Theory and Application of Homogenization
A Revisit with Recent Progress

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Abstract

The note is mainly for personal record, if you want to read it, please be careful. I followed Prof. Fang-Hua Lin's lecture in NCTS, Taiwan. He gave a series of lectures of homogenization theory during Dec 17-19, 2012. We separate this notes into three parts: Applied Analysis, Homogenization and Eigenvalue Problem.

1 Applied Analysis

What is applied analysis? We have two viewpoints of this: Concentrations and Oscillations.

1.1 Concentrations

Example 1.1. Sacks-Uhlenbeck bubbling (beyond conformal dimension (i.e. two dimension)): Harmonic maps and Minimal surfaces.

Example 1.2. Gromov (pseudo-holomorphic curves): Compactness $\Longrightarrow$Selberg-Witten theory.

Example 1.3. Yamabe problems: Studied in math biology (spikes, higher dimensional patterns).

Example 1.4. Particle like solutions: BEC, vortices (filaments), super-conductivity, codimention two concentration.

Example 1.5. Singularities (L. Simon), shocks (sharp interfaces), defects (motion by mean curvature), Ricci flows (black holes).

1.2 Oscillations

Example 1.6. Measures: Wigner measure (in quantum mechanism), Young measure, Defect measure, H-measure (homogenization measure, introduced by Luc Tartar and Gerand), microlocal measure.

Example 1.7. Averaging Principle: Large number laws, hydrodynamics, **Homogenization**.

Example 1.8. Weak continuity (conservation laws), compensate compactness, viscosity method (convexity and monotonicity).

2 Homogenization

We only consider elliptic equations with periodic coefficients and theory of homogenization.

2.1 Goals

1. Composite materials (in practical use).
2. A lot of classical physics can be explained by using homogenization.
3. (Pure) Mathematical reasons : Used in the Hopf conjecture.

Conjecture 2.1. *(Hopf conjecture) A Riemannian metric is defined on a torus without conjugate points, then the metric will be flat.*

It was solved by using the ideas in homogenization.

2.2 Elliptic Equations with Periodic Coefficients

First, we consider the ODE case as follows:

Lemma 2.2. *(Floquet Theory)*
$\frac{dx}{dt} = A(t)x$, *with* $A(x)$ *is periodic in* t, *then*

$$\phi(t) = P(t)\exp(\mathbb{C}t)$$

is the fundamental solution matrix, where $\mathbb{C}$ *is a constant matrix and* $P(t)$ *is periodic in* t.

Now, we consider the PDE case: Let L be a uniformly elliptic operator with periodic coefficients in $\mathbb{R}^n$. Suppose L is either of the form $L = div(A(x)\nabla)$ or of the form $L = -a^{ij}(x)\partial^2_{x_i x_j}$. Then

Theorem 2.3. *(Avellanda-Lin, 1989) If $Lu = 0$ in $\mathbb{R}^n$, and $\max_{B_R} |u(x)| \leq MR^m$, for a sequence of $R \to \infty$, then u is a polynomial of degree m with periodic coefficients. That is, $u(x) = \sum_{|\alpha|\leq m} a_\alpha(x) x^\alpha$ with $a_\alpha(x)$ is periodic in x.*

Remark 2.4. (a) This is a type of Liouville's theorem.

(b) Moser-Struwe (1992) proved that there are solutions of the PDE

$$-\Delta u + v(x,u) = 0 \text{ in } \mathbb{R}^n,$$

with $u(x) = \alpha \cdot x + B_\alpha(x)$, for any $\alpha \in \mathbb{R}^n$, and $B_\alpha(x)'s$ are bounded.

(c) Caffarelli-R. de Llave (2001): Planelike minimal surface in $(\mathbb{R}^n, g)$, $g(x)$ is periodic in x.

(d) P. Kuchment: Floquet Theory for Partial Differential Equations, Birkhauser, Basel (1993).

Theorem 2.5. *(M. Avellaneda and F. H. Lin, 1991)*

Let G and G_0 be Green functions of L and L_0, then

$$|G(x,y) - G_0(x,y)| \leq \frac{C_1}{|x-y|^{n-1}},$$

$$|\nabla_x G(x,y) - P(x)\nabla_x G_0(x,y)| \leq \frac{C_2}{|x-y|^n},$$

$$|\nabla_x \nabla_y G(x,y) - P(x)P(y)\nabla_x\nabla_y G_0(x,y)| \leq \frac{C_3}{|x-y|^{n+1}},$$

for some constants C_1, C_2, C_3 and periodic matrix $P(x)$, and for all $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$ with $|x-y| \geq 1$.

Consequences:

1. Operators $\frac{\partial}{\partial x^\alpha}(L)^{-1}\frac{\partial}{\partial^\beta}$, $\frac{\partial}{\partial x^\alpha}(L)^{-1/2}$, $(L)^{-1/2}\frac{\partial}{\partial x^\beta}$, $1 \leq \alpha, \beta \leq n$, are all bounded from $L^p(\mathbb{R}^n)$ into weak $L^1(\mathbb{R}^n)$ (Calderon-Zygmund theorem).

2. If $Lu = div\overrightarrow{F}$ in $\mathbb{R}^n$, then $\|u\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|\overrightarrow{F}\|_{L^p(\mathbb{R}^n)}$ (DeGiorgi's theorem).

3. The operator $\frac{\partial}{\partial x^\alpha}\frac{\partial}{\partial x^\beta}(L)^{-1}$ are bounded from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ (and from $L^1(\mathbb{R}^n)$ to weak $L^1(\mathbb{R}^n)$), $1 < p < \infty$, if and only if $divA = 0$.

4. If $divA$=0, then $Lu = f$ in $\mathbb{R}^n$ implies $||u||_{W^{2,p}(\mathbb{R}^n)} \leq C||f||_{L^p(\mathbb{R}^n)}$.

5. If $Lu = 0$ in H^n_+ (a half space) and $u = f$ on ∂H^n_+, then $||u^*||_{L^p(\partial H^n_+)} \leq C||f||_{L^p(\partial H^n_+)}$, $1 < p < \infty$. Here u^* is the usual Hardy-Littlewood maximal function (Fatou's theorem, introduced in the Stein's small book).

Quasilinear Elliptic Equations and Hamilton-Jacobi Equations, the corresponding studies were carried out by Lions- Papanicolaou-Varadhan, and by C. Evans. Fully nonlinear differential and fully nonlinear integral equations with periodic coefficients were studied by L. Cafferelli et al. The study of these equations with periodic 1 coefficients in large domain $\Longleftrightarrow$ the study of same type equations with periodic ϵ coefficients ($\epsilon \ll 1$) in a bounded domain. The latter is called homogenization problem. We show two problems in homogenization:

Problem 2.6. (The Inverse Problem)
Consider the equation

$$\begin{cases} \partial_t u^\epsilon(x,t) - div(a^{ij}(\frac{x}{\epsilon})u^\epsilon_{x_j}) = 0 & \text{in } \Omega \times [0,T],\ \Omega \subset \mathbb{R}^n \\ u^\epsilon(x,0) = \phi_0(x) & x \in \Omega \\ u^\epsilon(x,t) = 0 & (x,t) \in \partial\Omega \times [0,T] \end{cases} . \tag{2.1}$$

Assumptions: (1) $\lambda I \leq (a^{ij}(y)) \leq \Lambda I$, for some $0 < \lambda \leq \Lambda < \infty$. (2) $a^{ij}(y+z) = a^{ij}(y)\ \forall z \in \mathbb{R}^n$, $y \in \mathbb{R}^n$. (3) $0 < \epsilon \ll 1$.

One observes at time $t = T_0$ a possible solution of u^ϵ of (2.1) to find a function $f^\epsilon(x)$. How can one construct a solution of (2.1) for $0 < t < T_0$. We know the backward heat equation is ill-posed.

Question1: How can one assert that $f^\epsilon(x)$ is actually close to $u^\epsilon(x,T_0)$, for a solution of (2.1)? Is there a criteria?

Question2: If $f^\epsilon(x)$ is indeed close to some $u^\epsilon(x,T_0)$, then is it possible to construct $u^\epsilon(x,t)$ for $0 < t < T_0$? For question, it means, if $||f^\epsilon(x) - u^\epsilon(x,T_0)|| \leq \delta$, can one construct from $f^\epsilon(x)$ an approximate solution $V^\delta(x,t)$ such that

$$\|V^\delta(x,t) - u^\epsilon(x,t)\| \leq O(\delta^\alpha) \text{ for } 0 < t_0 \leq t \leq T_0 \ ?$$

Here $V^\delta(x,t)$ is distinct from the truly solution and in order to do so, one requires $t \geq t_0(\alpha) > 0$ and $t_0(\alpha) \to 0$ as $\alpha \to 0$, and all constants are independent of ϵ. We know that $\omega(\delta) \to 0$ as $\delta \to 0$, this is a Hadamard (sense) criterion.

J. Hadamard: Well-posed problems in PDEs - (a) Existence, (b) Uniqueness, (c) Stability (Continuous dependence). But this is not useful in numerical computations, we introduce John's idea.

F. John: Well-behaved problems in Numerical PDEs - Here given an error ϵ, in data (observation, grid size, $\cdots$), one wants the numerical solutions to be within $O(\epsilon^\alpha)$, $\alpha > 0$, error of the theoretical solution. e.g. When $\omega(\delta) = \dfrac{1}{\sqrt{\log \delta}} \leq 0.1$, this implies $\delta \leq e^{-100}$ is so small that the numerical method fails.

For simplicity, we first consider the Laplacian case:

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \Omega \times [0,T] \\ u = 0 & \text{on } \partial\Omega \times [0,T] \\ u(x,0) = \phi(x) \end{cases}.$$

Question1: $\Delta \phi_k + \lambda_k \phi_k = 0$ in Ω with $\int_\Omega \phi_k^2 dx = 1$ and λ_j's are eigenvalues with $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \to \infty$, by Weyl's asymptotic formula, we have $\lambda_k \approx C(n, |\Omega|)k^{2/n}$. Then $|\int f(x)\phi_k(x)dx| \leq ||f||_{L^2} e^{-\lambda_k T} + \delta$ and $\int_\Omega f(x)\phi_k(x)dx = a_k e^{-\lambda_k T}$, then we can solve $u(x,t) = \sum_{k=1}^N a_k e^{-\lambda_k t}\phi_k(x)$ and $||u(x,t) - u_*(x,t)|| \leq \delta + e^{-\lambda_N T} \leq 2\delta$ by choosing T large (by using spectrum method).

To estimate $|\lambda_k^\epsilon - \lambda_k| \leq c$ for some c, $\forall k \in \mathbb{N}, \epsilon > 0$, the difficulty is that we don't know how to choose N, and in theoretical sense, we want to k, ϵ are independent of the space variables. If c is small, this inverse problem can be solved.In early homogenization theory, we only known $\lambda_k^\epsilon \to \lambda_k$ as $\epsilon \to 0$ for all fixed k, and we couldn't estimate the speed of convergence; but now we can have the finer result of $|\lambda_k^\epsilon - \lambda_k|$ $\forall k \in \mathbb{N}$ and $\forall \epsilon > 0$ (we need more technical estimates for these results).

Look at the following equation:

$$(L) : \begin{cases} \partial_{x_i}(a^{ij}(\frac{x}{\epsilon})u^\epsilon_{x_j}) = 0 & \text{in } \Omega \\ u^\epsilon(x) = f & \text{on } \partial\Omega \end{cases} \text{ converges to } (R) : \begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega \end{cases}$$

Since the (R) is the homogenized convergence of the (L), w.l.o.g. we can assume the limit is Laplacian and denote this convergence to be the *homogenized convergence*. We have a lot of ideas of solving the (R)-equation (call it homogenized-problem), so we borrow these ideas and apply to the (L)-equation (call it ϵ-problem)!

In periodic homogenization: ϵ=periodic size, Δ=grid size, u_ϵ=solution to the ϵ-problem, u_0=solution to the homogenized problem, and u_Δ=solution to the Δ-problem (numerical) by using finite difference method to the homogenized problem.

Question: How close u_Δ to u_ϵ?

Total error bound$\leq O(\Delta^\beta)+O(\omega(\dfrac{\epsilon}{\Delta}))$, where $\omega(\delta)$=theoretical error bound of $||u_\Delta - u_0||$. Obviously, one wishes to have total error$\leq O(\Delta^\alpha)$, for some $\alpha > 0$. Thus, we need $\omega(\delta) \leq O(\delta^A)$ for some large A as possible, but in some statistical homogenization or nonlinear elliptic homogenization, the best known $\omega(\delta)$ may be given by

$$c|\log\delta|^{-a} \text{ for some small } a > 0.$$

Such estimate is often sufficient (or actually best possible) in the theory, but it is Ill-behaved for practical numerical computations. Moreover, in the computational situation, we always change the grid size in order to make the ϵ-problem more specific, and get a better estimate.

Problem 2.7. (Boundary Control by J. L. Lions)

Consider

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \Omega \times [0,T] \\ u = g & \text{on } \partial\Omega \times [0,T] \\ u(x,0) = u_0 & \\ u_t(x,0) = u_1 & \end{cases}$$

Question: $(u_0, u_1) \in L^2 \times L^2$, does there exist $T > 0$ and $g \in L^2(\partial\Omega \times [0,T])$ such that $u(x,T) = u_t(x,T) = 0$? Since this is a wave equation, having finite speed propagation, we only need to consider the minimal time T, it is interesting to consider the minimal time T(optimal control time) and the minimal $||g||_{L^2}$.

Consider the forward wave heat equation

$$\begin{cases} \phi_{tt} - \Delta\phi = 0 & \text{in } \Omega \times [0,T] \\ \phi = 0 & \text{on } \partial\Omega \times [0,T] \\ \phi(x,0) = \phi_0, \phi_t(x,0) = \phi_1 & \end{cases} \tag{2.2}$$

let $y = \dfrac{\partial\phi}{\partial\nu}$ on $\Omega \times [0,T]$ and define an operator L_t (a Dirichlet-to-Neumann map) by

$$L_t(\phi_0, \phi_1) = (-y_1, y_0),$$

where $y_0 = y(x,0)$ and $y_1 = y_t(x,0)$ and $y(x,t)$ solves the adjoint equation of (2.2)

$$\begin{cases} y_{tt} - \Delta y = 0 & \text{in } \Omega \times [0,T] \\ y(x,T) = y_t(x,T) = 0 & \\ y = \dfrac{\partial \phi}{\partial \nu} & \text{on } \partial\Omega \times [0,T] \end{cases} \tag{2.3}$$

and call (2.3) the backward equation (note that in general, the backward equation is il-posed), and $(L_t(\phi_0, \phi_1), (\phi_0, \phi_1)) = \int_0^T \int_{\partial\Omega} \left| \dfrac{\partial \phi}{\partial \nu} \right|^2 dSdt.$

Lemma 2.8. *(LafaHo, Pohozaev)*

For $T \geq T_0 > 0$ (T_0 is large) such that for ϕ a solution of (2.2), we have

$$c_T(\|\phi_0\|_{L^2}^2 + \|\phi_1\|_{L^2}^2) \leq \int_0^T \int_{\partial\Omega} \left| \frac{\partial \phi}{\partial \nu} \right|^2 dSdt \leq C_T(\|\phi_0\|_{L^2}^2 + \|\phi_1\|_{L^2}^2) \tag{2.4}$$

for some constants $c_T, C_T > 0$ depending on T.

The right hand side estimate of (2.4) is by using Pohozaev identity, but in general, the left hand side is wrong even if all the coefficients are smooth . When the left hand side holds, we have the coercive property to get L_t is invertible (i.e., we can use the Lax-Milgram theorem to inverse the problem for the product form $(L_t(\phi_0, \phi_1), (\phi_0, \phi_1))|_{t=0}$). For (u_0, u_1) given, $\exists (\phi_0, \phi_1)$ such that $L_t(\phi_0, \phi_1) = (-u_1, u_0)$ if $g = \left(\dfrac{\partial \phi}{\partial \nu} \right)$. The minimum time T is large than T_0, in order to find the optimality, we need to use the techniques in *microlocal analysis* and presumably assumed time T_0 to be unique.Now back to the homogenization problem, consider $L_\epsilon = \partial_{x_i}(a^{ij}(\dfrac{x}{\epsilon})\partial_{x_j})$ and $\int_\Omega u_\epsilon^2 = 1$ be such that $L_\epsilon u_\epsilon + \lambda_\epsilon u_\epsilon = 0$ in Ω.

Question: Is it true that $c\lambda_\epsilon \leq \int_{\partial\Omega \times [0,T]} \left| \dfrac{\partial u_\epsilon}{\partial \nu_\epsilon} \right|^2 d\sigma \leq C\lambda_\epsilon$?

In the control problem, we consider

$$\begin{cases} \dfrac{\partial^2}{\partial t^2} u^\epsilon(x,t) - \dfrac{\partial}{\partial x_i}(a^{ij}(\dfrac{x}{\epsilon}) u^\epsilon_{x_j}) = 0 & \text{in } \Omega \times [0,T] \\ u^\epsilon(x,0) = \phi_0 \in H^1, \ u^\epsilon_t(x,0) = \phi_1 \in L^2 & \\ u^\epsilon(x,t) = g & \text{on } \partial\Omega \times [0,T] \end{cases} \tag{2.5}$$

Question: Is there a time $T_0 > 0$ such that for $T \geq T_0$, $\exists g^\epsilon \in L^2(\partial\Omega \times [0,T])$ such that the solution $u^\epsilon(x,t)$ of (2.7) satisfies $u^\epsilon(x,T) = u^\epsilon_t(x,T) = 0$?

Answer: In general situations, we cannot find such g^ϵ, we need $\|g^\epsilon\|_{L^2} \leq C(\|\phi_0\|_{H^1} + \|\phi_1\|_{L^2})$ and $\int_{\partial\Omega \times [0,T]} \left| \dfrac{\partial u^\epsilon}{\partial \nu^\epsilon} \right|^2 \approx (\|\nabla \phi_0\|_{L^2}^2 + \|\phi_1\|_{L^2}^2)$. To get such estimates, the idea comes from considering the equation

$$\begin{cases} \partial_{tt}u - \Delta u = 0 & \text{in } \Omega \times [0,T] \\ u = 0 & \text{on } \partial\Omega \times [0,T] \\ u(x,0) = \phi_\lambda, u_t(x,0) = 0 \end{cases}$$

by separation of variable, it's easy to get a solution of the form $u(x,t) = \phi_\lambda(x) \cos \sqrt{\lambda} t$, where ϕ_λ satisfies $\Delta\phi_\lambda + \lambda\phi_\lambda = 0$ and $\int_\Omega \phi_\lambda^2 = 1$ and $\lambda = \int_\Omega |\nabla\phi_\lambda|^2 \approx_{C(\Omega)} \int_{\partial\Omega} \left|\frac{\partial\phi_\lambda}{\partial\nu}\right|^2$, the final step is the key point (it is dimensional balance) and apply this idea to solve the original homogenization problem.

2.3 G-Convergence (S. Spagnolo)

Consider

$$\begin{cases} -div(a(x)\nabla u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $a(x)$ satisfies $0 < \alpha \le a(x) \le \beta < \infty$ a.e. and $u \in H_0^1(\Omega)$ for any $f \in H^{-1}(\Omega)$, by Lax-Milgram lemma, we get

$$||u||_{H_0^1(\Omega)} \le C(n,\alpha,\beta)||f||_{H^{-1}(\Omega)}.$$

Question: Suppose there are sequence $a^{(m)}(x)$, $f^{(m)}(x)$ such that $||f^{(m)}||_{L^2} \le 1$ and $\alpha \le a^{(m)}(x) \le \beta$, then $||u^{(m)}||_{C^\gamma(\Omega)} + ||u^{(m)}||_{H^1} \le C_0$. Assume $u^{(m)} \to u^{(\infty)}$ in C^{α_0} for some α_0 ($G^{(m)} \to G^{(\infty)}$ in C^α off diagonal), also $a^{(m)} \to a^{(\infty)}$ in L^∞ weak* , is there $a^{(\infty)}(x)$ such that

$$\begin{cases} -div(a^{(\infty)}(x)\nabla u^{(\infty)}) = f^{(\infty)} & \text{in } \Omega \\ u^{(\infty)} = 0 & \text{on } \partial\Omega \end{cases}?$$

The answer is no! If is true for the question, $\exists A^\infty(x) = (a_{ij}^\infty(x))$ such that $\alpha I \le A^\infty(x) \le \beta I$ and $-div(A^\infty(x)\nabla u^{(\infty)}) = f^{(\infty)}(x)$, valued with $a^{(m)}(x)I$ replacing by $A^{(m)}(x)$, the problem will appear on the diagonal parts.

2.3.1 Essence of DeGiorgi-Spagnolo G-convergence theorem

1. Let $T_n : H \to H^*$, $\{T_n\}$ is a sequence of bounded, linear operators with $||T_n|| \le M$ for all $n = 1, 2, \cdots$. Assume $\langle T_n u, u\rangle = \lambda||u||^2$, $\lambda > 0$ $\forall n \ge 1$, this implies $\exists\{T_{n_j}\}$ and T_∞ in $L(H, H^*)$ such that $\forall f \in H^*$, $T_{n_j}u_{n_j} = f$, solutions $\{u_{n_j}\}$ converges weakly in H such that $T_\infty u_\infty = f$ and $\langle T_\infty u, u\rangle \ge \lambda||u||^2$ and $||T_\infty u||_* \le \frac{M^2}{\lambda}||u|| \ \forall u \in H$.

2. If $T_n = -div(A_n(x)\nabla)$, then $T_\infty = -div(A_\infty(x)\nabla)$. If $|A_n(x)\xi| \leq M|\xi|$ and $\langle A_n(x)\xi, \xi\rangle \geq \lambda|\xi|^2$, then $|A_\infty(x)\xi| \leq \frac{M^2}{\lambda}|\xi|$ and $\langle A_\infty(x)\xi, \xi\rangle \geq \lambda|\xi|^2$.

3. If $A_\epsilon(x) = A(\frac{x}{\epsilon})$ with ellpticity $A(y)$ and $A(y)$ is periodic, then $A_\infty(x) \equiv A_\infty$ is a constant (translation invariant) operator as before $(\frac{1}{\epsilon} \to \infty)$.

How to find A_∞? Note that the limit of diagonal operator T_n is not necessarily be diagonal, in homogenization theory, call $T_n \to T$ to be Γ-convergence.

Example 2.9. (ODE case)

Consider

$$\begin{cases} \frac{d}{dx}(a(\frac{x}{\epsilon})u^\epsilon_x) = f(x) \\ u^\epsilon(0) = u^\epsilon(1) = 0 \end{cases}$$

with $f \in L^2([0,1])$ and $\lambda \leq a(\frac{x}{\epsilon}) \leq \Lambda$.

(a) $u^\epsilon \rightharpoonup u_0$ in $H^1_0([0,1])$, $u^\epsilon \to u_0$ in L^2

(b) let $\xi_\epsilon(x) = a(\frac{x}{\epsilon})u^\epsilon_x$, ξ_ϵ converges to ξ_0 weakly in H^1 and strongly in L^2.

(c) $\frac{1}{a(\frac{x}{\epsilon})} \to \overline{(\frac{1}{a})}$ in L^∞-weak* (by using Riemann-Lebesgue lemma), $u^\epsilon_x = \frac{\xi_\epsilon}{a(\frac{x}{\epsilon})} \rightharpoonup \overline{(\frac{1}{a})}\xi_0$, $\frac{d}{dx}\xi_0 = 0$ (one of it is strong convergence, the other is weak convergence, and their product must be weak convergence).

Hence we get the homogenized equation $\left(\frac{u_{0x}}{\overline{(\frac{1}{a})}}\right)_x = f(x)$.

2.3.2 WKB-Analysis

$L_\epsilon u_\epsilon = -\frac{\partial}{\partial_{x_i}}(a^{ij}(\frac{x}{\epsilon})u^\epsilon_{x_j}) = f(x)$ in Ω and $u_\epsilon = 0$ on $\partial\Omega$. Suppose that $a^{ij}(y)$ is periodic with period 1 and $\lambda|\xi|^2 \leq a^{ij}(y)\xi_i\xi_j \leq \Lambda|\xi|^2$, $u^\epsilon \rightharpoonup u_0$ in $H^1_0(\Omega)$ and

$$\begin{cases} \hat{L}u_0 = \frac{\partial}{\partial x_i}(\hat{a^{ij}}u_{x_j}) = f & \text{in } \Omega \\ u_0 = 0 & \text{on } \partial\Omega \end{cases}.$$

Remark 2.10. If such $(\hat{a^{ij}})$ exists, then $\hat{a^{ij}}$ is a constant matrix.

Let $y = \dfrac{x}{\epsilon}$, we write $u_\epsilon(x) = u_0(x,y) + \sum_k \epsilon^k u_k(x,y)$, where $u_k(x,y)$'s are periodic in y with period 1. Note that given a smooth function $\phi(x,y)$, $y = \dfrac{x}{\epsilon}$, then

$$L_\epsilon(\phi(x,y)) = \epsilon^{-2}A_2\phi + \epsilon^{-1}A_1\phi + A_0\phi,$$

where

$$\begin{cases} A_0 \equiv a^{ij}(y)\dfrac{\partial^2}{\partial x_i \partial x_j} \\ A_1 \equiv \dfrac{\partial}{\partial y_i}(a^{ij}(y)\dfrac{\partial}{\partial x_j}) + \dfrac{\partial}{\partial x_j}(a^{ij}(y)\dfrac{\partial}{\partial y_j}) \\ A_2 \equiv \dfrac{\partial}{\partial y_i}(a^{ij}(y)\dfrac{\partial}{\partial y_j}) \end{cases}.$$

Therefore,

$$L_\epsilon u_\epsilon = (\epsilon^{-2}A_2 + \epsilon^{-1}A_1 + A_0)\sum_{k=0}^{\infty}\epsilon^k u_k(x,y) = f(x),$$

this implies $A_2u_0 = 0$, $A_2u_1 + A_1u_0 = 0$ and $A_2u_2 + A_1u_1 + A_0u_0 = f$ by comparison of ϵ-coefficients.

2.3.3 Conclusions

1. $u_0(x,y) \equiv u_0(x)$ (a solution on T^n of an elliptic operator), $L_1u_0 = 0$ (just think the Laplace equation).

2. $\sum_{i,j=1}^n \dfrac{\partial}{\partial y_i}(a^{ij}(y)\dfrac{\partial u_1}{\partial y_j}(x,y)) = -\sum_{i,j=1}^n \dfrac{\partial}{\partial y_i}(a^{ij}(y)\dfrac{\partial u_0}{\partial_j}(x)) \implies u_1(x,y) = \sum_{l=1}^n \chi_l(y)\dfrac{\partial u_0}{\partial x_l}$ and $\sum_{i,j=1}^n \dfrac{\partial}{\partial y_i}(a^{ij}(y)\dfrac{\partial \chi_l(y)}{\partial y_l}) = -\sum_{i=1}^n \dfrac{\partial}{\partial y_i}(a^{il}(y))$.

3. $A_2u_2 = f(x) - A_1u_1 - A_0u_0$ is solvable on $T^n \implies \int_{T^n}(f(x) - A_1u_1 - A_0u_0)dy = 0$ is a necessary condition by Fredholm alternative. The latter implies $\hat{L}u_0 = f(x)$, and $\hat{a^{ij}} = \langle a^{ij}\rangle + \sum_{l=1}^n \left\langle a^{ij}\dfrac{\partial \chi_j}{\partial y_l}\right\rangle$. Here the function χ is the standard corrector term in homogenization theory.

If $a^{ij}(x) = a^{ji}(x)$, then $\hat{a^{ij}} = \hat{a^{ji}}$; if $(a^{ij}(x)) > 0$, then $(\hat{a^{ij}}) > 0$. Indeed, $\hat{a^{ij}} = \int_{T^n} a^{kl}(y)\dfrac{\partial}{\partial y_k}\left(\chi^i(y) + y_i\right)\dfrac{\partial}{\partial y_l}\left(\chi^j(y) + y_j\right)dy$, and the integrand part is zero.

We cannot get the speed of convergence from the theory, but it is important to estimate the rate of convergence.

Remark 2.11. Even $(a^{ij}(x)) = a(x)I$, $(\hat{a^{ij}})$ may not be diagonal, where $\hat{a^{ij}} = \left\langle \sum_{k=1}^{n} a^{ik}(y) \right\rangle$ may not be zero when $i \neq j$.

Conjecture 2.12. *(via WKB expansions)*
(A) $u_\epsilon \to u_0$ *strongly in* L^2 *as* $\epsilon \to 0$.
(B) $u_\epsilon(x) - (u_0(x) + \epsilon\chi(\frac{x}{\epsilon})\nabla u_0(x)) \to 0$ *strongly in* H^1 *as* $\epsilon \to 0$.

2.4 Method of oscillating test functions

We start with $L_\epsilon u^\epsilon = f$ in Ω with zero boundary data $\Leftrightarrow - \int_\Omega a^{ij}(\frac{x}{\epsilon})u^\epsilon_{x_i} v_{x_j} dx = \int_\Omega f(x)v(x)dx$ $\forall v \in H^1_0(\Omega)$. u^ϵ converges to u^0 weakly in $H^1_0(\Omega)$ and $L\chi = -divA$ on T^n, as before, $\hat{a^{ij}} = \langle a^{ij} \rangle + \left\langle a^{il}\frac{\partial \chi_j}{\partial y_l} \right\rangle$.

Lemma 2.13. *We may assume* $a^{ij}(\frac{x}{\epsilon})u^\epsilon_{x_j} \to \xi_i$ *in* $L^2(\Omega; \mathbb{R}^n)$*, hence*

$$-\int_\Omega \xi \cdot \nabla v dx = \int_\Omega f v dx \tag{2.6}$$

$\forall v \in H^1_0(\Omega)$.

Proof. Let $\eta \in C^\infty_0(\Omega)$ and let $v_\epsilon(x) = x + \epsilon\chi(\frac{x}{\epsilon})$ and apply $v = \eta \cdot v_\epsilon$ in (2.6), note that $\eta v_\epsilon \in H^1_0(\Omega)$, then

$$-\int_\Omega a^{ij}(\frac{x}{\epsilon})u^\epsilon_{x_i}(v_\epsilon \eta_{x_j} + v_{\epsilon,x_j}\eta)dx = \int_\Omega f v_\epsilon \eta dx,$$

as $\epsilon \to 0$, $\int_\Omega f v_\epsilon \eta dx \to \int_\Omega f x \eta dx$. Note

$$\int_\Omega a^{ij}(\frac{x}{\epsilon})u^\epsilon_{x_i} v^l_{\epsilon x_j}\eta dx = \int_\Omega (a^{ij}(\frac{x}{\epsilon})v^l_{\epsilon x_j})\eta_{x_i} u^\epsilon dx \to \int_\Omega \eta_{x_i} u^0 \hat{a^{ik}} dx$$

as $\epsilon \to 0$. Therefore,

$$-\int_\Omega (\hat{a^{ij}}\eta_{x_i}u^0 + \xi_j \chi_l \eta)dx = \int_\Omega f \eta_{x_l} dx,$$

but $\int_\Omega f\eta\chi_l dx = -\int_\Omega (\xi^l \eta + \xi \cdot \chi_l \nabla \eta)dx$, hence

$$\int_\Omega \hat{a^{ij}} u^0_{x_i} \eta dx = \int_\Omega \xi^l \eta.$$

□

The original proof is by using Div-Curl lemma, $a^{ij}(\frac{x}{\epsilon})u^{\epsilon}_{x_j}u^{\epsilon}_{x_i} \to \hat{a^{ij}}u^0_{x_j}u^0_{x_i}$ as an energy convergence, the interesting part is $a^{ij}(\frac{x}{\epsilon})u^{\epsilon}_{x_j} \rightharpoonup \hat{a^{ij}}u^0_{x_j}$ and $u^{\epsilon}_{x_i} \rightharpoonup u^0_{x_i}$ weakly, but $a^{ij}(\frac{x}{\epsilon})u^{\epsilon}_{x_j}u^{\epsilon}_{x_i}$ still converges to $\hat{a^{ij}}u^0_{x_j}u^0_{x_i}$ (in general, the product of weak convergences does not imply the weak convergence).

2.5 Elliptic Operators with Oscillating Periodic Coefficients

Consider a family of elliptic operators $L_\epsilon = -\frac{\partial}{\partial x_i}\left(a^{\alpha\beta}_{ij}(\frac{x}{\epsilon})\frac{\partial}{\partial x_j}\right)$, $\epsilon > 0$. Let $A = A(y) = (a^{\alpha\beta}_{ij}(y))$, $1 \le i, j \le d$, $1 \le \alpha, \beta \le m$. Assume that A is real and uniformly elliptic, symmetric (i.e. $a^{\alpha\beta}_{ij} = a^{\beta\alpha}_{ji}$), A is Holder continuous and periodic w.r.t $\mathbb{Z}^d$ (i.e. $A(y+z) = A(y)$, $\forall z \in \mathbb{Z}^d$).

Consider the boundary value problem

$$\begin{cases} L_\epsilon(u_\epsilon) = div(f) \text{ in } \Omega \\ u_\epsilon \text{ subject to some kind of boundary condition.} \end{cases}$$

describes a stationary process in strongly inhomogeneous medium with periodic structure and $\epsilon > 0$ is the inhomogeneous scale, ϵ is very small with respect to the other length scales in the problem (if ϵ and the length scale are compatible, in phiycal situations, anything can happen and it is not easy to predict). As $\epsilon \to 0$, $u_\epsilon \to u_0$ strongly in $L^2(\Omega)$ and weakly in $H^1(\Omega)$, where u_0 is a solution of an elliptic system with constant coefficients,

$$\begin{cases} L_0(u_0) = div(f) \text{ in } \Omega \\ u_0 \text{ subject to some kind of boundary condition.} \end{cases}$$

where $L_0 = -div(\hat{A}\nabla)$ and $\hat{A}$ may be computed explicitly by using $A(y)$. There are three problems:

1. Uniform regularity estimates.
2. Uniform solvabilities of boundary value problems.
3. Convergence rates of $||u_\epsilon - u_0||$.

We give interior gradient estimate for ϵ-problem:

Theorem 2.14. *(Abellaneda and F. H. Lin, 1987)*
Suppose that $L_\epsilon(u_\epsilon) = 0$ *in* Ω *and* $B(x, 2r) \subset \Omega$. *Then*

$$|\nabla u_\epsilon(x)| \leq \frac{C}{r^{d+1}} \int_{B_r} |u_\epsilon(y)| dy,$$

where C *is independent of* ϵ.

Let $D(x,r) = B(x,r) \cap \Omega$ and $\Delta(x,r) = B(x,r) \cap \partial\Omega$, where $x \in \partial\Omega$ and $0 < r < r_0$, then we have Lipschitz estimates with Dirichlet condition:

Theorem 2.15. *(Avellaneda-Lin)*
Let Ω *be a* $C^{1,\alpha}$ *domain. Suppose that*

$$\begin{cases} L_\epsilon(u_\epsilon) = 0 & \text{in } D(x, 2r) \\ u_\epsilon = 0 & \text{on } \Delta(x, 2r). \end{cases}$$

Then

$$||\nabla u_\epsilon||_{L^\infty(D(x,r))} \leq \frac{C}{r} \left(\frac{1}{|D(x,2r)|} \int_{D(x,2r)} |u_\epsilon(y)|^2 dy \right)^{1/2},$$

where C *is independent of* ϵ.

Remark 2.16. There is no uniform Holder estimate for ∇u_ϵ (we can see it in the following example). There is no Lipschitz estimate on C^1 domains (even for constant coefficients) and the condition of symmetric is not needed.

Example 2.17. Look at $\frac{d}{dx}\left(\frac{u^\epsilon_x}{2 + \cos\frac{x}{\epsilon}}\right) = 0$. It's easy to find a solution $u^\epsilon(x) = 2x + \epsilon \sin\frac{x}{\epsilon}$ converges to $u^0(x) = 2x$ as $\epsilon \to 0$ with $u^0(x)$ satisfying $(u^0)''(x) = 0$. $\frac{d}{dx}u^\epsilon(x) = 2 + \cos\frac{x}{\epsilon} \rightharpoonup 2$ weakly by Riemann-Lebesgue lemma and it can never be a strong convergence. Note that the first derivative of u^ϵ is bounded (not C^1 since when ϵ is small, the derivative has highly oscillation in x-variable), so we can only expect the best possible regularity for the homogenization theory in PDE is the Lipschitz regularity (Similar to the KAM theory). In the following, we consider the Neumann boundary data problem:

Lipschitz Estimates for Solutions with Neumann boundary conditions

Theorem 2.18. *(Kenig-Lin-Shen, 2010)*
Let Ω be a $C^{1,\alpha}$ domain, suppose that

$$\begin{cases} L_\epsilon(u_\epsilon) = 0 & \text{in } D(x,2r) \\ \dfrac{\partial u_\epsilon}{\partial \nu_\epsilon} = 0 & \text{on } \Delta(x,2r). \end{cases}$$

Then

$$||\nabla u_\epsilon||_{L^\infty(D(x,r))} \le \frac{C}{r}\left(\frac{1}{|D(x,2r)|}\int_{D(x,2r)} |u_\epsilon(y)|^2 dy\right)^{1/2},$$

where C is independent of ϵ and note that $\left(\dfrac{\partial u_\epsilon}{\partial \nu_\epsilon}\right)^\alpha = n_i(x) a_{ij}^{\alpha\beta}(\dfrac{x}{\epsilon})\dfrac{\partial u_\epsilon^\beta}{\partial x_j}$.

Main Steps of the Proof:
First, we need a type of Maximum Principle as follows:

Lemma 2.19. *(Miranda Maximum Principle)*
Let u_ϵ satisfy

$$\begin{cases} L_\epsilon u_\epsilon = 0 & \text{in } \Omega \\ u_\epsilon = \phi & \text{on } \partial\Omega \end{cases},$$

then

$$||u_\epsilon||_{L^\infty(\Omega)} \le C||\phi||_{L^\infty(\partial\Omega)}.$$

Here $C(> 1)$ is not necessarily to be 1.

Construction and estimates of boundary correctors for Neumann conditions. Let $P_j^\beta = P_j^\beta(x) = x_j(0,\ldots,1,\ldots 0)$ with 1 in the β-th position and let $\phi_\epsilon = (\phi_{\epsilon,j}^{\alpha\beta})$, where for each $1 \le j \le d$, $1 \le \beta \le m$, $\phi_{\epsilon,j}^\beta = (\phi_{\epsilon,j}^{1\beta},\cdots,\phi_{\epsilon,j}^{m\beta})$ is a solution to the Neumann problem

$$\begin{cases} L_\epsilon(\phi_{\epsilon,j}^\beta) = 0 & \text{in } \Omega \\ \dfrac{\partial}{\partial \nu_\epsilon}(\phi_{\epsilon,j}^\beta) = \dfrac{\partial}{\partial \nu_0}(P_j^\beta) & \text{on } \partial\Omega. \end{cases}$$

Theorem 2.20. *(Kenig-Lin-Shen, 2010)*
Let Ω be a $C^{1,\alpha}$ domain. Then

$$||\nabla \phi_\epsilon||_{L^\infty(\Omega)} \le C.$$

Now, we need to estimate the boundary correctors:

Let $w = \phi - x - \epsilon\chi(\dfrac{x}{\epsilon})$, where the function χ is the corrector appeared in the previous sections. Then $L_\epsilon(w) = 0$ in Ω, write

$$w(x) = \int_{\partial\Omega} N_\epsilon(x,y)\frac{\partial w}{\partial \nu_\epsilon} d\sigma(y),$$

where $N_\epsilon(x,y)$ is a matrix of Neumann (kernel) functions for L_ϵ in Ω (just like the Poisson formula for Neumann boundary problem) and $\frac{\partial w}{\partial \nu_\epsilon}$ can be written as tangential derivatives of some g_{ij} with $||g_{ij}||_{L^\infty(\partial\Omega)} \leq C\epsilon$ (we borrow the ideas of the representation of the Poisson kernel to write the solution in the explicit form and estimate the solution by the representation formula). One can prove Holder estimates for solutions with Neumann conditions by a compactness argument which does not involve correctors. Hence, $N_\epsilon(x,y)$ is Holder continuous. This, together with the uniform Rellich estimates (Kenig-Shen, 2009), gives

$$\int_{\partial\Omega} |\nabla_y \{N_\epsilon(x,y) - N_\epsilon(z,y)\}| d\sigma(y) \leq C,$$

if $|x-z| \leq cr$ and $r = \text{dist}(x, \partial\Omega)$. Moreover, one can obtain

$$|\nabla \phi_\epsilon(x)| \leq C + \frac{C\epsilon}{\text{dist}(x,\partial\Omega)}.$$

Use a standard blow-up argument to finish the proof.

2.6 A compactness argument

Recall that we have introduced the notions of $D(x,r)$ and $\Delta(x,r)$, we give the following lemma:

Lemma 2.21. *There exist $\epsilon_0, \kappa, \theta$ and C with the following property. Suppose that $L_\epsilon(u_\epsilon) = 0$ in $D(0,1)$, $\frac{\partial u_\epsilon}{\partial \nu_\epsilon} = g$ on $\Delta(0,1)$ and $u_\epsilon(0) = g(0) = 0$. Assume that $\epsilon < \theta^{l-1}\epsilon_0$ for some $l \geq 1$. Then there exist constants $B_\epsilon^j \in \mathbb{R}^{dm}$ for $j = 0, 1, \cdots, l-1$ such that*

$$\left\langle n(0)\hat{A}, B_\epsilon^j \right\rangle = 0, \ |B_\epsilon^j| \leq CJ$$

and

$$||u_\epsilon - \sum_{j=0}^{l-1} \theta^{\kappa j} \left\langle n_\epsilon^j, B_\epsilon^j \right\rangle ||_{L^\infty(D(0,\theta^l))} \leq \theta^{l(1+\kappa)} J,$$

where $n_\epsilon^j(x) = \theta^j \phi_{\frac{\epsilon}{\theta^j}}(\theta^{-j}x, \Omega_{\psi_j})$, $\psi_j(x) = \theta^{-j}\psi(\theta^j x)$ and
$J = \max\{||g||_{C^{0,\eta}(\Delta(0,1))}, ||u_\epsilon||_{L^\infty(D(0,1))}\}$.

There are two steps proving this lemma: 1. $j = 0$ by contradiction; 2. $j \geq 1$ by induction. Use the above results, we can get the "real" Lipschitz Estimates for Neumann Problems:

Theorem 2.22. *Let Ω be a bounded $C^{1,\alpha}$ domain, $0 < \eta < \alpha < 1$ and $q > d$. Then for any $g \in C^{\eta}(\partial\Omega)$ and $F \in L^q(\Omega)$ with $\int_\Omega F + \int_{\partial\Omega} g = 0$, the weak solution to*

$$\begin{cases} L_\epsilon(u_\epsilon) = F & \text{in } \Omega \\ \dfrac{\partial u_\epsilon}{\partial \nu_\epsilon} = g & \text{on } \partial\Omega \end{cases}$$

satisfy the estimate

$$||\nabla u_\epsilon||_{L^\infty(\Omega)} \le C(||g||_{C^\eta(\partial\Omega)} + ||F||_{L^q(\Omega)}).$$

Remark 2.23. In the case of elliptic equations ($m = 1$) in Lipschitz domains, the L^p Dirichlet problem for $L_\epsilon(u_\epsilon) = 0$ for $2 - \delta < p < \infty$ was solved by B. Dalhberg (1990, unpublished). The L^p Neumann and regularity problems for $1 < p < 2 + \delta$ were solved by Kenig and Shen (2009).

In the case of elliptic systems ($m \ge 1$), the L^2 Dirichlet, Neumann, and regularity problems for $L_\epsilon(u_\epsilon) = 0$ in Lipschiz domains were solved by Kenig and Shen (2009), using the method of layer potentials.

If $m \ge 1$ and is $C^{1,\alpha}$, the L^p Dirichlet problem was solved by Avellaneda-Lin (1987), and the L^p Neumann and regularity problems as well as representations by layer potentials were solved by Kenig-Lin-Shen (2010), for $1 < p < \infty$.

2.7 Another approach

Recall two functional spaces: Morrey space and Campanato space.

Definition 2.24. We call Morrey's space, denoted by

$$M^{p,\lambda} = \left\{ u \in L^p(\Omega) | \sup_{x_0 \in \Omega, \rho < \text{diam}\Omega} \frac{1}{\rho^\lambda} \int_{B_\rho(x_0)} |u|^p dx < \infty \right\}$$

and Campanato's space is denoted by

$$C^{p,\lambda} = \left\{ u \in L^p(\Omega) | \sup_{x_0 \in \Omega, \rho < \text{diam}\Omega} \frac{1}{\rho^\lambda} \int_{B_\rho(x_0)} |u - u_{x_0,\rho}|^p dx < \infty \right\},$$

where $u_{x_0,\rho} = \fint_{B_\rho(x_0)} u(y)dy$, there are a lot of properties of these spaces, we don't give details here.

Lemma 2.25. *(Morrey's lemma)*
Suppose $u \in H^1(B_1^n)$ such that

$$\frac{1}{r^{n-2}} \int_{B_r(x_0)} |\nabla u|^2 \le M r^{2\alpha}$$

for some $\alpha \in (0,1)$, $\forall x_0 \in B_{1/2}(0)$ and $\forall r \in (0, \frac{1}{2})$, then $u \in C^\alpha(B_{1/2})$ and

$$||u||_{C^\alpha(B_{1/2})} \le c_0(n)M.$$

Lemma 2.26. *(Campanato's lemma)*
Let $u \in L^2(B_1^n)$ be such that

$$\frac{1}{r^n}\int_{B_r(x_0)} |u - u_{x_0,r}|^2 dx \le M r^{2\alpha}$$

$\forall x_0 \in B_{1/2}(0)$ and $\forall r \in (0, \frac{1}{2})$, then $u \in C^\alpha(B_{1/2})$ and

$$||u||_{C^\alpha(B_{1/2})} \le C(n)M.$$

Remark 2.27. Let $A(x)$ be such that $\lambda I \le A(x) \le \Lambda I$, period 1 and $||A||_{C^\beta} \le M$ for some constant M and $\beta \in (0,1)$. $L = \{L_\epsilon = div(A(\frac{x}{\epsilon})\nabla),\ A$ satisfies the previous assumptions$\}$. Consider $L_\epsilon u_\epsilon = 0$, $u_\epsilon \in H^1(B_1)$ and $L_\epsilon \in L$, then we have

$$||u_\epsilon||_{C^\alpha(B_{1/2})} \le C(n,\lambda,\Lambda,\alpha,\beta,M)||u_\epsilon||_{L^2(B_1)},\ \forall \alpha \in (0,1).$$

This estimate is distinct from the standard DeGiorgi's theorem (recall that in DeGiorgi theorem states the C^α-estimate for some $\alpha \in (0,1)$ and the right hand side constant C is universal). Moreover, the above three lemmas are all dimensional balance on the left hand side.

Lemma 2.28. *There are positive constants $\theta_0 = \theta_0(n,\lambda,\Lambda) \in (0,1)$ (is computable), $\epsilon_0 = \epsilon_0(n,\lambda,\Lambda,M,\alpha,\beta) \in (0,1)$ (not computable) such that $\forall L_\epsilon = L_{A_\epsilon} \in L$ and $L_\epsilon u_\epsilon = 0$ in B_1 with $u_\epsilon \in H^1(B_1)$ and $\int_{B_1} u_\epsilon^2 = 1$. Then*

$$\frac{1}{\theta_0^n}\int_{B_{\theta_0}(0)} |u_\epsilon - u_{\epsilon,\theta_0}|^2 dx \le \theta_0^{2\alpha},$$

here $\alpha \in (0,1)$ is given $\forall \epsilon \le \epsilon_0$.

Proof. By contradiction: Suppose to the contrary that there is a sequence $A^i_{\epsilon_i} = A^i(\frac{x}{\epsilon_i})$, $\epsilon_i \to 0$ and A^i's satisfy the conditions mentioned before, then $L_{\epsilon_i} u^{\epsilon_i} = 0$ in B_1 and $\int_{B_1} |u^{\epsilon_i}|^2 dx = 1$ such that

$$\frac{1}{\theta_0^n}\int_{B_{\theta_0}(0)} |u^{\epsilon_i} - u^{\epsilon_i}_{\theta_0}|^2 dx > \theta_0^{2\alpha}$$

for $i = 1, 2, \cdots$. We may assume $A^i(y) \to A(y)$ uniformly for $y \in \mathrm{T}^n$ (n-dimensional tori) and $A(y)$ satisfies previous conditions again. Then

$$L_{A_{\epsilon_i}} = div(A(\frac{x}{\epsilon_i})\nabla) \to L_0 = div(A_0\nabla)$$

homogenized as $\epsilon \to 0$ since $A^i(y) \to A(y)$ in L^∞ (uniformly). WLOG, say $u^{\epsilon_i} \to u_0$ weakly in $H^1_{loc}(B_1)$ and $u^{\epsilon_i} \to u_0$ strongly in $L^2(B_1)$ (after extracting a subsequence of $\{u^{\epsilon_i}\}$) with $L_0 u_0 = 0$ in B_1, then $\int_{B_1} |u_0|^2 \leq 1$ by $\int_{B_1} |u^{\epsilon_i}|^2 \leq 1$. Therefore,

$$\theta^2 \geq \frac{1}{\theta_0^n} \int_{B_{\theta_0}(0)} |u_0 - u_{0,\theta_0}|^2 dx > \theta_0^{2\alpha}$$

is impossible by taking θ sufficiently small (Note that $L_0 u_0 = 0$, then
$|\nabla u_0|_{L^\infty(B_{1/2})} \leq C(n, \lambda, \Lambda)$ with $||u_0||^2_{L^2(B_1)} = 1$). □

Lemma 2.29. *(Iteration)*
Let $L_\epsilon \in L$, $L_\epsilon u_\epsilon = 0$ in B_1 with $||u_\epsilon||_{L^2(B_1)} \leq 1$, for $k = 1, 2, \cdots K$, for some K. If $\frac{\epsilon}{\theta_0^k} \leq \epsilon_0$, then

$$\int_{B_{\theta^k}(0)} |u^\epsilon - u^\epsilon_{\theta^k}| dx \leq (\theta_0^k)^{2\alpha}.$$

Proof. For $k = 1$, let $v_\epsilon = \frac{u_\epsilon(\theta_0 x) - u_{\epsilon,\theta_0}}{\theta_0^\alpha}$, then $\int_{B_1} |v_\epsilon|^2 \leq 1$ and $L_\epsilon v_\epsilon = 0$ in B_1. By lemma 1, $\int_{B_{\theta_0}} |v_\epsilon - v_{\epsilon,\theta_0}|^2 dx \leq \theta_0^{2\alpha}$, then get the conclusion in lemma 2 for $k = 2$, then continue this process until some K such that $\frac{\epsilon}{\theta_0^K} \leq \epsilon_0$ but $\frac{\epsilon}{\theta_0^{K+1}} > \epsilon_0$ ($\theta_0^k \approx \frac{\epsilon}{\epsilon_0}$). □

Lemma 2.30. *If u^ϵ satisfies before conditions, then*

$$\fint_{B_r} |u^\epsilon - u^\epsilon_r|^2 dx \leq N_0 r^{2\alpha} \ \forall r \in (0, \frac{1}{2}].$$

Recall that if $Lu = 0$ in B_1 and $u \in L^2(B_1)$, u satisfies $\frac{1}{r^{n+2}} \int_{B_r(x_0)} |u(x) - l^r(x)|^2 dx \leq M r^{2\delta}$ for some $\delta \in (0, 1)$, where $l^r(x) = u_{x_0,\theta_0} + (x - x_0)(\nabla u)_{x_0,r}$, then $u \in C^{1,\delta}$. Recall a corrector $\chi(y)$ introduced before: $div(A(y)\nabla\chi(y)) = -divA(y)$ on T^n, then $v_\epsilon = x + \epsilon\chi(\frac{x}{\epsilon})$ is a solution of $L_\epsilon v_\epsilon = 0$.

Lemma 2.31. *(Lemma 2.28')*
All the assumptions hold as before. Then

$$\frac{1}{\theta_0^{n+2}}\int_{B_{\theta_0}}|u^\epsilon-(u^\epsilon_{\theta_0}+v_\epsilon(\nabla u^\epsilon)_{\theta_0})|^2\leq\theta_0^{2\delta},\ \textit{for } 0<\delta<\alpha.$$

It can be regarded as a Taylor expansion starting from the quadratic terms, for smoothness case, the power $n+2$ can be replaced by $n+4$. The way to prove this lemma is similar as lemma 2.28, assuming the contrary and let θ_0 small to get a contradiction.

Lemma 2.32. *(Lemma 2.29')*
If $\frac{\epsilon}{\theta_0^k}\leq\epsilon_0$, *then*

$$\frac{1}{\theta_0^{k(n+2)}}\int_{B_{\theta_0^k}}|u^\epsilon(x)-l_\epsilon^k(x)|^2dx\leq\theta_0^{2\delta k},$$

where $l_\epsilon^k(x)=x+\epsilon\chi(\frac{x}{\epsilon})$.

Lemma 2.33. *(Lemma 2.30')*
Elliptic regularity $C^{1,\delta}$ *for* $\delta<\alpha$ *and*

$$\frac{1}{r^{n+2}}\int_{B_r(0)}|u^\epsilon(x)-[a_\epsilon^r+\overrightarrow{b_\epsilon^r}(x+\epsilon\chi(\frac{x}{\epsilon}))]|^2dx\leq N_0r^{2\delta},\ \textit{for } 0<r\leq\frac{1}{2}.$$

The homogenization theory did not tell you the speed of convergence, but it is rather important; the corrector function χ we considered before did not involve the boundary data, so we need to get a "new" corrector in order to deal with the boundary estimate. We consider these problems in the following section.

2.8 Convergence rate in periodic homogenization

For a single equation, $L_\epsilon u^\epsilon=f$ in Ω with zero boundary condition, $f\in L^p(\Omega)$,$p>n$. Let

$$w_\epsilon(x)=u^\epsilon(x)-(u^0(x)+\epsilon\chi(\frac{x}{\epsilon})\nabla u^0(x)),$$

where u^0 is the solution of $L_0u^0=0$ and L_0 is the limit of L_ϵ as $\epsilon\to 0$, w.l.o.g., call $\Delta u^0=L_0u^0$ and $u^0=0$ on $\partial\Omega$. By standard elliptic regularity,

$u^0 \in W_0^{2,p}(\Omega)$ and $\nabla u^0 \in W^{1,p}(\Omega) \hookrightarrow C^\alpha(\overline{\Omega}) \subset L^\infty(\Omega)$ by Sobolev embedding theorem (remember $p > n$).

$$\begin{cases} L_\epsilon w_\epsilon = div(F_\epsilon) = O(\epsilon) & \text{in } \partial(L^p) \\ w_\epsilon|_{\partial\Omega} = -\epsilon\chi(\frac{x}{\epsilon})\nabla u_0 = O(\epsilon) & \text{in } L^\infty, \end{cases}$$

then $||w_\epsilon||_{L^\infty(\Omega)} \le O(\epsilon) \Longrightarrow ||u_\epsilon - u_0||_{L^\infty(\Omega)} \le C_0(\epsilon)$ by the setting of w_ϵ.

Theorem 2.34. *(Kenig-Lin-S., 2012)*

Suppose that A is elliptic, periodic, and if $m \ge 2$, Hölder continuous. Let Ω be $C^{1,1}$. Let $F \in L^2(\Omega)$ and $u_\epsilon \in H_0^1(\Omega)$ be the unique weak solution to $L_\epsilon(u_\epsilon) = F$ in Ω. Then

$$||u_\epsilon - u_0||_{L^2(\Omega)} \le C\epsilon||F||_{L^2(\Omega)}, \tag{2.7}$$

$$||u_\epsilon - u_0 - \{\phi^\beta_{\epsilon,j} - P^\beta_j\}\frac{\partial u_0^\beta}{\partial x_j}||_{H_0^1(\Omega)} \le C\epsilon||F||_{L^2(\Omega)}. \tag{2.8}$$

where C depends only on A and . $P^\beta_j(x) = x_j(\delta^{\alpha\beta})$, and $\phi^\beta_{\epsilon,j}$ is the Dirichlet corrector, defined by $L_\epsilon(\phi^\beta_{\epsilon,j}) = 0$ in Ω and $\phi^\beta_{\epsilon,j} = P^\beta_j$ on $\partial\Omega$.

The key formula: Let $w_\epsilon = u_\epsilon - u_0 - \{\phi^\beta_{\epsilon,j} - P^\beta_j\}\dfrac{\partial u_0^\beta}{\partial x_j}$, suppose that $L_\epsilon(u_\epsilon) = L_0(u_0)$ and $L_\epsilon(\phi^\beta_{\epsilon,j}) = 0$. Then

$$\begin{aligned} L_\epsilon(w_\epsilon) &= \epsilon\frac{\partial}{\partial x_i}\left\{F^{\alpha\gamma}_{jik}(\frac{x}{\epsilon})\frac{\partial^2 u_0^\gamma}{\partial x_j \partial x_k}\right\} \\ &+ \frac{\partial}{\partial x_i}\left\{a^{\alpha\beta}_{ij}(\frac{x}{\epsilon})[\phi^{\beta\gamma}_{\epsilon,k}(x) - x_k\delta^{\beta\gamma}]\frac{\partial^2 u_0^\gamma}{\partial x_j \partial x_k}\right\} \\ &+ a^{\alpha\beta}_{ij}(\frac{x}{\epsilon})\frac{\partial}{\partial x_j}[\phi^{\beta\gamma}_{\epsilon,k}(x) - x_k\delta^{\beta\gamma} - \epsilon\chi^{\beta\gamma}_k(\frac{x}{\epsilon})]\frac{\partial^2 u_0^\gamma}{\partial x_j \partial x_k}. \end{aligned}$$

Remark 2.35. Since $||\phi^\beta_{\epsilon,j} - P^\beta_j||_{L^\infty(\Omega)} \le C\epsilon$, the H^1 estimate (2.9) implies the L^2 estimate (2.7). If $m = 1$, the L^2-estimate (2.7) is known (Moskow-Vogelius, 1996; G. Grisco, 2006). Also, it is known that if $\Omega_1 \Subset \Omega$,

$$||u_\epsilon - u_0 - \epsilon\chi^\beta_j(\frac{x}{\epsilon})\frac{\partial u_0^\beta}{\partial x_j}||_{H^1(\Omega_1)} \le C\epsilon||F||_{L^2(\Omega)},$$

$$||u_\epsilon - u_0 - \epsilon\chi^\beta_j(\frac{x}{\epsilon})\frac{\partial u_0^\beta}{\partial x_j}||_{H^1(\Omega)} \le C\epsilon^{1/2}||F||_{L^2(\Omega)},$$

where (χ_j^β) are correctors for L_ϵ in $\mathbb{R}^d$.

Suppose A is symmetric. Let $\{\lambda_{\epsilon,k}\}$ denote the sequence of Dirichlet eigenvalues in an increasing order for L_ϵ in Ω. One may use the H^1 estimate (2.9) to show that

$$|\lambda_{\epsilon,k} - \lambda_{0,k}| \leq C\epsilon(\lambda_{0,k})^{3/2},$$

where C is independent of ϵ and k.

Theorem 2.36. *(Convergence rate in $H^{1/2}$, Kenig-Lin-S., 2011)*

Suppose that $A \in C^\alpha(\Omega)$ is elliptic, periodic and symmetric and let Ω be $C^{1,1}$. Let $u_\epsilon \in H_0^1(\Omega)$ be the unique solution to $L_\epsilon(u_\epsilon) = F$ in Ω and

$$w_\epsilon = u_\epsilon - u_0 - \epsilon\chi(\frac{x}{\epsilon})\frac{\partial u_0^\beta}{\partial x_j}.$$

Then

$$||w_\epsilon||_{H^{1/2}(\Omega)} + \left(\int_\Omega |\nabla w_\epsilon(x)|^2 dist(x, \partial\Omega)dx\right)^{1/2} \leq C\epsilon||F||_{L^2(\Omega)},$$

where C depends only on A and Ω.

What about the Lipschitz domain?

Theorem 2.37. *(Kenig-Lin-S., 2011)*

Let Ω be a bounded Lipschitz domain. Suppose that A is elliptic, periodic, symmetric, and Hölder continuous. Suppose that

$$\begin{cases} L_\epsilon(u_\epsilon) = F & in\ \Omega \\ u_\epsilon = f & on\ \partial\Omega \end{cases}$$

where $F \in L^2(\Omega)$ and $f \in H^1(\partial\Omega)$. Then

$$||u_\epsilon - u_0||_{L^2(\Omega)} \leq C\epsilon(|\ln\epsilon| + 1)^{\frac{1}{2}+\sigma}\left\{||F||_{L^2(\Omega)} + ||f||_{H^1(\partial\Omega)}\right\}$$

for any $\sigma > 0$, where C depends only on A, Ω and σ.

Proof. Replace u_0 by v_ϵ, where v_ϵ solves a Dirichlet problem for L_0 in Ω_ϵ, a slightly larger domain such that dist$(\partial\Omega, \partial\Omega_\epsilon)\approx \epsilon$. The interior estimate for ϵ-periodic domain will get the term $\ln\epsilon$ and the boundary estimate is similar to the before estimate. □

Remark 2.38. Nothing comes for free. The spirit of the homogenization problem is the limiting of control is not the control of limiting.

Theorem 2.39. *(Convergence rate in L^p, Kenig-Lin-S., 2012)*

Suppose that $A \in C^\alpha$ is elliptic, periodic, $m \geq 2$ and let $\Omega \in C^{1,1}$. Suppose that

$$\begin{cases} L_\epsilon(u_\epsilon) = F & \text{in } \Omega \\ u_\epsilon = 0 & \text{on } \partial\Omega \end{cases}.$$

Then

$$||u_\epsilon - u_0||_{L^q(\Omega)} \leq C\epsilon ||F||_{L^p(\Omega)}$$

holds if $1 < p < d$ and $\dfrac{1}{q} = \dfrac{1}{p} - \dfrac{1}{d}$, or $p > d$ and $q = \infty$. Moreover,

$$||u_\epsilon - u_0||_{L^\infty(\Omega)} \leq C\epsilon[\ln(\epsilon^{-1} + 2)]^{1-\frac{1}{d}} ||F||_{L^d(\Omega)}.$$

In the paper Avellaneda-Lin, 1987, they proved the following fact:
Let $G_\epsilon(x,y)$ denote the matrix of Green functions for L_ϵ in Ω. Then

$$|G_\epsilon(x,y)| \leq C|x-y|^{2-d},$$

$$|\nabla_x G_\epsilon(x,y)| + |\nabla_y G_\epsilon(x,y)| \leq C|x-y|^{1-d},$$
$$|\nabla_x \nabla_y G_\epsilon(x,y)| \leq C|x-y|^{-d}.$$

The theorem follows from the asymptotic expansion of Green functions:

$$|G_\epsilon(x,y) - G_0(x,y)| \leq C\epsilon|x-y|^{1-d} \tag{2.9}$$

for any $x, y \in \Omega$ and $x \neq y$.

The proof of (2.9) relies on the following:

Lemma 2.40. *(Boundary L^∞ estimate)*

Suppose that

$$\begin{cases} L_\epsilon(u_\epsilon) = L_0(u_0) & \text{in } D_{2r} \\ u_\epsilon = u_0 & \text{on } \Delta_{2r} \end{cases},$$

where D_r and Δ_r are introduced before. Then for $p > d$,

$$\begin{aligned} ||u_\epsilon - u_0||_{L^\infty(D_r)} &\leq C \fint_{D_{2r}} |u_\epsilon - u_0| + C\epsilon ||\nabla u_0||_{L^\infty(D_{2r})} \\ &+ C_p \epsilon r^{1-\frac{d}{p}} ||\nabla^2 u_0||_{L^p(D_{2r})}. \end{aligned}$$

Proof. Use the representation by Green functions. Since $G_\epsilon(x,y)$ and $G_0(x,y)$ are Green functions of L_ϵ and L_0, respectively, we have

$$-L_\epsilon G_\epsilon(x,y) = \delta_y \text{ and } -L_0 G_0(x,y) = \delta_y.$$

Note that u_ϵ and u_0 satisfy

$$\begin{cases} L_\epsilon u_\epsilon = f \in L^p(\mathbb{R}^n) \\ u_\epsilon(\infty) = 0 \end{cases} \quad \text{and} \quad \begin{cases} L_0 u_0 = f \\ u_0(\infty) = 0 \end{cases},$$

then we can represent u_ϵ and u_0 as $u_\epsilon(x) = \int G_\epsilon(x,y)f(y)dy$ and $u_0(x) = \int G_0(x,y)f(y)dy$ and $\|u_\epsilon - u_0\|_{L^\infty} \leq C\epsilon||f||_{L^p}$ for $n < p < \infty$. Moreover, $\|G_\epsilon(x,y) - G_0(x,y)\|_{L^q_y} \leq C\epsilon$ for $1 \leq q < \dfrac{n}{n-1}$. For x fixed, let $A = \left\{\frac{1}{2} \leq |x-y| \leq 2\right\}$, for y in this region, we have $L_\epsilon G_\epsilon(x,y) = 0$ and $L_0 G_0(x,y) = 0$ and denote $v_\epsilon(y)$ and $v_0(y)$ to be solutions of these equations, respectively. Then we have $\|v_\epsilon(y) - v_0(y)\|_{L^q(A)} \leq C\epsilon$ and by DeGiorgi's theorem, we get

$$\|G_0(x,y) - G_\epsilon(x,y)\| \leq C\epsilon.$$

After rescaling back, it is easy to get

$$\|G_0(x,y) - G_\epsilon(x,y)\| \leq \frac{C\epsilon}{|x-y|^{n-1}}.$$

Finally, scan all the annulus region A to get the whole domain estimate. □

Theorem 2.41. *(Convergence rate in $W^{1,p}$, Kenig-Lin-S., 2012)*
Suppose A satisfies the previous conditions and $\Omega \in C^{2,\alpha}$. Suppose that

$$\begin{cases} L_\epsilon(u_\epsilon) = L_0(u_0) & \text{in } \Omega \\ u_\epsilon = u_0 & \text{on } \partial\Omega. \end{cases}$$

Then

$$||u_\epsilon - u_0 - \{\phi^\beta_{\epsilon,j} - P^\beta_j\}\frac{\partial u^\beta_0}{\partial x_j}||_{W^{1,p}(\Omega)} \leq C\epsilon[\ln(\epsilon^{-1}+2)]^{4|\frac{1}{2}-\frac{1}{p}|}||F||_{L^p(\Omega)},$$

where $1 < p < \infty$ and C depends on p, A and Ω.

The proof of the theorem uses the asymptotic expansion for $\nabla_x G_\epsilon(x,y)$:

$$|\frac{\partial}{\partial_{x_i}} G_\epsilon(x,y) - \frac{\partial}{\partial x_i}\phi_{\epsilon,j}(x) \cdot \frac{\partial}{\partial x_j} G_0(x,y)| \leq C\epsilon\frac{\ln[\epsilon^{-1}|x-y|+2]}{|x-y|^d} \tag{2.10}$$

for any $x, y \in \Omega$ and $x \neq y$. We also obtain asymptotic expansions for $\nabla_y G_\epsilon(x,y)$ and $\nabla_x\nabla_y G_\epsilon(x,y)$.

The proof of (2.10) relies on the following:

Lemma 2.42. *Suppose that*

$$\begin{cases} L_\epsilon(u_\epsilon) = L_0(u_0) & \text{in } D_{2r} \\ u_\epsilon = u_0 & \text{on } \Delta_{2r}. \end{cases}$$

Then for any $\rho \in (0,1)$,

$$\begin{aligned} \|\frac{\partial u_\epsilon}{\partial x_i} - \frac{\partial}{\partial x_i}\phi_{\epsilon,j} \cdot \frac{\partial u_0}{\partial x_j}\|_{L^\infty(D_r)} &\leq \frac{C}{r} \fint_{D_{2r}} |u_\epsilon - u_0| + C\epsilon r^{01} \|\nabla u_0\|_{L^\infty(D_{2r})} \\ &+ C\epsilon \ln[\epsilon^{-1} r + 2] \|\nabla^2 u_0\|_{L^\infty(D_{2r})} \\ &+ C\epsilon r^\rho \|\nabla^2 u_0\|_{C^{0,\rho}(D_{2r})}. \end{aligned}$$

What about Neumann Boundary Conditions? Convergence rates are also obtained for solutions with Neumann boundary conditions. Let $N_\epsilon(x,y)$ denote the matrix of Neumann functions for L_ϵ in Ω. Then

$$|N_\epsilon(x,y)| \leq C|x-y|^{2-d},$$

$$|\nabla_x N_\epsilon(x,y)| + |\nabla_y N_\epsilon(x,y)| \leq C|x-y|^{1-d},$$
$$|\nabla_x \nabla_y N_\epsilon(x,y)| \leq C|x-y|^{-d}.$$

Theorem 2.43. *(Kenig-Lin-S., 2012)*
Suppose A satisfies the previous conditions and $\Omega \in C^{1,1}$, *then*

$$|N_\epsilon(x,y) - N_0(x,y)| \leq C\epsilon \frac{\ln[\epsilon^{-1}|x-y| + 2]}{|x-y|^{d-1}}$$

for any $x, y \in \Omega$ *and* $x \neq y$, *where C depends on* A, Ω.

Theorem 2.44. *(Kenig-Lin-S., 2012)*
Suppose A satisfies the previous conditions and $\Omega \in C^{2,\alpha}$. *Then for any* $\rho \in (0,1)$,

$$|\frac{\partial}{\partial x_i} N_\epsilon(x,y) - \frac{\partial}{\partial x_i}\psi_{\epsilon,j}(x) \cdot \frac{\partial}{\partial x_j} N_0(x,y)| \leq C\epsilon^{1-\rho} \frac{\ln[\epsilon^{-1} r_0 + 2]}{|x-y|^{d-\rho}},$$

where $r_0 = diam\Omega$ *and C depends on* ρ, A *and* Ω.

Note that $\psi_{\epsilon,j}$ is the Neumann corrector for L_ϵ in Ω, defined by

$$\begin{cases} L_\epsilon(\psi^\beta_{\epsilon,j}) = 0 & \text{in } \Omega \\ \dfrac{\partial}{\partial \nu_\epsilon}\psi^\beta_{\epsilon,j} = \dfrac{\partial}{\partial \nu_0} P^\beta_j & \text{on } \partial\Omega. \end{cases}$$

Lemma 2.45. *(Boundary L^∞ estimate)*
Suppose that $L_\epsilon(u_\epsilon) = L_0(u_0)$ in D_{2r} and $u_\epsilon = u_0$ on Δ_{2r}. Then for $p > d$,

$$\begin{aligned} \|u_\epsilon - u_0\|_{L^\infty(D_r)} &\leq C \fint_{D_{2r}} |u_\epsilon - u_0| \\ &+ C\epsilon \ln[\epsilon^{-1} r + 2] \|\nabla u_0\|_{L^\infty(D_{2r})} \\ &+ C\epsilon r^{1-\frac{d}{p}} \|\nabla^2 u_0\|_{L^\infty(D_{2r})}. \end{aligned}$$

Lemma 2.46. *(Boundary Lipschitz estimate)*
Suppose that $L_\epsilon(u_\epsilon) = L_0(u_0)$ in D_{2r} and $u_\epsilon = u_0$ on Δ_{2r}. Then, if $0 < \epsilon < \frac{r}{2}$ and $\rho > 0$,

$$\begin{aligned} \|\frac{\partial u_\epsilon}{\partial x_i} - \frac{\partial}{\partial x_i}\phi_{\epsilon,j} \cdot \frac{\partial u_0}{\partial x_j}\|_{L^\infty(D_r)} &\leq \frac{C}{r} \fint_{D_r} |u_\epsilon - u_0| \\ &+ C\epsilon r^{-1} \ln[\epsilon^{-1} r_0 + 2] \|\nabla u_0\|_{L^\infty(D_{2r})} \\ &+ C\epsilon^{1-\rho} r^{-\rho} \ln[\epsilon^{-1} r_0 + 2] \|\nabla^2 u_0\|_{L^\infty(D_{2r})} \\ &+ C\epsilon r^{\rho} \ln[\epsilon^{-1} r_0 + 2] \|\nabla^2 u_0\|_{L^\infty(D_{2r})}, \end{aligned}$$

where $r_0 = diam\Omega$.

Remark 2.47. The asymptotic expansion of $\nabla_x G_\epsilon(x, y)$ leads to

$$P_\epsilon(x, y) = P_0(x, y)\omega_\epsilon(y) + R_\epsilon(x, y),$$

where $P_\epsilon(x, y)$ is the Poisson kernel for L_ϵ on Ω, and

$$|R_\epsilon(x, y)| \leq C\epsilon \frac{\ln[\epsilon^{-1}|x - y| + 2]}{|x - y|^d}$$

for any $x \in \Omega$, $y \in \partial\Omega$. This improves a result of Avellaneda and F. H. Lin. We also obtain asymptotic results for the Dirichlet-to-Neumann Map as well as for the operator $\nabla(L_\epsilon)^{-1}\nabla$.

Theorem 2.48. *Suppose that A is elliptic, periodic, and Holder continuous. Let Ω be $C^{2,\alpha}$. Suppose that*

$$\begin{cases} L_\epsilon(u_\epsilon) = 0 & in\ \Omega \\ u_\epsilon = f_\epsilon & on\ \partial\Omega \end{cases} \quad and \quad \begin{cases} L_0(v_\epsilon) = 0 & in\ \Omega \\ v_\epsilon = \omega_\epsilon f_\epsilon & on\ \partial\Omega. \end{cases}$$

Then for $1 < p < \infty$,

$$\|u_\epsilon - v_\epsilon\|_{L^p(\Omega)} \leq C \left\{\epsilon[\ln(\epsilon^{-1} r_0 + 2)]^2\right\}^{1/p} \|f_\epsilon\|_{L^p(\partial\Omega)}$$

where $r_0 = diam\Omega$.

3 Eigenvalue problem and nondivergence operator

3.1 Asymptotic of eigenvalues and eigenfunctions in periodic homogenization

Consider the eigenvalue problem

$$\begin{cases} L_\epsilon \phi_{\epsilon,k} + \lambda_{\epsilon,k}\phi_{\epsilon,k} = 0 & \text{in } \Omega \\ \phi_{\epsilon,k} = 0 & \text{on } \partial\Omega \end{cases} \quad \text{for } k = 1, 2, \cdots,$$

where $\int_\Omega \phi_{\epsilon,k}^2 dx = 1$ and $L_\epsilon = \dfrac{\partial}{\partial x_i}(a^{ij}(\dfrac{x}{\epsilon})\dfrac{\partial}{\partial x_j})$, it is known that

$$0 < \lambda_{\epsilon,1} < \lambda_{\epsilon,2} \leq \lambda_{\epsilon,3} \leq \cdots \to \infty.$$

If we consider $-L_\epsilon u_\epsilon = f$ in Ω with $u_\epsilon = 0$ on $\partial\Omega$ and $f \in L^2(\Omega)$, then we can let $T_\epsilon = (-L_\epsilon)^{-1}$ such that $u_\epsilon = T_\epsilon(f)$. Note that $T_\epsilon : L^2(\Omega) \to H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ and $T_\epsilon \geq 0$ is a self-adjoint linear operator. Let $\mu_{\epsilon,k} = \dfrac{1}{\lambda_{\epsilon,k}}$, then $\mu_{\epsilon,1} > \mu_{\epsilon,2} \geq \cdots \to 0$. Say L_ϵ converges homogenized to L_0, a constant elliptic operator and $\{\phi_{0,k}\}$ are eigenfunctions with respect to $\lambda_{0,k}$ are eigenvalues of L_0. For all k fixed, $\lambda_{\epsilon,k} \to \lambda_{0,k}$ as $\epsilon \to 0$.

Question: Can we find an upper bound of $|\lambda_{\epsilon,k} - \lambda_{0,k}|$ (There are many mathematicians considered this problem such as Zhikon, Kozlov, Oleinik, Mostow, Vogelius, Santosa, Castro, Zuazua, Bardos, Rauch, Cristo,...)? They found $|\lambda_{\epsilon,k} - \lambda_{0,k}| \leq C_0\epsilon^{1/3}\lambda_{0,k}^5$!

Theorem 3.1. $|\lambda_{\epsilon,k} - \lambda_{0,k}| \leq C_0\epsilon\lambda_{0,k}^{3/2}$.

Remark 3.2. Weyl's asymptotic formula tells us $\lambda_{0,k} \approx C(n, |\Omega|)k^{2/n}$ and by using minimax principle, we can get $\lambda_{\epsilon,k} \approx \lambda_{0,k}$ (the constant is universal). By minimax principle, we know

$$\lambda_{\epsilon,k} = \max_{X_{k-1}} \min_{v \in H_0^1(\Omega)\cap X_{k-1}, ||v||_{L^2}=1} \langle -L_\epsilon v, v\rangle .$$

Moreover, if $\epsilon\sqrt{\lambda_{0,k}} \geq 1$, the theorem holds trivial; it is interesting only when $\epsilon\sqrt{\lambda_{0,k}} < 1$. In general, the estimate is optimal. Prove this theorem, we need a lemma:

Lemma 3.3. *(Minimax)* $|\mu_{\epsilon,k} - \mu_{0,k}| \leq \max\{a_\epsilon, b_\epsilon\} \leq c_0\sqrt{\mu_{0,k}}\epsilon$, *where*

$$a_\epsilon = \min_{||f||_{L^2}=1, f\perp V_{0,k-1}} |\langle (T_\epsilon - T_0)f, f\rangle|,$$

$$b_\epsilon = \min_{||f||_{L^2}=1, f\perp V_{\epsilon,k-1}} |\langle (T_\epsilon - T_0)f, f\rangle|.$$

Since you don't know what space you will take, both need to be considered.

Proof. For a_ϵ: Let $u_0 = T_0 f$ and $u_\epsilon = T_\epsilon f$, if $f \perp V_{0,k-1}$ and $\|f\|_{L^2} = 1$, then $\langle u_0, f\rangle = \langle T_0 f, f\rangle \le \mu_{0,k}$ and $\|\nabla u_0\|_{L^2}^2 \le c\int_\Omega \hat{a^{ij}} u_{0x_i} u_{0x_j} = c\langle f, u_0\rangle \le c\mu_{0,k}$, then we have

$$\|f\|_{H^{-1}} \le c\|\nabla u_0\|_{L^2} \le c\mu_{0,k}^{1/2}.$$

Moreover, we have

$$\begin{aligned} |\langle u_\epsilon - u_0, f\rangle| &\le |\langle u_\epsilon - u_0 - (\phi_\epsilon(x) - x)\nabla u_0, f\rangle| + |\langle (\phi_\epsilon(x) - x)\nabla u_0, f\rangle| \\ &\le C\epsilon\mu_{0,k}^{1/2} + \|\langle (\phi_\epsilon(x) - x)\nabla u_0, f\rangle\|_{L^2} \\ &\le C\epsilon\mu_{0,k}^{1/2}, \end{aligned}$$

where $\phi_\epsilon(x)$ is the boundary corrector to the Dirichlet boundary problem. □

Example 3.4. Look at 1-dimensional case:

$$\begin{cases} \frac{d}{dx}(a(\frac{x}{\epsilon})u_x^{\epsilon,k}) + \lambda_{\epsilon,k}u^{\epsilon,k} = 0 & \text{in } [0,1] \\ u^{\epsilon,k}(0) = u^{\epsilon,k}(1) = 0 \end{cases}.$$

By Strum-Liouville's theorem, we know there exist k-nodal domains, say the first nodal domain is $[0, l_\epsilon^k]$ with $l_\epsilon^k = \frac{1}{k} + O(\epsilon)$ and $u^{\epsilon,k}|_{[0,l_\epsilon^k]}$ is the first eigenfunction. $\lambda_{\epsilon,k} \approx k^2$ by $\epsilon\sqrt{\lambda_{0,k}} \approx k\epsilon$. Call $\widetilde{\lambda}_{\epsilon,1}$ the first eigenvalue of $L_{k\epsilon}$ on $[0,1]$, where $L_\epsilon u_x^{\epsilon,k} = \frac{d}{dx}(a(\frac{x}{\epsilon})u_x^{\epsilon,k})$ and $\widetilde{\lambda}_{\epsilon,1}$ converges to $\widetilde{\lambda}_1$ as $\epsilon \to 0$ with $\widetilde{\lambda}_1$ is the first eigenvalue of the homogenized operator $L_0 = \lim_{\epsilon\to 0} L_\epsilon$ (the limit is under the homogenization sense as we mentioned in section 2). And for the k-th eigenvalue, we have $|\widetilde{\lambda}_{\epsilon,k} - \widetilde{\lambda}_k| \ge C_0(k\epsilon)$. Therefore,

$$|\lambda_{\epsilon,k} - \lambda_{0,k}| \ge C_0\epsilon k^3 = c_0\epsilon\lambda_{0,k}^{3/2}.$$

Consider the homogenization problem

$$\begin{cases} L_\epsilon(u_\epsilon) = f & \text{in } \Omega \\ u_\epsilon = 0 & \text{on } \partial\Omega \end{cases} \text{ converges to } \begin{cases} L_0 u_0 = f & \text{in } \Omega \\ u_0 = 0 & \text{on } \partial\Omega \end{cases}$$

then as before, $\|u_\epsilon - u_0\|_{L^2(\Omega)} \le C_0\epsilon||f||_{L^2(\Omega)}$, i.e., $\|(T_\epsilon - T_0)f\|_{L^2(\Omega)} = \|T_\epsilon(f) - T_0(f)\|_{L^2(\Omega)} \le C_0\epsilon||f||_{L^2(\Omega)}$, then $\|T_\epsilon - T_0\|_{L^2\to L^2} \le C_0\epsilon$. By the functional analysis, we have $|\mu_{\epsilon,k}-\mu_{0,k}| \leq C_0\epsilon$ (need to check it). i.e., $|\frac{1}{\lambda_{\epsilon,k}}-\frac{1}{\lambda_{0,k}}| \leq C_0\epsilon$, then $|\lambda_{\epsilon,k}-\lambda_{0,k}| \leq C_0\epsilon\lambda_{0,k}^2$ (by Weyl's asymptotic again).

The key estimate (H^1-estimate) is similar as before: (recall that ϕ_ϵ is the corrector)

$$\|u_\epsilon - u_0 - (\phi_\epsilon(x) - x)\frac{\partial u_0}{\partial x}\|_{H_0^1(\Omega)} \leq C_1\epsilon\|f\|_{L^2(\Omega)},$$

$$\|\phi_\epsilon(x) - x\|_{L^\infty(\Omega)} \leq C\epsilon \text{ and } \|\nabla\phi_\epsilon\|_{L^\infty(\Omega)} \leq C.$$

Theorem 3.5. *(Dirichlet-to-Neumann control)*

$$\int_{\partial\Omega} |\frac{\partial\phi_{\epsilon,k}}{\partial\nu}| ds \leq C_0\lambda_{\epsilon,k}^{3/2},\ \forall 0 \leq \epsilon \leq 1, \forall k,$$

$$\int_{\partial\Omega} |\frac{\partial\phi_{\epsilon,k}}{\partial\nu}|^2 ds \leq C_*\lambda_{\epsilon,k} \textit{ if } \epsilon\lambda_{\epsilon,k} \leq 1 \textit{(not optimal)}.$$

Castro-Zuazua proved (in fact, they consider the 1-dimensional problem):

$$\int_{\partial\Omega} |\frac{\partial\phi_{\epsilon,k}}{\partial\nu}|^2 ds \approx \lambda_{\epsilon,k}^{3/2} \text{ when } \epsilon^2 \approx \frac{1}{\lambda_{\epsilon,k}}.$$

Theorem 3.6. $\int_{\partial\Omega} |\frac{\partial\phi_{\epsilon,k}}{\partial\nu}|^2 ds \geq C_0\lambda_{\epsilon,k}$ *if* $\epsilon\lambda_{\epsilon,k} \leq 1$.

Proof. Step1: Require $\epsilon\lambda_{\epsilon,k} \leq 1$, by Rellich identity, we know that

$$\frac{1}{\epsilon}\int_{\Omega_\epsilon} |\nabla\phi_{\epsilon,k}|^2 dx \approx \lambda_{\epsilon,k},$$

where $\Omega_\epsilon = \{x \in \Omega; \mathrm{dist}(x, \partial\Omega) \leq 3\epsilon\}$ (in general, the above estimate is not true, but here it gives a compatibility).

Step2: Jacobian of $\phi_\epsilon(x) \geq c_0 > 0$ if $\mathrm{dist}(x, \partial\Omega) \leq 3\epsilon$, here we assume $\partial\Omega \in C^{1,1}$.

Step3: Consider a cube Q_2 and its subcube Q_1 inside Q_2 with the boundary Γ. On this region, we impose a Cauchy problem

$$\begin{cases} \Delta u = 0 & \text{in } Q_2, \\ u = 0 & \text{on } \Gamma. \end{cases}$$

If

$$\int_{Q_1} u^2 dx \geq c_0 \int_{Q_2} u^2 dx$$

($\int_{Q_1} |\nabla u|^2 dx \geq c_0 \int_{Q_2} |\nabla u|^2 dx$), then

$$\int_\Gamma |\frac{\partial u}{\partial \nu}|^2 ds \geq c_* \int_{Q_1} u^2 dx \tag{3.1}$$

($\int_\Gamma |\frac{\partial u}{\partial \nu}|^2 ds \geq c_* \int_{Q_1} |\nabla u|^2 dx$, respectively). Finally, we divide the annulus region Ω_ϵ into many small "cubes", there are some good cubes which can give the compatibility as the above estimate, but there are still some bad cubes cannot satisfy the estimate (3.1), but we can prove the fact that these bad cubes only have small portions of Ω_ϵ, them add up all of them to get the desired estimate. □

In the final section, we consider the nondivergent operator:

3.2 Nondivergent operator

Let (a^{ij}) be a positive definite and periodic, consider $A = a^{ij} \frac{\partial^2}{\partial x_i \partial x_j}$.

Lemma 3.7. *There is a smooth, periodic $p(x) > 0$ such that*

$$A^* p = \frac{\partial^2}{\partial x_i \partial x_j}(a^{ij} p) = 0, \ \langle p \rangle = 1.$$

Proof. Any solution of $Au = 0$ on T^n is a constant by the maximum principle. Fredholm alternative gives $A^* p = 0$ has a unique (to a constant factor) periodic solution $p \neq 0$. Normalize p so that $\langle p \rangle = 1$, this $p > 0$ on T^n. If p changes sign, then $\exists f \geq 0$ and $f \neq 0$ such that $\langle p, f \rangle = 0$ and therefore $Au == f$ wold have a periodic solution. But this is not possible via maximum principle ($\Rightarrow p \geq 0$)! Let $u \geq 0$ be a smooth solution of the elliptic equation of $a^{ij} u_{ij} + b^i u_i + cu = 0$, if $u(x_0) = 0$ for an interior point x_0, then $u \equiv 0$. □

Now, let $b_j = \frac{\partial}{\partial x_i}(p a_{ij})$, by lemma, we have div $\overrightarrow{b}$=0. Thus $\exists (b_{ij})$ such that $\overrightarrow{b}$ = div(b_{ij}) with $b_{ij} = -b_{ji}$and $\langle b_{ij} \rangle = 0$. Hence $a_{ij} u_{ij} = f$ if and only if $p a_{ij} u_{ij} = pf$ or div$[(p a_{ij} - b_{ij}) u_{x_j}] = pf$, and set $\widetilde{A}_j(y) = p(y) a_{ij}(y) - b_{ij}(y)$. In this case, the corrector $\chi = 0$, i.e. div$(A \nabla \chi) = -$div$A = 0$.

Now, consider the homogenization of nondivergence solution of PDEs:

$$\begin{cases} -a_{ij}(x, \frac{x}{\epsilon}, Du^\epsilon) u^\epsilon_{x_i x_j} = b(x, \frac{x}{\epsilon}, Du^\epsilon) & \text{in } \Omega \\ u^\epsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

where $a^{ij}, b : \mathbb{R} \times Y \times \mathbb{R}^n \to \mathbb{R}$, with $Y = \mathbb{Z}^n$, periodic in y variable. Assume $\xi^T A(x, y, p)\xi \geq \lambda_0|\xi|^2$, $|A(x, y, p| \leq C$ and $b(x, y, p) \leq C(1 + |p|^2)$.

Fix $x \in \Omega$, $p \in \mathbb{R}^n$, consider the adjoint problem

$$-(a_{ij}(x, y, p)m(p, x, y))_{y_i y_j} = 0 \text{ in } \mathbb{R}^n,$$

where m is Y-periodic. It has a positive solution m unique subject to normalization $\int_Y m(p, x, y)dy = 1$. Define average coefficients

$$\widetilde{a^{ij}}(x, p) = \int_Y m a_{ij}(x, y, p)dy,$$

$$\widetilde{b}(x, p) = \int_Y b(x, y, p)m(x, y, p)dy.$$

Homogenized problem

$$\begin{cases} -\widetilde{a^{ij}}(x, Du)u_{x_i x_j} = \widetilde{b}(x, Du) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Theorem 3.8. *Assume $\{u^\epsilon\}_{\epsilon>0}$ is bounded, then there is a sequence $\{u^{\epsilon_i}\}$ and $u \in C(\overline{\Omega})$ such that $u^{\epsilon_i} \to u$ uniformly on $\overline{\Omega}$ and u is a weak solution of the homogenized problem.*

Proof. Elliptic estimate under the quadratic growth of $b \Rightarrow \{u^\epsilon\} \subset C^\beta(\overline{\Omega})$ for some $\beta > 0$. $\exists u^{\epsilon_i} \to u$. Fix $v \in C^2(\Omega)$ and if $u - v$ has a strict maximum at $x_0 \in \Omega$. Using Fredholm alternative, we find a solution w of the corrector problem

$$\begin{aligned} -a^{ij}(x_0, y, Dv(x_0))w_{y_i y_j} &= a^{ij}(x_0, y.Dv(x_0)) - \widetilde{a^{ij}}(x_0, Dv(x_0))v_{x_i x_j}(x_0) \\ &- [b(x_0, y, Dv(x_0)) - \widetilde{b}(x_0, Dv(x_0))] \end{aligned}$$

with w is Y-periodic. Define the perturbed test function $v^\epsilon(x) = v(x) + \epsilon^2 w(\frac{x}{\epsilon})$, then $v^\epsilon(x) \to v(x)$ uniformly as $\epsilon \to 0$. Then $u^{\epsilon_k} - v^{\epsilon_k}$ has a local maximum at a point x_{ϵ_k} with $x_{\epsilon_k} \to x_0$ as $k \to \infty$. By the maximum principle, $-a^{ij}(x_\epsilon, \frac{x_\epsilon}{\epsilon}, Dv^\epsilon(x_\epsilon))v^\epsilon_{x_i x_j} \leq b(x_\epsilon, \frac{x_\epsilon}{\epsilon}, Dv^\epsilon(x_\epsilon))$ for $\epsilon = \epsilon_k$ and $Dv^\epsilon(x_\epsilon) = Dv(x_\epsilon) + \epsilon Dw(\frac{x_\epsilon}{\epsilon})$, $D^2 v^\epsilon(x_\epsilon) = D^2 v(x_\epsilon) + D^2 w(\frac{x_\epsilon}{\epsilon})$. This implies

$$-a^{ij}(x_0, y_*, Dv(x_0))(v_{x_i x_j}(x_0) + w_{ij}(y_*)) \leq b(x_0, y_*, Dv(x_0)) + o_\epsilon(1).$$

□

3.3 Homogenization of Hamilton-Jacobi equations

$$\begin{cases} H(Du^\epsilon, u^\epsilon, x, \frac{x}{\epsilon}) = 0 & \text{in } \Omega \\ u^\epsilon(x) = 0 & \text{on } \partial\Omega \end{cases} \tag{3.2}$$

where $\Omega \subset \mathbb{R}^n$ bounded smooth. Assumptions on H:

1. $y \to H(p, u, x, y)$ is periodic (period 1).
2. $\lim_{|p|\to\infty} H(p, u, x, y) = \infty$ uniformly on $[-L, L] \times \Omega \times \mathbb{R}^n$ $\forall L$.
3. $u \to H(p, u, x, y) - \mu u$ increasing in u.
4. H is Lipschitz on any $B_L(0) \times [-L, L] \times \Omega \times \mathbb{R}^n$ $\forall L > 0$.

The vanishing viscosity solution method implies that $\forall \epsilon > 0$, $\exists$ a viscosity solution u^ϵ of (3.2). Moreover, such u^ϵ is unique and $\|u^\epsilon\|_{C^{0,1}(\overline{\Omega})} \le M < \infty$ $\forall \epsilon \in (0, 1)$. We introduce Lions-Papanicolaou-Varadhan lemma.

Lemma 3.9. *For each $p \in \mathbb{R}^n$, $u \in \mathbb{R}$, $x \in \Omega$, there is a unique real number λ for which the PDE*

$$\begin{cases} H(D_y v_p, u, x, y) = \lambda & \text{in } \mathbb{R}^n \\ v \text{ is periodic in } y \end{cases}$$

has a solution $v \in C^{0,1}(\mathrm{T}^n)$. Denote $\lambda = \overline{H}(p, u, x)$ for $p \in \mathbb{R}^n$, $u \in \mathbb{R}$, $x \in \Omega$.

Effected Hamiltonina $\overline{H}$ and the Cell Problem

$$\begin{cases} H(D_y v + p, u, x, y) = \overline{H}(p, u, x) & \text{in } \mathbb{R}^n \\ v \text{ is periodic,} \end{cases}$$

$\overline{H}$ has the same property as H.

Now, we introduce the result proved by Caffarelli and L. C. Evans.

Consider $F(D^2u, y) = 0$, F is uniformly elliptic in D^2u and 1-periodic in $y \in \mathbb{R}^n$. Let $u_\epsilon$be a viscosity solution of $F(D^2u_\epsilon(x), \frac{x}{\epsilon}) = 0$ in Ω and $u_\epsilon = 0$ on $\partial\Omega$.Let $S = \{A \in M_s^{n\times n} : F(A + D^2w, y) \equiv 0$ has a solution on $\mathrm{T}^n\}$, $\underline{S}$ describes the zero set of a uniformly elliptic equation $\overline{F}(D^2u) = 0$. This $\overline{F}$ is called the homogenized limit which introduced by Evans.

Fact 3.10. *Let u_ϵ be a bounded solution of $F(D^2u_\epsilon, \frac{x}{\epsilon}) = 0$ in B_1, then $u_\epsilon|_{B_{1/2}} \in C^\alpha$ for some $\alpha > 0$ and $\|u_\epsilon\|_{C^\alpha(B_{1/2})} \le C\|u_\epsilon\|_{L^\infty(B_1)}$.*

Proof. $F(D^2u_\epsilon, \frac{x}{\epsilon}) = F(0, \frac{x}{\epsilon}) + \int_0^1 \frac{d}{dt}F(tD^2u_\epsilon, \frac{x}{\epsilon})dt = F(0, \frac{x}{\epsilon}) + \int_0^1 F_{ij}(tD^2u_\epsilon, \frac{x}{\epsilon})dt \cdot D^2_{ij}u_\epsilon = 0$, this forms $a^\epsilon_{ij}(x)u^\epsilon_{x_ix_j}(x) + f_\epsilon(x) = 0$, by using Krylov-Safanov theorem, we can get the conclusion. □

Theorem 3.11. *Let $\{u_{\epsilon_k}\}$ be a sequence of solutions of $F(D^2u_{\epsilon_k}, \frac{x}{\epsilon_k}) = 0$, $\epsilon_k \to 0$ and $u_{\epsilon_k} \to u_0$ in C^α as $k \to \infty$. Then u_0 is a viscosity solution of $\overline{F}(D^2u_0) = 0$.*

In the end of this lecture, we give some open problems.

3.4 Open problems

1. Let $\{x_i\}_{i=1}^\infty \subset \mathbb{R}^n$ be such that
   (a) $|x_i - x_j| \geq 3$ whenever $i \neq j$.
   (b) $\cup_{i=1}^\infty B_5(x) \supseteq \mathbb{R}^n$.
   Let $0 \leq \phi \leq 1$, $\phi \in C_0^\infty(B_1)$ and $A_{kl}(x) = \delta_{kl} + a^0_{kl}\sum_{i=1}^\infty \phi(x - x_i)$, hence $I \leq A(x) \leq I + C_n(a^0_{kl})$.
   What are asymptotic behavior of its Green function $G_A(x, y)$? Are there similar results as in periodic cases?

2. Rate of Convergence in
   (a) Statistical homogenization ?
   (b) Homogenization of Hamilton-Jacobi equation?
   (c) Nonlinear equations?