# Partial regularity for the Navier-Stokes equation and higher derivatives estimates

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**Abstract**

**The note is mainly for personal record, if you want to read it, please be careful.** This short note was given by Prof. Alexis Vasseur in NCTU, 2014. For more regularity result for NS-equation, we refer readers to G. Seregin's lecture notes.

## 1 Introduction

For the incompressible Navier-Stokes equation

$$\begin{cases} \partial_t u - u \cdot \nabla u + \nabla p - \Delta u = 0 \\ \nabla \cdot u = 0 \end{cases}, \tag{1.1}$$

rewrite the equation as $-\Delta p = \sum_{i,j} \partial_{ij}^2 u_i u_j$, then we have $p = \sum_{i.j}[(-\Delta)^{-1}\partial_{ij}^2]u_i u_j$ and

$$\|p\|_{L^{p/2}L^{q/2}} \leq C_q \|u\|_{L^pL^q}^2.$$

From the $(NS) \cdot u$, we can get

$$\partial_t \frac{|u|^2}{2} + \nabla \cdot (u\frac{|u|^2}{2} + p)) + |\nabla u|^2 - \Delta \frac{|u|^2}{2} \leq 0. \tag{1.2}$$

**Theorem 1.1.** *(Leray-Hopf) For all initial value $u^0 \in L^2(\mathbb{R}^3)$, $\exists u \in L^\infty L^2$, $\nabla u \in L^2L^2$ such that $u$ is a solution to (NS) and satisfies (*).*

What can we get from the energy ?

1. Local study for small energy. Smallness on some quantity (or $u$) can imply regularity.
2. Global property on the flow (without the smallness condition). Canonical scaling of the equation: If $u$ is a solution to (1.1) and (1.2) in $\mathbb{R}^+ \times \mathbb{R}^3$, then for $\epsilon \leq 1$, we set

   $$\begin{cases} u_\epsilon(t,x) = \epsilon u(t_0 + \epsilon^2 t, x_0 + \epsilon x) \\ p_\epsilon(t,x) = \epsilon^2 p(t_0 + \epsilon^2 t, x_0 + \epsilon x) \end{cases}$$

   is still a solution to $(NS)$ (it is a good idea to get Prodi-Serrin condition).

Now we set $Q_r = (-r, 0) \times B_r$ in the following lecture.

**Theorem 1.2.** *$\exists \eta > 0$ such that any solution of (1.1) and (1.2) verifying*

$$\int_{Q_2} |u|^3 dxdt + \int_{Q_2} |p|^{3/2} dxdt \leq \eta,$$

*then we have $|u| \leq 1$ in $Q_1$ and $|\nabla^n u| \leq C_n$ in $Q_{1/2}$.*

## 2 DeGiorgi Method

In 1957, DeGiorgi got a very useful theorem for the elliptic regularity (call it DeGiorgi-Nash-Moser iteration). For the standard elliptic equation

$$\nabla \cdot (A(x)\nabla u) = 0, \tag{2.1}$$

$A(x)$ only satisfies $\frac{1}{\Lambda}I \leq A(x) \leq \Lambda I$. If $u \in L^2(\Omega)$ solves this equation, then $u \in C^{\alpha}_{loc}(\Omega)$. There are two steps:

1. $L^2$ implies $L^\infty$.
2. $L^\infty$ implies $C^{\alpha}_{loc}$.

**Lemma 2.1.** *(Energy method) $\exists \eta > 0$ such that for any solution $u$ solves (2.1), if $\int_{B_2} |\nabla u|^2 dx \leq \eta$, then $|u| \leq 1$ in $B_1$.*

Use the DeGiorgi's method, let $c_k = 1 - 2^{-k}$, $u_k = (u - c_k)_+$, $U_k = \int |\nabla u_k|^2 dx$ and $\widetilde{B_k} = B_{1+2^{-k}}(0)$, then we want to show

$$U_k \leq C^k U_{k-1}^{\beta}$$

for some $\beta > 1$. If $U_0 \ll 1$, then $U_k \to 0$ as $k \to \infty$. Now we let $(u - c_k)\varphi_k$ be the test function with $\varphi_k$ as a cutoff function of $\widetilde{B_{k-1}}$ and $\varphi_k|_{\widetilde{B_k}} = 1$, then we can obtain

$$U_k \leq \int |\nabla(\varphi_k u_k)|^2 dx \leq C^k \int_{\widetilde{B_k}} u_k^2 dx.$$

From the Sobolev embedding theorem, we have $u_{k-1} \geq C\|u_k\|^2_{\frac{2N}{N-2}}$. Moreover, from the Tchebyshev's inequality, we can get $\chi_{\{u_k>0\}} \leq \frac{u_{k-1}}{c_k - c_{k-1}}$ and $\chi_{\{u_k>0\}} \leq (2^k u_{k-1})^{\gamma}$, for some $\gamma \geq 0$. Then

$$U_k \leq C^k \int_{\widetilde{B_{k-1}}} u_{k-1}^{\frac{2N}{N-2}} dx \leq C^k U_{k-1}^{\frac{N}{N-2}},$$

where $\beta = 1 + \frac{2}{N-2} = \frac{N}{N-2}$.

## 3 Prodi-Serrin Criteria

**Theorem 3.1.** *$3 \leq p < \infty$ with $\frac{2}{p} + \frac{3}{q} \leq 1$, for any $u$ solves (1.1) and (1.2) in $\mathbb{R}^+ \times \mathbb{R}^3$ verifying $\|u\|_{L^pL^q} < \infty$, then $u$ is smooth for $t \geq t_0$.*

*Remark* 3.2. Theorem 3.1 is still true for $1 \leq p < 3$. For $p = \infty$, we need other techniques.

*Proof.* We set $(u_\epsilon, p_\epsilon)$ to be as before, then we calculate

$$\|u_\epsilon\|_{L^pL^q(Q_2)} = \epsilon^{1-\frac{3}{p}-\frac{2}{q}} \|u\|_{L^pL^q((t_0-\epsilon^2, t_0)\times B_\epsilon(x_0))}.$$

Then for $\epsilon$ small enough, $\|u_\epsilon\|_{L^pL^q(Q_2)} + \|p_\epsilon\|_{L^{p/2}L^{q/2}(Q_2)} \leq \eta$, by the before lemma, we can get $|u_\epsilon| \leq 1$ (which means $|u| \leq \frac{1}{\epsilon}$ and note that we need $t$ far away from 0 since if $t$ is very small, then $\epsilon$ will be very tiny so that $1/\epsilon$ will be very large). □

## 4 Partial Regularity

**Theorem 4.1.** *(CKN) $u$ is a solution to (1.1) and (1.2), $\Omega = \{(t,x)|u$ is not locally bounded at $(t,x)\}$, then $dim_H(\Omega) \leq 1$ and $H^1(\Omega) = 0$.*

By Scheffer's early result, we have

**Theorem 4.2.** $dim_H(\Omega) \leq \frac{5}{3}$.

For $\epsilon$ fixed, we set

$$F_\epsilon(t,x) = \fint_{\widetilde{Q_{2\epsilon}}(t,x)} |u|^{\frac{10}{3}}(s,y)dsdy + \fint_{\widetilde{Q_{2\epsilon}}(t,x)} |p|^{\frac{5}{3}}(s,y)dsdy,$$

where $\widetilde{Q_\epsilon}(t,x) = (t-\epsilon^2, t)\times B_\epsilon(x)$, and we have $\|F_\epsilon\|_{L^1} \leq C$. If $F_\epsilon(t,x) \leq \eta\epsilon^{-\frac{10}{3}}$, then $(t,x) \notin \Omega$. Use $(u_\epsilon, p_\epsilon)$ to verify the hypothesis and $u_\epsilon$ is smooth near $(t,x)$, we obtain

$$\begin{aligned} |\Omega| &\leq |\{F_\epsilon(t,x) \geq \frac{\eta}{\epsilon^{\frac{10}{3}}}\}| \\ &\leq \frac{\epsilon^{\frac{10}{3}}}{\eta}\int F_\epsilon(t,x)dxdt \leq C\epsilon^{5-\frac{5}{3}}, \end{aligned}$$

and note that the exponent $\frac{5}{3}$ is dimension.

**Theorem 4.3.** *$\exists\eta > 0$ such that any solution of (1.1) and (1.2) verifying*

$$\int |M(\nabla u)|^2 dxdt + \int_{Q_2} |\nabla^2 p|dxdt \leq \eta$$

*and $\forall t \in (-2,0)$, $\int_{B_1} u(t,x)dx = 0$. Then $|u| \leq 1$ in $Q_1$.*

For more details, we refer readers to see *The De Giorgi method for regularity of solutions of elliptic equations and its applications to fluid dynamics. Luis A. Caffarelli and Alexis F.*