^4 \\ &= \Pi_{j=1}^4 (\xi - \lambda_j(\eta_2,s)) \end{aligned}$$

where $\eta_j(\eta_2,s)$ are the four roots of the characteristic polynomial $p(\xi,\eta_2,s)$. Then

$$\begin{aligned} \hat{V} &= \frac{1}{p(\xi,\eta_2,s)} A(\xi,\eta_2,s)^* \left(-V_x^0 + \begin{pmatrix} -1-\Lambda-\xi & 0 \\ 0 & 1-\Lambda-\xi \end{pmatrix} V^0 \right) \\ &= \frac{f}{p} V^0 + \frac{g}{p} V_x^0 \\ &= \frac{f(\xi,\eta_2,s)}{\Pi_{j=1}^4 (\xi - \lambda_j(\eta_2,s))} V^0 + \frac{g(\xi,\eta_2,s)}{\Pi_{j=1}^4 (\xi - \lambda_j(\eta_2,s))} V_x^0, \end{aligned}$$

where f, g are 2 by 2 matrices and the inverse Laplace transform in ξ is

$$V(x,\eta_2,s) = \frac{1}{2\pi i} \left[V^0 \int_{-i\infty}^{i\infty} e^{\xi x} \frac{f}{p} d\xi + V_x^0 \int_{-i\infty}^{i\infty} e^{\xi x} \frac{g}{p} d\xi \right].$$

Note that $\deg_\xi f = 3$, $\deg_\xi g = 2$ and $\deg_\xi p = 4$, then we can do the contour integral to the infinity and

$$\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{\xi x} \frac{f}{p} d\xi = \sum_{j=1}^4 \left(Res_{\xi=\lambda_j} \frac{f}{p} \right) e^{\lambda_j x}.$$

By using mathematica, we have distinct four roots and rearrange their order to $\lambda_4 < \lambda_3 < 0 < \lambda_2 < \lambda_1$, by the stability we get that

$$V^0(Res_{\xi=\lambda_j}\frac{f}{p}) + V_x^0(Res_{\xi=x_j}\frac{g}{p}) = 0$$

for $j = 1, 2$. We have the Dirichlet-Neumann Relation for $j = 1, 2$. On the other hand, for a well-posed problem, one relation should suffice well-posed situation have means $|\Lambda| < 1$. The two relations $j = 1, 2$ are identical and we actually get only one relation. Dirichlet-Neumann relations give that

$$V_x^0(\eta_2, s) = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} V^0(\eta_2, s) = \alpha V^0(\eta_2, s),$$

where $\alpha_{11} = -\dfrac{\{1 - s - \eta_2^2 - \Lambda^2 + 2\lambda_1 + \lambda_1^2 + 2\lambda_2 + \lambda_1\lambda_2 + \lambda_2^2\}}{1 - \Lambda + \lambda_1 + \lambda_2}$,

$\alpha_{12} = -\dfrac{i}{1 - \Lambda + \lambda_1 + \lambda_2}$,

$\alpha_{21} = -i\dfrac{\{s + \eta_2^2 + (-1 + \Lambda)\lambda_1 - \lambda_1^2\}\{s + \eta_2^2 + (-1 + \Lambda)\lambda_2 - \lambda_2^2\}}{1 - \Lambda + \lambda_1 + \lambda_2}$ and

$\alpha_{22} = -\dfrac{\{-1 + s + \eta_2^2 + 2\Lambda - +\Lambda^2 + (-1 + \Lambda)\lambda_2 + (-1 + \Lambda + \lambda_2)\lambda_1\}}{1 - \Lambda + \lambda_1 + \lambda_2}$. If we invert the transform of λ_1,λ_2 and $\dfrac{1}{1 - \Lambda + \lambda_1 + \lambda_2}$, then we invert α. Now, we (A) compute $\dfrac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{st}\lambda_j(\eta_2, s)ds$ would not involve simply residue calculation and (B) then $\dfrac{1}{1 - \Lambda + \lambda_1 + \lambda_2}$ worse.

To solve the case (B), by using (A), we invert $\dfrac{\partial\lambda_2}{\partial s}$ instead and use "Fourier path method". Recall that λ_j are the roots of p, then we have the roots relations as follows: $\sum\lambda_j = 2\Lambda = C_1$ and $\Pi\lambda_j = s^2 + 2\eta_2^2 s + \eta_2^2 + \eta_2^4 = C_4$and $\dfrac{\sum\lambda_j}{\lambda_k} = -2\Lambda s - 2\eta_2^2\Lambda = C_3$. If we let $\sigma : \{1, 2, 3, 4\} \to \{1, 2, 3, 4\}$ be a permutation and the relations C_j depending on $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, we have $C_j(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = C_j(\lambda_{\sigma(1)}, \lambda_{\sigma(2)}, \lambda_{\sigma(3)}, \lambda_{\sigma(4)})$, invariant under all the permutation σ. There is a theorem in algebra can help us solve the question.

Theorem 6. *If a polynomial $p(\lambda_1, \lambda_2\lambda_3, \lambda_4)$ is invariant under all permutation σ, then $p(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = Q(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ for some polynomial Q.*

Back to the example and by using the above theorem, the polynomial p becomes a polynomial in 2Λ, $-2\Lambda s - 2\eta_2^2\Lambda$, ..., $s^2 + 2\eta_2^2 s + \eta_2^2 + \eta^4$, i.e., the polynomial in these coefficients of the characteristic polynomial p and we write

$$\frac{1}{1 - \Lambda + \lambda_1 + \lambda_2} = \left(\Pi_\sigma \frac{1}{1 - \Lambda + \lambda_{\sigma(1)} + \lambda_{\sigma(2)}}\right)\left(\Pi_{\sigma\neq I}(1 - \Lambda + \lambda_{\sigma(1)} + \lambda_{\sigma(2)}\right)$$

and $\Pi_\sigma(1 - \Lambda + \lambda_{\sigma(1)} + \lambda_{\sigma(2)})$ is invariant under all permutations. This implies that the polynomial of the coefficients of the characteristic polynomial of p is equal to the polynomial of the coefficients 2Λ,..., $s^2 + 2\eta_2^2 s + \eta_2^2 + \eta_2^4$.

Recall the case $\Lambda = 0$, we have calculated that $\lambda_1 = \sqrt{\frac{1}{2} + s + \frac{1}{2}\sqrt{1 + 4s} + \eta_2^2}$ with $|\lambda_1| \nrightarrow 0$ as $|s| \to \infty$, it is convenient for varying the contour integral in the complex plane, i.e., instead, we compute $\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{st}\frac{\partial\lambda_1}{\partial s}ds$ and $\left|\frac{\partial\lambda_1}{\partial s}\right| \to 0$ as $|s| \to \infty$. For the general case $|\Lambda| < 1$, we also consider the integral

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{st}\frac{\partial\lambda_j(\eta_2, s)}{\partial s}ds.$$

where $\lambda_j(\eta_2, s)$ is a root of $p(\xi, \eta_2, s) = 0$. Instead, going back to the case $\Lambda = 0$, then $\lambda_2 = \sqrt{\frac{1}{2} + s - \frac{1}{2}\sqrt{1+4s} + \eta_2^2}$varying s and fix $\eta_2 \in \mathbb{R}$. It is bad when $\frac{1}{2}+s-\frac{1}{2}\sqrt{1+4s}+\eta_2^2 < 0$ since the Riemann surfaces will appear. So we can set $\frac{1}{2}+s-\frac{1}{2}\sqrt{1+4s}+\eta_2^2 = -\eta_1^2$ and look at the branching curves for λ_2 ($\lambda_2 = i\lambda_1$). Since $p(\lambda_2(\eta_2, s), \eta_2, s) = 0$, then set $p(i\eta_1, \eta_2, s(\eta_1, \eta_2)) = 0$, where $s(\eta_1, \eta_2)$ is a solution of p which is a polynomial in s of degree 2. Then

$$s = i\left(\Lambda\eta_1 \pm \sqrt{\eta_1^2 + \eta_2^2}\right) - \left(\eta_1^2 + \eta_2^2\right).$$

Therefore, we use the change of variable, then we have

$$\begin{aligned}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{st}\frac{\partial\lambda_j}{\partial s}ds &= \frac{1}{2\pi i}\int e^{s(\eta_1,\eta_2)t}\frac{\partial i\eta_1}{\partial s}\frac{ds}{d\eta_1}d\eta_1 \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{s(\eta_1,\eta_2)t}d\eta_1.\end{aligned}$$

Further inversion in η_2, we can get

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int e^{i\eta_2 y}e^{s(\eta_1,\eta_2)t}d\eta_1 d\eta_2 = \frac{1}{2\pi}\int\int e^{i\eta_1 x + i\eta_2 y}e^{s(\eta_1,\eta_2)t}d\eta_1 d\eta_2$$

and rewrite $s(\eta_1, \eta_2)$ as the form $s(\eta_1, \eta_2) = \pm i\sqrt{}t + [\ ]t$, as a combination of Huygens and dissipation terms, note that we have used the previous skill that we integrate over a region in $\mathbb{R}^2 - C$, where C is the singularities occurring. Finally, use Fourier transform in η_1 and η_2.