# Introduction to Homogenization Theory - Periodic Second Order Elliptic Equations

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**Abstract**

**The note is mainly for personal record, if you want to read it, please be careful.** This notes was given by Prof. Zhongwei Shen's lecture in the National Taiwan University, 2015.03. For more details we refer readers to his draft lecture (with a similar title as this lecture). For simplicity, the following arguments hold for the linear elliptic system, however, we only discuss the linear scalar elliptic equation case in this lecture.

## 1 Introduction

We consider the following elliptic equations

$$\begin{cases} -\Delta u = F & \text{in } \Omega \subset \mathbb{R}^d, \\ -\nabla\cdot(A(x)\nabla u) = F & \text{in } \Omega \subset \mathbb{R}^d, \\ F(D^2u, Du, u, x) = F & \text{in } \Omega \subset \mathbb{R}^d, \end{cases} \text{ with suitable boundary conditions.}$$

Medium with a fine self-similar microscopic structure. Let $\epsilon > 0$ be a small parameter, and consider the following homogenization problem

$$F(D^2u_\epsilon, Du_\epsilon, u_\epsilon \frac{x}{\epsilon}, x) = 0 \text{ in } \Omega, \text{ with suitable boundary conditions.}$$

In the following, we only consider the simplest linear elliptic homogenization problem. Let $A(x) = (a_{ij}(x))_{d\times d}$, $A^\epsilon(x) = A(\frac{x}{\epsilon})$ and consider

$$\begin{cases} -\nabla\cdot(A(\frac{x}{\epsilon})\nabla u_\epsilon) = F & \text{in } \Omega, \\ \text{Suitable boundary condition on } \partial\Omega. \end{cases} \tag{1}$$

What can we say about $u_\epsilon$ as $\epsilon \to 0$. In fact, we have

$$u_\epsilon \rightharpoonup u_0 \text{ weakly in } H^1(\Omega) \text{ and } u_\epsilon \to u_0 \text{ strongly in } L^2(\Omega).$$

As a result, for $F \in H^{-1}(\Omega)$, we have

$$\begin{cases} -\nabla\cdot(A(\frac{x}{\epsilon})\nabla u_\epsilon) = F & \text{in } \Omega, \\ \text{Suitable B.Cs} \end{cases} \longrightarrow \begin{cases} -\nabla\cdot(\overline{A}\nabla u_0) = F & \text{in } \Omega, \\ \text{Same B.Cs} \end{cases}$$

as $\epsilon \to 0$ and $\overline{A} = (\overline{a_{ij}})_{d\times d}$ is a constant matrix. Here is the problem setting. Assume the second order elliptic operator

$$L_\epsilon := -\frac{\partial}{\partial x_i}\{a_{ij}(\frac{x}{\epsilon})\frac{\partial}{\partial x_j}\}$$

with the coefficients satisfying

$$\begin{cases} \text{Ellipticity: } \mu|\xi|^2 \le a_{ij}(y)\xi_i\xi_j \le \frac{1}{\mu}|\xi|^2, \text{ for all } \xi \in \mathbb{R}^d, \\ \text{1-Periodicity: } A(y+z) = A(y) \text{ for all } y \in \mathbb{R}^d,\ z \in \mathbb{Z}^d. \end{cases}$$

Consider the *Dirichlet problem* to be

$$\begin{cases} L_\epsilon u_\epsilon = F & \text{in } \Omega, \\ u_\epsilon = g & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where $F \in H^{-1}(\Omega)$, $g \in H^{1/2}(\partial\Omega)$, and $\Omega$ is a bounded Lipschitz domain in $\mathbb{R}^d$. By the Lax-Milgram lemma, we have

$$\|u_\epsilon\|_{H^1(\Omega)} \le C\{\|F\|_{H^{-1}(\Omega)} + \|g\|_{H^{1/2}(\partial\Omega)}\}, \tag{3}$$

where the constant $C$ independent of $\epsilon$. Note that $u_\epsilon$ is a weak solution of (2) if for all $\varphi \in H_0^1(\Omega)$, we have

$$\int_\Omega (A(\frac{x}{\epsilon})\nabla u_\epsilon)\cdot\nabla\varphi dx = \langle F, \varphi\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)}.$$

Consider the *Neumann problem* to be

$$\begin{cases} L_\epsilon u_\epsilon = F & \text{in } \Omega, \\ \frac{\partial u_\epsilon}{\partial \nu_\epsilon} = g & \text{on } \partial\Omega, \end{cases} \tag{4}$$

where $F \in H_0^{-1}(\Omega) = (H^1(\Omega))'$, $g \in H^{-1/2}(\partial\Omega) = (H^{1/2}(\partial\Omega))'$, $\frac{\partial u_\epsilon}{\partial \nu_\epsilon} = \nu_i(x)a_{ij}(\frac{x}{\epsilon})\frac{\partial u_\epsilon}{\partial x_j}$, and $\nu = (\nu_1, \nu_2, \cdots, \nu_d)$ is a unit outer normal on $\partial\Omega$. Similarly, we call $u_\epsilon$ to be a weak solution if for all $\varphi \in H^1(\Omega)$, we have

$$\int_\Omega (A(\frac{x}{\epsilon})\nabla u_\epsilon)\cdot\nabla\varphi dx = \langle F, \varphi\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)} - \langle g, \varphi\rangle_{H^{-1/2}(\partial\Omega)\times H^{1/2}(\partial\Omega)}.$$

Moreover, by the Lax-Milgram lemma again, we have

$$\|\nabla u_\epsilon\|_{L^2(\Omega)} \le C\{\|F\|_{H_0^{-1}(\Omega)} + \|g\|_{H^{-1/2}(\partial\Omega)}\}, \tag{5}$$

where $C$ is independent of $\epsilon$. For the second order elliptic system, we consider the homogenization problem to be

$$\begin{cases} L_\epsilon u_\epsilon = -\frac{\partial}{\partial x_i}\left[a_{ij}^{\alpha\beta}(\frac{x}{\epsilon})\frac{\partial u_\epsilon^\beta}{\partial x_j}\right] = F_\alpha \text{ in } \Omega \\ u_\epsilon \text{ satisfies suitable boundary conditions} \end{cases}$$

with $1 \le i, j \le d$ and $1 \le \alpha, \beta \le m$. In order to demonstrate ideas, we consider $m = 1$ in most situations.

## 2 Derivation of the homogenized equation

The ideas of the derivation of the homogenized equation is quite clever. We consider $u_\epsilon$ to be the perturbation of $u_0$ with respect to $\epsilon$-parameter. Moreover, by observing the elliptic operator $L_\epsilon$, we introduce the *two-scaled method*, which means we consider $x = x$, and $y = \frac{x}{\epsilon}$ to be two independent parameters. Let

$$u_\epsilon := u_0 + \epsilon u_1 + \epsilon^2 u_2 + \cdots$$

be the asymptotic expansion of $u_\epsilon$, where

$$u_j := u_j(x, y) = u_j(x, \frac{x}{\epsilon}).$$

In addition,

$$\nabla u_j = \nabla_x u_j(x, y) + \frac{1}{\epsilon}\nabla_y u_j(x, y), \text{ as } y = \frac{x}{\epsilon},$$

which means under our two-scaled method, the operator $\nabla = \nabla_x + \frac{1}{\epsilon}\nabla_y$. Therefore, (1) will become

$$-(\nabla_x + \frac{1}{\epsilon}\nabla_y) \cdot \{A(y)[(\nabla_x + \frac{1}{\epsilon}\nabla_y)(u_0 + \epsilon u_1 + \epsilon^2 u_2 + \cdots)]\} = F(x) \text{ in } \Omega, \quad (6)$$

where we are not concerned about the boundary condition for the equation. Expand (6) and compare it with the same orders, so we get

$$\begin{aligned}
O(\frac{1}{\epsilon^2}): &\quad -\nabla_y \cdot (A(y)\nabla_y u_0(x, y)) = 0,\\
O(\frac{1}{\epsilon}): &\quad -\nabla_y \cdot (A(y)\nabla_y u_1(x, y)) = \nabla_y \cdot (A(y)\nabla_x u_0) + \nabla_x \cdot (A(y)\nabla_y u_0),\\
O(1): &\quad -\nabla_y \cdot (A(y)\nabla_y u_2(x, y)) = \nabla_y \cdot (A(y)\nabla_x u_1) + \nabla_x \cdot (A(y)\nabla_y u_1)\\
&\qquad\qquad + \nabla_x \cdot (A(y)\nabla_x u_0) + F(x).
\end{aligned}$$

Recall that for the periodic elliptic equation

$$-\nabla \cdot (A(y)\nabla u(y)) = h(y), \ A(y) \text{ is 1-periodic},$$

then we have

$$\int_{\mathbb{T}^d} h(y)dy = 0$$

by using the Stokes formula. For $O(\frac{1}{\epsilon^2})$ term, this equation is solvable because the right hand side is zero. In further, we multiply $u_0(x, y)$ on both sides and integrate by parts, which will imply

$$0 = \int_{\mathbb{T}^d} A(y)\nabla_y u_0 \cdot \nabla_y u_0 \geq \mu \int_{\mathbb{T}^d} |\nabla_y u_0(x, y)|^2 dy \geq 0,$$

which gives us the information that

$$u_0(x, y) \equiv u_0(x)$$

independent of $y$.

Now, for the second term $O(\frac{1}{\epsilon})$, the second term on the right hand side should be zero since $\nabla_y u_0(x) = 0$. Solve the equation

$$\begin{aligned} -\nabla_y \cdot (A(y)\nabla_y u_1(x,y)) &= \nabla_y \cdot (A(y)\nabla_x u_0) \\ &= (\nabla_y \cdot A(y))(\nabla_x u_0) \end{aligned}$$

formally. Note that since $A(y)$ is 1-periodic, then the equation is solvable for $u_1$ if

$$\int_{\mathbb{T}^d} (\nabla_y \cdot A(y))\nabla_x u_0) dy = \int_{\partial \mathbb{T}^d} (A(y)\nabla_x u_0) \cdot \nu(y) dS(y) = 0.$$

By using the separation of variables, we put the *ansatz*

$$u_1(x,y) = \chi(y)(\nabla_x u_0(x))$$

into the equation with $\chi(y)$ being 1-periodic, we will find that

$$-\nabla \cdot (A(y)\nabla_y \chi(y))(\nabla_x u_0) = (\nabla_y \cdot A(y))(\nabla_x u_0).$$

Moreover, we call

$$-\nabla \cdot (A(y)\nabla_y \chi(y)) = (\nabla_y \cdot A(y))$$

to be the *cell problem* and $\chi(y)$ to be the *corrector.*

Finally, we observe the last equation carefully. Put $u_1(x,y) = \chi(y)\nabla_x u_0$ into the $O(1)$ equation and examine the solvability condition for $u_2(x,y)$, we have

$$\begin{aligned} 0 &= \int_{\mathbb{T}^d} [\nabla_y \cdot (A(y)\nabla_x u_1) + \nabla_x \cdot (A(y)\nabla_y u_1) + \nabla_x \cdot (A(y)\nabla_x u_0) + F(x)] dy \\ &= \nabla_x \cdot \{[\int_{\mathbb{T}^d} A(y)(\nabla_y \chi(y)) dy]\nabla_x u_0\} + \nabla_x \cdot \{[\int_{\mathbb{T}^d} A(y) dy]\nabla_x u_0\} + F(x), \end{aligned}$$

where the first term vanishes by the periodicity of $A$ and $\chi$. Thus, we can obtain

$$-\nabla \cdot (\overline{A}\nabla u_0) = F(x) \text{ in } \Omega, \tag{7}$$

where

$$\overline{A} = \int_{\mathbb{T}^d} \{A(y) + A(y)(\nabla_y \chi(y))\} dy$$

is the (constant) homogenized operator and we call (7) to be the *homogenized equation.* For the rigorous derivation of the homogenized equation, we will give the serious proof later by using the famous tool: The Div-Curl lemma.

# 3 Basic properties

Now, let us recall the corrector again. $\chi = (\chi_1, \chi_2, \cdots, \chi_d) \in H^1_{per}(\mathbb{Z}^d)$ is the corrector, where $H^1_{per}(\mathbb{Z}^d)$ is the closure of $C^\infty$ 1-periodic function under the standard $H^1$-norm. We rewrite the cell problem in the following way:

$$\begin{cases} -\dfrac{\partial}{\partial y_i}\{a_{ij}(y)\dfrac{\partial \chi_k}{\partial y_j}\} = \dfrac{\partial}{\partial y_i}(a_{ik}(y)) \quad \text{in } \mathbb{R}^d, \\ \chi_k \in H^1_{per}(\mathbb{Z}^d),\ 1 \le k \le d. \end{cases}$$

Note that $\frac{\partial}{\partial y_i}(a_{ik}(y)) = \frac{\partial}{\partial y_i}(a_{ij}(y)\frac{\partial}{\partial y_j}(y_k))$, then the cell-problem becomes

$$L_1(\chi_k + x_k) = 0 \text{ in } \mathbb{R}^d.$$

In addition, we consider the equivalence space $H^1_{per}(\mathbb{Z}^d)/\sim$ and define the bilinear form

$$a_{per}(\phi, \psi) := \int_{\mathbb{T}^d} a_{ij}(y)\frac{\partial \phi}{\partial y_j}\frac{\partial \psi}{\partial y_i}dy,$$

then we also can prove the Lax-Milgram lemma under this equivalence Sobolev space. Moreover, when $\phi \in H^1_{per}(\mathbb{Z}^d)$, then $a_{per}(u, \phi) = 0$ for all $u \in H^1(\Omega)$.

For convenience, we set $L_0 := -\nabla \cdot (\overline{A}\nabla)$ be the homogenized second order elliptic operator with respect to $A$.

**Proposition 3.1.** *The homogenized operator $\overline{A} = (\overline{a_{ij}})$can be rewritten in the following form*

$$\overline{a_{ij}} = a_{per}(y_j + \chi_j, y_i).$$

*Moreover, use the above relation, we can write $\overline{A}$ as*

$$\overline{a_{ij}} = a_{per}(y_j + \chi_j, y_i + \chi_i).$$

Use the above proposition, it is not hard to obtain the following theorem.

**Theorem 3.2.** *$L_0$ is elliptic, which means*

$$\mu|\xi|^2 \leq \overline{a_{ij}}\xi_i\xi_j \leq \mu_1|\xi|^2,$$

*where the lower bound comes from the original $A$, and $\mu_1 = \mu_1(\mu, d)$.*

*Proof.* $\overline{a_{ij}} = a_{per}(y_j + \chi_j, y_i + \chi_i)$ and $a_{per}(y_j + \chi_j, \phi) = 0$ when $\phi \in H^1_{per}(\mathbb{Z}^d)$. Therefore, we have

$$\begin{aligned}
\overline{a_{ij}}\xi_i\xi_j &= a_{per}((y_j + \chi_j)\xi_j, (y_i + \chi_i)\xi_i) \\
&\geq \mu \int_{\mathbb{Z}^d} |\nabla(\chi_i + y_i)\xi_i|^2 dy \\
&= \int_{\mathbb{T}^d} \{|\nabla(\chi_i\xi_i)|^2 + 2\nabla(\chi_i\xi_i) \cdot \nabla(y_i\xi_i) + |\nabla(y_i\xi_i)|^2\} \\
&\geq \mu|\xi|^2,
\end{aligned}$$

since $\nabla(\chi_i\xi_i) \cdot \nabla(y_i\xi_i) = 0$. □

For the rigorous proof of the $L_\epsilon \to L_0$ in a suitable sense, we use the following two useful lemmas.

**Lemma 3.3.** *Let $h \in L^2_{loc}(\mathbb{R}^d)$ and 1-periodic, then*

$$h(\frac{x}{\epsilon}) \rightharpoonup c_0 := \int_{\mathbb{T}^d} h(y)dy$$

*weakly in $L^2(\Omega)$.*

*Proof.* We gave a hint for readers and leave details of proof to an exercise. Hint: Try to solve

$$\begin{cases} \Delta u = h \text{ in } \mathbb{T}^d, \\ u \text{ is 1-periodic.} \end{cases}$$

□

**Lemma 3.4.** *(Div-Curl lemma) Let $\{u_k\}$, $\{v_k\}$ be two bounded sequences in $L^2(\Omega, \mathbb{R}^d)$. Suppose that $u_k \rightharpoonup u_0$, $v_k \rightharpoonup v_0$ weakly in $L^2(\Omega, \mathbb{R}^d)$ and $\nabla \times u_k = 0$ for all $k$, $\nabla \cdot v_k \to f$ strongly in $H^{-1}(\Omega)$. Then*

$$\int_\Omega u_k \cdot v_k \varphi dx \to \int_\Omega u_0 \cdot v_0 \varphi dx$$

*for all $\varphi \in C_0^1(\Omega)$.*

*Proof.* Leave this lemma to an exercise. □

**Theorem 3.5.** *(Homogenization Theorem) Suppose $A(y)$ is elliptic and 1-periodic. Let $\Omega$ be a bounded Lipschitz domain. If $u_\epsilon$ is a weak solution of the Dirichlet problem (2). Then $u_\epsilon \rightharpoonup u_0$ weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$, where $u_0 \in H^1(\Omega)$ is the weak solution to the homogenized equation*

$$\begin{cases} L_0 u_0 = F & \text{in } \Omega, \\ u_0 = g & \text{on } \partial\Omega. \end{cases}$$

*Moreover, we have $A(\frac{x}{\epsilon})\nabla u_\epsilon \rightharpoonup \overline{A}\nabla u_0$ weakly in $L^2(\Omega, \mathbb{R}^d)$.*

*Remark* 3.6. In general, $u_\epsilon$ does not converge to $u_0$ strongly in $H^1(\Omega)$.

Before giving proofs for the homogenization theorem, we introduce the dual problem for the homogenization problem. Let $L_\epsilon^* := -\nabla \cdot (A^*(\frac{x}{\epsilon})\nabla)$ be the adjoint operator of $L_\epsilon$. Then we have the following properties.

1. $a_{ij}^* = a_{ji}$.
2. $a_{per}^*(\phi, \psi) = a_{per}(\psi, \phi)$.
3. Let $\chi^*$ be the corrector of $L_\epsilon^*$, then we have

$$a_{per}^*(y_k + \chi_k^*, \psi) = 0, \text{ for all } \psi \in H_{per}^1(\mathbb{Z}^d).$$

**Proposition 3.7.** *We have $\overline{a_{ij}} = a_{per}^*(y_i + \chi_i, y_i)$ and $\overline{A}^* = \overline{A^*}$. If $A = A^*$, then $\overline{A}^* = \overline{A}$.*

*Proof.* Exercise. □

In addition, we have

$$\begin{aligned} \overline{a_{ij}} &= a_{per}^*(y_i + \chi_i, y_j) = a_{per}(y_j, y_i + \chi_i^*) \\ &= \int_{\mathbb{T}^d} a_{\ell k}(y) \frac{\partial y_j}{\partial y_k} \frac{\partial (y_i + \chi_i^*)}{\partial y_\ell} dy \\ &= \int_{\mathbb{T}^d} a_{\ell j}(y) \{\delta_{\ell k} + \frac{\partial \chi_i^*}{\partial y_\ell}\} dy \\ &= \int_{\mathbb{T}^d} \{a_{ij}(y) + a_{\ell j}(y) \frac{\partial \chi_i^*}{\partial y_\ell}\} dy. \end{aligned}$$

Now we prove the theorem.

*Proof.* By using the Lax-Milgram theorem, we have

$$\|u_\epsilon\|_{H^1(\Omega)} \leq C\{\|F\|_{H^{-1}(\Omega)} + \|g\|_{H^{1/2}(\partial\Omega)}\},$$

which means $\{u_\epsilon\}$ is bounded in $H^1(\Omega)$. Moreover, $u_\epsilon$ satisfies (2) and

$$\int_\Omega A(\frac{x}{\epsilon})\nabla u_\epsilon \cdot \varphi dx = \langle F, \varphi \rangle .$$

By the duality argument, we have $\|A(\frac{x}{\epsilon})\nabla u_\epsilon\|_{L^2(\Omega)} \leq C < \infty$. WLOG, we say $\{u_\epsilon\}$ be the sequence such that $u_\epsilon \rightharpoonup v$ weakly in $H^1(\Omega)$ and $A(\frac{x}{\epsilon})\nabla u_\epsilon$ to $p$ weakly in $L^2(\Omega)$. Then remaining work is to prove

$$\begin{cases} v = u_0 \text{ and} \\ p = \overline{A}\nabla u_0. \end{cases}$$

Consider

$$\int_\Omega A(\frac{x}{\epsilon})\nabla u_\epsilon \cdot \nabla(x_k + \epsilon\chi_k^*(\frac{x}{\epsilon}))\psi dx = \int_\Omega \nabla u_\epsilon \cdot A^*(\frac{x}{\epsilon})\nabla(x_k + \epsilon\chi(\frac{x}{\epsilon}))\psi dx,$$

for all $\psi \in C_0^1(\Omega)$. For the left side, we have $A(\frac{x}{\epsilon})\nabla u_\epsilon \rightharpoonup p$ weakly in $L^2(\Omega; \mathbb{R}^d)$.

$$\nabla(x_k + \epsilon\chi_k^*(\frac{x}{\epsilon})) = \delta_{ik} + \frac{\partial \chi_k^*}{\partial x_i}(\frac{x}{\epsilon}) \overset{L^2}{\rightharpoonup} \delta_{ik} + \int_{\mathbb{T}^d} \frac{\partial \chi_k^*}{\partial y_i} dy = \delta_{ik},$$

and

$$\nabla \times (\nabla(x_k + \epsilon\chi_k^*(\frac{x}{\epsilon})) = 0 \text{ and } \nabla \cdot (A(\frac{x}{\epsilon})\nabla u_\epsilon) = F.$$

The left hand side satisfies conditions of the Div-Curl lemma, then the left hand side will converge to

$$\int_\Omega p_i \delta_{ik} \psi dx = \int_\Omega p_k \psi dx.$$

For the right side, $\nabla u_\epsilon \rightharpoonup \nabla u_0$ in $L^2$,

$$\begin{aligned} A^*(\frac{x}{\epsilon})\nabla(x_k + \epsilon\chi_k^*(\frac{x}{\epsilon})) &= A^*(\frac{x}{\epsilon})\{\nabla x_k + \nabla\chi_k^*(\frac{x}{\epsilon})\} \\ &\rightharpoonup \int_{\mathbb{T}^d} a_{ij}^*(y)\{\delta_{jk} + \frac{\partial}{\partial y_j}(\chi_k^*)\}dy \\ &= \int_{\mathbb{Z}^d} \{a_{ki}(y) + a_{ji}(y)\frac{\partial \chi_k^*}{\partial y_j}\}dy \\ &= \overline{a_{ki}}. \end{aligned}$$

Then the right side converges to $\int_\Omega \frac{\partial v}{\partial x_i}\overline{a_{ki}}\psi dx$. Thus, $\int_\Omega p_k \psi dx = \int_\Omega \frac{\partial v}{\partial x_i}\overline{a_{ki}}\psi dx$, which implies

$$p_k = \overline{a_{ki}}\frac{\partial v}{\partial x_i} \text{ and } A(\frac{x}{\epsilon})\nabla u_\epsilon \rightharpoonup \overline{A}\nabla v$$

and

$$\begin{cases} -\nabla \cdot (\overline{A}\nabla v) = F & \text{in } \Omega, \\ v = g & \text{on } \partial\Omega, \end{cases}$$

and $v = u_0$ (uniqueness for Dirichlet problem). □

*Remark* 3.8. The homogenization theorem still holds for the Neumann problem

$$\begin{cases} L_\epsilon(u_\epsilon) = F & \text{in } \Omega, \\ \dfrac{\partial u_\epsilon}{\partial \nu_\epsilon} = g & \text{on } \partial\Omega, \end{cases}$$

where $F \in H_0^{-1}(\Omega)$, $g \in H^{-1/2}(\partial\Omega)$, and $\int_\Omega u_\epsilon = 0$. The proof is followed by using

$$\int_\Omega A(\frac{x}{\epsilon})\nabla u_\epsilon \cdot \nabla\psi dx = \langle F, \psi\rangle_{H_0^{-1}\times H^1} - \langle g, \psi\rangle_{H^{-1/2}\times H^{1/2}},$$

and use the Div-Curl lemma argument, then we can obtain the desired results.

# 4 Rates of convergence

In the homogenization theory, it is interesting that how fast for $u_\epsilon$ converging to $u_0$.

**Theorem 4.1.** *(Dirichlet) Suppose $A$ is elliptic and 1-periodic. Let $\Omega$ be a bounded Lipschitz domain. Then*

$$\|u_\epsilon - u_0 - \epsilon\chi(\frac{x}{\epsilon}) \cdot \nabla u_0\|_{H^1(\Omega)} \leq C\sqrt{\epsilon}\|u_0\|_{W^{2,d}(\Omega)}.$$

*If $d \geq 3$ and $\chi$ is Hölder continuous (if $m = 1$),*

$$\|u_\epsilon - u_0 - \epsilon\chi(\frac{x}{\epsilon}) \cdot \nabla u_0\|_{H^1(\Omega)} \leq C\sqrt{\epsilon}\|u_0\|_{W^{2,2}(\Omega)}.$$

From $\|u_\epsilon - u_0 - \epsilon\chi(\frac{x}{\epsilon})\nabla u_0\| \leq C\sqrt{\epsilon}\|u_0\|_{W^{2,d}(\Omega)}$, we can derive

$$\begin{aligned} \|u_\epsilon - u_0\|_{L^2(\Omega)} &\leq \epsilon\|\chi(\frac{x}{\epsilon})\nabla u_0\|_{L^2(\Omega)} + C\sqrt{\epsilon}\|u_0\|_{W^{2,d}(\Omega)} \\ &\leq C\sqrt{\epsilon}\|u_0\|_{W^{2,d}(\Omega)}. \end{aligned}$$

Notice that the second inequality is nontrivial, we leave it as an exercise to readers. Moreover, we have

$$\|u_\epsilon - u_0\|_{L^2(\Omega)} \leq C\epsilon\|u_0\|_{W^{2,2}(\Omega)} \text{ if } \begin{cases} \Omega \text{ is Lipschitz}, \ m = 1, \\ \Omega \text{ is } C^{1,1}, \ m \geq 2, \end{cases}$$

and

$$\|u_\epsilon - u_0\|_{H^{1/2}(\Omega)} \leq C\epsilon\|u_0\|_{W^{2,2}(\Omega)}, \text{ when coefficients are Hölder continuous.}$$

**Definition 4.2.** Define $b_{ij}(y) := a_{ij}(y) + a_{ik}(y)\dfrac{\partial \chi_j}{\partial y_k} - \overline{a_{ij}}$.

Note that $b_{ij} \in L^2_{loc}$, $b_{ij}$ is 1-periodic, $\int_{\mathbb{T}^d} b_{ij}(y)dy = 0$, and $\dfrac{\partial b_{ij}}{\partial y_i} = 0$.

**Lemma 4.3.** *There exists $\phi_{kij} \in H^1_{per}(\mathbb{Z}^d)$ such that $b_{ij} = \dfrac{\partial}{\partial y_k}\phi_{kij}$ and $\phi_{kij} = -\phi_{ikj}$.*

*Proof.* We solve

$$\begin{cases} \Delta f_{ij} = b_{ij} \text{ in } \mathbb{R}^d, \\ f_{ij} \text{ is 1-periodic, } f_{ij} \in H^2_{per}(\mathbb{Z}^d). \end{cases}$$

Let $\phi_{kij} := \dfrac{\partial}{\partial y_k} f_{ij} - \dfrac{\partial}{\partial y_i} f_{kj} \in H^1_{per}(\mathbb{Z}^d)$, then $\phi_{kij} = -\phi_{ikj}$. Since $\Delta \dfrac{\partial f_{ij}}{\partial y_i} = \dfrac{\partial b_{ij}}{\partial y_i} = 0$, by the Liouville's theorem, we can get $\dfrac{\partial f_{ij}}{\partial y_i}$ is a constant. Hence

$$\frac{\partial}{\partial y_k}(\phi_{kij}) = \Delta f_{ij} - \frac{\partial^2}{\partial y_k \partial y_i} f_k = \Delta f_{ij} = b_{ij}.$$

□

*Remark* 4.4. If $\chi \in C^\alpha$ for some $\alpha \in (0,1)$, then $\phi = (\phi_{kij})$ is bounded.

**Lemma 4.5.** *Let $w_\epsilon := u_\epsilon - u_0 - \epsilon\chi_k(\frac{x}{\epsilon})\frac{\partial u_0}{\partial x_k}$ in $\Omega$. Then*

$$L_\epsilon(w_\epsilon) = -\epsilon\frac{\partial}{\partial x_i}\{\phi_{kij}(\frac{x}{\epsilon})\frac{\partial^2 u_0}{\partial x_j \partial x_k}\} + \epsilon\frac{\partial}{\partial x_i}\{a_{ij}(\frac{x}{\epsilon})\chi_k(\frac{x}{\epsilon})\frac{\partial^2 u_0}{\partial x_j \partial x_k}\}.$$

*Proof.* From direct calculation, we have

$$\begin{aligned}
L_\epsilon(w_\epsilon) &= L_\epsilon(u_\epsilon) - L_\epsilon(u_0) - L_\epsilon\{\epsilon\chi_k(\frac{x}{\epsilon})\frac{\partial u_0}{\partial x_k}\} \\
&= L_0(u_0) - L_\epsilon(u_0) - L_\epsilon\{\epsilon\chi_k(\frac{x}{\epsilon})\frac{\partial u_0}{\partial x_k}\} \\
&= -\frac{\partial}{\partial x_i}\{[\overline{a_{ij}} - a_{ij}(\frac{x}{\epsilon})]\frac{\partial u_0}{\partial x_j}\} + \frac{\partial}{\partial x_i}\{a_{ij}(\frac{x}{\epsilon})\frac{\partial \chi_k}{\partial x_i}(\frac{x}{\epsilon})\frac{\partial u_0}{\partial x_k}\} \\
&\quad + \epsilon\frac{\partial}{\partial x_i}\{a_{ij}(\frac{x}{\epsilon})\chi(\frac{x}{\epsilon})\frac{\partial^2 u_0}{\partial x_j \partial x_k}\} \\
&= \frac{\partial}{\partial x_i}\{b_{ij}(\frac{x}{\epsilon})\frac{\partial u_0}{\partial x_j}\} + I \\
&= \epsilon\frac{\partial}{\partial x_i}\{\frac{\partial}{\partial x_k}[\phi_{kij}(\frac{x}{\epsilon})\cdot\frac{\partial u_0}{\partial x_j}]\} + I \\
&= \epsilon\frac{\partial^2}{\partial x_i \partial x_k}\{\phi_{kij}(\frac{x}{\epsilon})\frac{\partial u_0}{\partial x_j}\} - \epsilon\frac{\partial}{\partial x_i}\{\phi_{kij}\frac{\partial^2 u_0}{\partial x_j \partial x_k}\} + I \\
&= -\epsilon\frac{\partial}{\partial x_i}\{\phi_{kij}\frac{\partial^2 u_0}{\partial x_j \partial x_k}\} + I,
\end{aligned}$$

where $I = \epsilon\dfrac{\partial}{\partial x_i}\{a_{ij}(\frac{x}{\epsilon})\chi(\frac{x}{\epsilon})\dfrac{\partial^2 u_0}{\partial x_j \partial x_k}\}$ and $\dfrac{\partial^2}{\partial x_i \partial x_k}\{\phi_{kij}(\frac{x}{\epsilon})\dfrac{\partial u_0}{\partial x_j}\} = 0$ by $\phi_{kij} = -\phi_{ikj}$ (after summing over $i, j, k$). □

For more delicate rates of convergence, we introduce the boundary corrector. Let $v_\epsilon$ be a solution satisfying

$$\begin{cases} L_\epsilon(v_\epsilon) = 0 & \text{in } \Omega, \\ v_\epsilon = -\chi_k(\frac{x}{\epsilon})\frac{\partial u_0}{\partial x_k} & \text{on } \partial\Omega, \end{cases}$$

then we have the following theorem.

**Theorem 4.6.** *For $d \geq 3$, we have*

$$\|u_\epsilon - u_0 - \epsilon\chi_k(\frac{x}{\epsilon})\frac{\partial u_0}{\partial x_k} - \epsilon v_\epsilon\|_{H_0^1(\Omega)} \leq C\epsilon\|\nabla^2 u_0\|_{L^d(\Omega)}.$$

*Proof.* (Sketch) Let $\widetilde{w_\epsilon} := u_\epsilon - u_0 - \epsilon\chi_k(\frac{x}{\epsilon})\frac{\partial u_0}{\partial x_k} - \epsilon v_\epsilon$, then it is easy to see that $\widetilde{w_\epsilon} \in H_0^1(\Omega)$ and $L_\epsilon(\widetilde{w_\epsilon}) = L_\epsilon(w_\epsilon)$. Hence by the Lax-Milgram lemma, we have

$$\begin{aligned} \|\widetilde{w_\epsilon}\|_{H_0^1(\Omega)} &\leq C\|L_\epsilon(\widetilde{w_\epsilon})\|_{H^{-1}(\Omega)} \\ &\leq C\epsilon\{\|\phi(\frac{x}{\epsilon})\nabla^2 u_0\|_{L^2(\Omega)} + \|\chi(\frac{x}{\epsilon})\nabla^2 u_0\|_{L^2(\Omega)}\}. \end{aligned}$$

The remaining task is to estimate

$$\|\phi(\frac{x}{\epsilon})\nabla^2 u_0\|_{L^2(\Omega)} + \|\chi(\frac{x}{\epsilon})\nabla^2 u_0\|_{L^2(\Omega)} \leq C\epsilon\|\nabla^2 u_0\|_{L^d(\Omega)}.$$

For the first term, we have

$$\begin{aligned} \int_\Omega |\phi(\frac{x}{\epsilon})\nabla^2 u_0|^2 dx &\leq \int_\Omega |\phi(\frac{x}{\epsilon})|^2 |\nabla^2 u_0|^2 dx \\ &\leq (\int_\Omega |\phi(\frac{x}{\epsilon})|^{\frac{2d}{d-2}} dx)^{\frac{d-2}{d}} (\int_\Omega |\nabla^2 u_0|^d dx)^{\frac{2}{d}}. \end{aligned}$$

Note that $\nabla\phi \in L^2(\mathbb{T}^d)$ and the Sobolev embedding theorem gives $\int_{\mathbb{T}^d} |\phi|^{\frac{2d}{d-2}} dx \leq C\int_{\mathbb{T}^d} |\nabla\phi|^2 dx \leq C$ and

$$\int_\Omega |\phi(\frac{x}{\epsilon})|^{\frac{2d}{d-2}} dx \leq \int_{\frac{1}{\epsilon}\Omega} |\phi(y)|^{\frac{2d}{d-2}} dy \cdot \epsilon^d \leq C.$$

For the other term, we leave it to readers. □

*Remark* 4.7. If $d = 2$, $\chi \in C^\alpha$ for some $\alpha \in (0,1)$, then we can use the Meyers $L^p$ estimate to derive $\phi \in L^\infty(\mathbb{T}^d)$.

Now, we want to prove the fact that

$$\|u_\epsilon - u_0 - \epsilon\chi(\frac{x}{\epsilon})\nabla u_0\|_{H^1(\Omega)} \leq C\sqrt{\epsilon}\|u_0\|_{W^{2,d}(\Omega)}. \tag{8}$$

To prove (8), only need to check the following lemma.

**Lemma 4.8.** $\|v_\epsilon\|_{H^1(\Omega)} \leq \frac{C}{\sqrt{\epsilon}}\|u_0\|_{W^{2,d}(\Omega)}$.

*Proof.* Recall that $v_\epsilon$ satisfies

$$\begin{cases} L_\epsilon(v_\epsilon) = 0 & \text{in } \Omega, \\ v_\epsilon = -\chi_k(\frac{x}{\epsilon})\frac{\partial u_0}{\partial x_k} & \text{on } \partial\Omega, \end{cases}$$

and by the standard elliptic regularity, we have

$$\|v_\epsilon\|_{H^1(\Omega)} \leq C\|\chi_k(\frac{x}{\epsilon})\frac{\partial u_0}{\partial x_k}\|_{H^{1/2}(\partial\Omega)} \leq C\|\eta_\epsilon(x)\chi_k(\frac{x}{\epsilon})\frac{\partial u_0}{\partial x_k}\|_{H^1(\Omega)},$$

where $\eta_\epsilon \in C_0^1(\mathbb{R}^d)$ with $\eta_\epsilon(x) = \begin{cases} 1, & \text{if } \operatorname{dist}(x, \partial\Omega) \leq \epsilon \\ 0, & \text{if } \operatorname{dist}(x, \partial\Omega) \geq 2\epsilon \end{cases}$, and $\|\nabla\eta_\epsilon\|_{L^\infty} \leq \dfrac{C}{\epsilon}$. Therefore, we have

$$\begin{aligned} &\int_\Omega |\nabla\eta_\epsilon|^2|\chi(\frac{x}{\epsilon})|^2|\nabla u_0|^2 dx \\ \leq &\frac{C}{\epsilon^2}\int_{\operatorname{dist}(x,\partial\Omega)\leq 2\epsilon} |\chi(\frac{x}{\epsilon})|^2|\nabla u_0|^2 dx \\ \leq &\frac{C}{\epsilon^2}(\int_{\operatorname{dist}(x,\partial\Omega)\leq 2\epsilon} |\chi(\frac{x}{\epsilon})|^{\frac{2d}{d-2}}|dx)^{\frac{d-2}{d}}(\int_{\operatorname{dist}(x,\partial\Omega)\leq 2\epsilon} |\nabla u_0|^d)^{\frac{2}{d}}. \end{aligned}$$

**Claim**: $\int_{\operatorname{dist}(x,\partial\Omega)\leq 2\epsilon} |\chi(\frac{x}{\epsilon})|^{\frac{2d}{d-2}} dx \leq C\epsilon$ and $\int_{\operatorname{dist}(x,\partial\Omega)\leq 2\epsilon} |\nabla u_0|^d dx \leq C\epsilon\|u_0\|^d_{W^{2,d}(\Omega)}$ (this is true $\forall u_0 \in W^{2,d}(\Omega)$).

For the first part, we choose cubes $Q_j^\epsilon$ with $\operatorname{diam} Q_j^\epsilon \sim O(\epsilon)$ and

$$\{\operatorname{dist}(x, \partial\Omega) \leq 2\epsilon\} \subset \cup_{j=1}^N Q_j^\epsilon \subset \{\operatorname{dist}(x, \partial\Omega) \leq 5\epsilon\}.$$

We can estimate $\chi$ in each cubes and use the periodicity of $\chi$, we can find the uniform estimate for the first term.

For the second part, let $\Omega_t := \{x \in \Omega; \operatorname{dist}(x, \partial\Omega) > t\}$ and $n$ be the unit outer normal on $\partial\Omega_t$. Let $C_0 > 0$ be a positive constant and $\overrightarrow{\alpha}$ be the vector such that $\langle n, \overrightarrow{\alpha}\rangle \geq C_0 > 0$. Then

$$\begin{aligned} C_0 &\int_{\partial\Omega_t} |\nabla u_0|^q dS \leq \int_{\Omega_t} |\nabla u_0|^q \langle n, \overrightarrow{\alpha}\rangle\, dS \\ \leq &\int_{\Omega_t} \operatorname{div}\overrightarrow{\alpha}|\nabla u_0|^q dx + \alpha_k\frac{\partial}{\partial x_k}(|\nabla u_0|^q)dx \\ \leq &C\int_{\Omega_t} |\nabla u_0|^q dx + C\int_{\Omega_t} |\nabla u_0|^{q-1}|\nabla^2 u_0|dx \\ \leq &C\|u_0\|^q_{W^{2,q}}. \end{aligned}$$

Finally, integrate the above inequalities by $\int_0^{2\epsilon} dt$, then we can obtain the desired estimate and the constant $C$ is independent of $\epsilon$. Combine the above estimates together, we will find

$$\|u_\epsilon - u_0\|_{L^2(\Omega)} \leq C\sqrt{\epsilon}\|u_0\|_{W^{2,d}(\Omega)}.$$

□

*Remark* 4.9. In fact, we can prove

$$\|u_\epsilon - u_0\|_{L^2(\Omega)} \leq C\epsilon\|u_0\|_{W^{2,2}(\Omega)} \text{ if } \begin{cases} \Omega \text{ is Lipschitz, } m = 1, \\ \Omega \text{ is } C^{1,1},\ m \geq 2. \end{cases}$$

## 5 Construction of Dirichlet corrector

Let $\Phi_\epsilon = (\Phi_{\epsilon,j}) \in H^1(\Omega)$ be the Dirichlet corrector satisfying

$$\begin{cases} L_\epsilon(\Phi_{\epsilon,j}) = 0 & \text{in } \Omega, \\ \Phi_{\epsilon,j} = x_j & \text{on } \partial\Omega. \end{cases} \tag{9}$$

Note that $x_j$ itself also satisfies (9).

**Proposition 5.1.** *Assume that $m = 1$, then*

$$\|\Phi_{\epsilon,j} - x_j\|_{L^\infty(\Omega)} \le C\epsilon.$$

*Proof.* Let $u_\epsilon = x_j + \epsilon\chi_j(\frac{x}{\epsilon}) - \Phi_{\epsilon,j}(x)$, then

$$\begin{cases} L_\epsilon(u_\epsilon) = L_\epsilon(x_j + \epsilon\chi_j(\frac{x}{\epsilon})) - L_\epsilon(\Phi_{\epsilon,j}) = 0 & \text{in } \Omega, \\ u_\epsilon = \epsilon\chi_j(\frac{x}{\epsilon}) & \text{on } \partial\Omega. \end{cases}$$

By the maximum principle, we have

$$\begin{aligned} \|u_\epsilon\|_{L^\infty(\Omega)} \le \|u_\epsilon\|_{L^\infty(\partial\Omega)} &= \epsilon\|\chi_j(\frac{x}{\epsilon})\|_{L^\infty(\partial\Omega)} \\ &\le \epsilon\|\chi_j(\frac{x}{\epsilon})\|_{L^\infty(\partial\Omega)} \le C\epsilon, \end{aligned}$$

which implies

$$\|\Phi_{\epsilon,j} - x_j\|_\infty \le \|u_\epsilon\|_\infty + \|\epsilon\chi_j(\frac{x}{\epsilon})\|_\infty \le C\epsilon.$$

□

*Remark* 5.2. When $m \ge 2$, we consider

$$w_\epsilon = u_\epsilon - u_0 - \epsilon\chi_k(\frac{x}{\epsilon})S_\epsilon(\frac{\partial\widetilde{u_0}}{\partial x_k}),$$

where $\widetilde{u_0}$ is an extension of $u_0$ to $\mathbb{R}^d$ by 0 outside $\Omega$, and $S_\epsilon(\widetilde{u_0})(x) = \fint_{B_\epsilon(x)} \widetilde{u_0}$ is called the Steklov smoothing operator (the ideas were discovered by A. Suslina).

## 6 Uniform interior estimates

The uniform estimates are given by the compensated compactness argument.

**Theorem 6.1.** *Suppose $A = A(y)$ is elliptic and periodic. Let $u_\epsilon \in H^1(B(x_0, R))$ be a weak solution of $L_\epsilon(u_\epsilon) = F$ in $B = B(x_0, R)$. Suppose that $F \in L^p(B)$, $p > d$ and $0 < \epsilon < R$, then*

$$(\fint_B |\nabla u_\epsilon|^2)^{1/2} \le C\{(\fint_B |\nabla u_\epsilon|^2)^{1/2} + R(\fint_B |F|^p)^{1/2}\},$$

*where $C_p = C(d, p, \mu)$.*

**Theorem 6.2.** *(M. Avellenda, F. H. Lin, 1987) Suppose $A = A(y)$ is elliptic and periodic. $A(y)$ satisfies*

$$|A(y) - A(z)| \leq \lambda |y - z|^\tau, \ \forall y, z \in \mathbb{R}^d, \text{ for some } \tau > 0.$$

*Let $u_\epsilon \in H^1(B)$ be a weak solution of $L_\epsilon(u_\epsilon) = F$ in $B = B(x_0, R)$, where $F \in L^p(B)$, $p > d$. Then*

$$|\nabla u_\epsilon(x_0)| \leq C_p\{(\fint_B |\nabla u_\epsilon|^2)^{1/2} + R(\fint_B |F|^p)^{1/p}\},$$

*or*

$$\|\nabla u_\epsilon\|_{L^\infty(B_{R/2})} \leq C_p\{(\fint_B |\nabla u_\epsilon|^2)^{1/2} + R(\fint_B |F|^p)^{1/p}\},$$

*where $C_p = C(p, d, \mu\lambda, \tau)$.*

**Observations**:

1. Translation.
   If $-\nabla \cdot (A(\frac{x}{\epsilon})\nabla u_\epsilon) = F$ and $v_\epsilon = u_\epsilon(x - x_0)$, then

   $$-\nabla \cdot (\widetilde{A}(\frac{x}{\epsilon})\nabla v_\epsilon) = \widetilde{F},$$

   where $\widetilde{A}(y) = A(y + \frac{x_0}{\epsilon})$ and $\widetilde{F} = F(x - x_0)$.

2. Dilation.
   If $-\nabla \cdot (A(\frac{x}{\epsilon})\nabla u_\epsilon) = F$ and $v_\epsilon(x) = u_\epsilon(rx)$, $r > 0$, then

   $$L_{\frac{\epsilon}{r}}(v_\epsilon) = G,$$

   where $G(x) = r^2 F(rx)$.

Note that Theorem 1 implies Theorem 2 by using the blow-up arguments. By translation and dilation, we may assume $x_0 = 0$ and $R = 1$. Also, we may assume $\epsilon < \frac{1}{2}$ (if $\epsilon \geq \frac{1}{2}$, then $A(\frac{x}{\epsilon})$ is good, we only need to use the standard argument for elliptic regularity).

Let $w_\epsilon(x) = \epsilon^{-1}u_\epsilon(\epsilon x)$, then $L_1(w) = \epsilon F(\epsilon x)$ in $B(0, 1)$ (note that $\frac{1}{\epsilon} > 1$). By the interior Lipschitz estimates for $L_1$, we have

$$|\nabla w(0)| \leq C\{(\fint_{B(0,1)} |\nabla w|^2)^{1/2} + (\fint_{B(0,1)} |\epsilon F(\epsilon x)|^p)^{1/p}\}.$$

This implies

$$\begin{aligned}
|\nabla u_\epsilon(0)| &\leq C\{(\fint_{B(0,\epsilon)} |\nabla u_\epsilon|^2)^{1/2} + \epsilon(\fint_{B(0,\epsilon)} |F|^p)^{1/p}\} \\
&\leq C\{(\fint_{B(0,\epsilon)} |\nabla u_\epsilon|^2)^{1/2} + \epsilon^{1-\frac{d}{p}}(\fint_{B(0,1)} |F|^p)^{1/p}\} \\
&\leq C\{(\fint_{B(0,1)} |\nabla u_\epsilon|^2)^{1/2} + (\fint_{B(0,1)} |F|^p)^{1/p}\}.
\end{aligned}$$

*Remark* 6.3. Local property needs the regularity, but the global one does not in the homogenization theory.

**Theorem 6.4.** *(Compactness) Let $\{A^\ell(y)\}$ be a sequence of 1-periodic matrices satisfying the elliptic condition with the same $\mu$. Suppose $-\nabla\cdot(A^\ell(\frac{x}{\epsilon_\ell})\nabla u_\ell) = F_\ell$ in $\Omega$, where $\epsilon_\ell \to 0$. Also assume that $u_\ell \to u$ in $H^1(\Omega)$, $\overline{A^\ell} \to A^0$, and $F_\ell \to F$ in $H^{-1}(\Omega)$. Then $A^\ell(\frac{x}{\epsilon_\ell})\nabla u_\ell \rightharpoonup A^0\nabla u$ weakly in $L^2(\Omega)$ and $-\nabla\cdot(A^0\nabla u) = F$ in $\Omega$.*

*Proof.* The proof is similar to the proof of the homogenization theorem. □

# 7 Compensated compactness

Here we give the compactness argument, which contains three steps. The first is one-step improvement, the second is iteration, the last is blow-up argument.

**Lemma 7.1.** *(One-step improvement) Let $0 < \sigma < \rho < 1$ and $\rho = 1 - \frac{d}{p}$. There exist $\epsilon_0 \in (0, \frac{1}{2})$ and $\theta \in (0, \frac{1}{4})$ depending on $\mu, \sigma, \rho, d$ such that*

$$\begin{aligned}
&\Big(\fint_{B(0,\theta)} |u_\epsilon - \fint_{B(0,\theta)} u_\epsilon - x_j + \epsilon\chi_j(\frac{x}{\epsilon}))\fint_{B(0,\theta)} \frac{\partial u_\epsilon}{\partial x_j}|^2\Big)^{1/2} \\
\le& \theta^{1+\sigma}\max\{(\fint_{B(0,1)} |u_\epsilon|^2)^{1/2}, (\fint_{B(0,1)} |F|^p)^{1/p}\},
\end{aligned}$$

*where $0 < \epsilon < \epsilon_0$, $u_\epsilon \in H^1(B(0,1))$ is a solution of $L_\epsilon(u_\epsilon) = F$ in $B(0,1)$.*

*Proof.* Prove it by contradiction. Use the fact that if $-\nabla\cdot(A^0\nabla u) = F$ in $B(0,\frac{1}{2})$, then

$$\begin{aligned}
\sup_{|x|<\theta} |u - \fint_{B(0,\theta)} u - x_j \fint_{B(0,\theta)} \frac{\partial u}{\partial x_j}| &\le C\theta^{1+\rho}\|\nabla u\|_{C^{0,\rho}(B(0,\theta))} \\
&\le C\theta^{1+\rho}\|\nabla u\|_{C^{0,\rho}(B(0,\frac{1}{4}))} \\
&\le C_0\theta^{1+\rho}\max\{(\fint_{B(0,\frac{1}{2})} |u|^2)^{1/2}, (\fint_{B(0,\frac{1}{2})} |F|^p)^{1/p}\}.
\end{aligned}$$

Since $\sigma < \rho$, choose $\theta \in (0, \frac{1}{4})$ such that $2^{d+1}C_0\theta^{1+\rho} < \theta^{1+\sigma}$.

**Claim**: Suppose not, $\exists$ sequences $\{\epsilon_\ell\} \subset (0, \frac{1}{2})$ and $\{A^\ell(y)\}$ satisfying periodicity and ellipticity with $\mu$. $\{F_\ell\} \subset L^p(B(0,1))$, $\{u_\ell\{\subset H^1(B(0,1))$ such that for $\epsilon_\ell \to 0$, we have $-\nabla\cdot(A^\ell(\frac{x}{\epsilon_\ell})\nabla u_\ell) = F_\ell$, with $(\fint_{B(0,1)} |u_\ell|^2)^{1/2} \le 1$, $(\fint_{B(0,1)} |F_\ell|^p)^{1/p} \le 1$ and

$$\Big(\fint_{B(0,\theta)} |u_\ell - \fint_{B(0,\theta)} u_\ell - (x_j + \epsilon_\ell\chi_j^\ell(\frac{x}{\epsilon_\ell})\fint_{B(0,\theta)} \frac{\partial u_\ell}{\partial x_j}|^2\Big)^{1/2} > \theta^{1+\sigma},$$

where $\chi_j^\ell$ are correctors for $L_\epsilon^\ell = -\nabla \cdot (A^\ell(\frac{x}{\epsilon_\ell})\nabla)$. By Caccioplli's inequality, $\{u_\ell\}$ is bounded in $H^1(B(0,\frac{1}{2}))$. By passing to subsequences, we may assume that

$$\begin{aligned} &u_\ell \rightharpoonup u \text{ weakly in } L^2(B(0,1)),\ u_\ell \rightharpoonup u \text{ weakly in } H^1(B(0,1)), \\ &F_\ell \rightharpoonup F \text{ weakly in } L^p(B(0,1)),\ \text{and } \overline{A^\ell} \to A^0, \end{aligned}$$

which will imply $F_\ell \to F$ strongly in $H^{-1}(B(0,1))$ (since $L^p$ compactly embedded to $H^{-1}$ when $p > d$). By the compactness theorem, $-\nabla \cdot (A^0 \nabla u) = F$ in $B(0,\frac{1}{2})$. Also, $(\fint_{B(0,1)} |u|^2)^{1/2} \le 1$, $(\fint_{B(0,1)} |F|^p)^{1/p} \le 1$, and

$$(\fint_{B(0,\theta)} |u - \fint_{B(0,\theta)} u - x_j \fint_{B(0,\theta)} \frac{\partial u}{\partial x_j}|^2)^{1/2} \ge \theta^{1+\sigma},$$

which will lead to a weird conclusion. □

**Lemma 7.2.** *(Iteration) Let $0 < \sigma < \rho < 1$ and $\rho = 1 - \frac{d}{p}$. Let $(\epsilon_0, \theta)$ be given by the previous lemma. Suppose $0 < \epsilon < \theta^{k-1}\epsilon_0$ for some $k \ge 1$. Let $L_\epsilon(u_\epsilon) = F$ in $B(0,1)$. Then $\exists$ constants $E(\epsilon,\ell) = (E_j(\epsilon,\ell))$ $(1 \le \ell \le k)$ such that if $v_\epsilon = u_\epsilon - (x_j + \epsilon\chi_j(\frac{x}{\epsilon}))E_j(\epsilon,\ell)$, then*

$$(\fint_{B(0,\theta^\ell)} |v_\epsilon - \fint_{B(0,\theta^\ell)} v_\epsilon|^2)^{1/2} \le C\theta^{\ell(1+\sigma)} \max\{(\fint_{B(0,1)} |u_\epsilon|^2)^{1/2}, (\fint_{B(0,1)} |F|^p)^{1/p}\}.$$

*Moreover,*

$$|E(\epsilon,\ell)| \le C \max\{(\fint_{B(0,1)} |u_\epsilon|^2)^{1/2}, (\fint_{B(0,1)} |F|^p)^{1/p}\}$$

*and*

$$|E(\epsilon,\ell+1) - E(\epsilon,\ell)| \le C\theta^{\ell\sigma} \max\{(\fint_{B(0,1)} |u_\epsilon|^2)^{1/2}, (\fint_{B(0,1)} |F|^p)^{1/p}\}.$$

*Proof.* Prove by induction on $\ell$, for $1 \le \ell \le k$. When $\ell = 1$, by the previous lemma, $E_j(\epsilon,1) = \fint_{B(0,\theta)} \frac{\partial u_\epsilon}{\partial x_j}$. Suppose that the constants $E_j(\epsilon,i)$ exists, for all $1 \le i \le \ell$, $\ell \le k-1$. To construct $E(\epsilon,\ell+1)$, consider

$$\begin{aligned} w(x) = u_\epsilon(\theta^\ell x) &- \{\theta^\ell x_j + \epsilon\chi_j(\frac{\theta^\ell x}{\epsilon}\}E_j(\epsilon,\ell) \\ &- \fint_{B(0,\theta^\ell)} \{u_\epsilon - (x_j + \epsilon\chi_j(\frac{x}{\epsilon})\}E_j(\epsilon,\ell). \end{aligned}$$

Then $L_{\frac{\epsilon}{\theta^\ell}}(w) = \theta^{2\ell}F(\theta^{2\ell}x)$ in $B(0,1)$. Since $\frac{\epsilon}{\theta^\ell} \le \frac{\epsilon}{\theta^{k-1}} < \epsilon_0$, then we may apply the previous lemma to $w$ in $B(0,1)$ and repeat all the arguments again, which will finish Lemma 7.2. □

**Proof of the compactness theorem**: By translation and dilation, we may assume $x_0 = 0$ and $R = 1$, $0 < \epsilon < 1$. Also we assume that $0 < \epsilon < \epsilon_0\theta$ and choose $k \geq 2$ such that $\epsilon_0\theta^k \leq \epsilon < \epsilon_0\theta^{k-1}$. By the iteration lemma, we can get

$$(\fint_{B(0,\theta^{k-1})} |u_\epsilon - \fint_{B(0,\theta^{k-1})} u_\epsilon|^2)^{1/2} \leq C\theta^k\{(\fint_{B(0,1)} |u_\epsilon|^2)^{1/2} + (\fint_{B(0,1)} |F|^p)^{1/p}\},$$

after rescaling, the above can be rewritten as

$$(\fint_{B(0,\epsilon)} |u_\epsilon - \fint_{B(0,\epsilon)} u_\epsilon|^2)^{1/2} \leq C\epsilon\{(\fint_{B(0,1)} |u_\epsilon|^2)^{1/2} + (\fint_{B(0,1)} |F|^p)^{1/p}\},$$

for some constant $C$ independent of $\epsilon$. Therefore,

$$\begin{aligned}(\fint_{B(0,\epsilon)} |\nabla u_\epsilon|^2)^{1/2} &\leq \{\frac{C}{\epsilon}(\fint_{B(0,2\epsilon)} |u_\epsilon - \fint_{B(0,2\epsilon)} u_\epsilon|^2)^{1/2} + \epsilon(\fint_{B(0,2\epsilon)} |F|^2)^{1/2}\} \\ &\leq C\{(\fint_{B(0,1)} |u_\epsilon|^2)^{1/2} + (\fint_{B(0,1)} |F|^p)^{1/p}\}.\end{aligned}$$

**Theorem 7.3.** *(Dirichlet problem, Avellenda-Lin, 87) Assume that $A$ is elliptic, periodic and $|A(y) - A(z)| \leq \lambda|y - z|^\tau$. Let $\Omega \Subset \mathbb{R}^d$ be $C^{1,\alpha}$ Suppose*

$$\begin{cases} L_\epsilon(u_\epsilon) = F & \text{in } \Omega, \\ u_\epsilon = g & \text{on } \partial\Omega. \end{cases}$$

*Then*

$$\|\nabla u_\epsilon\|_{L^\infty(\Omega)} \leq C_{p,\sigma}\{\|F\|_{L^p(\Omega)} + \|g\|_{C^{1,\sigma}(\partial\Omega)}\},$$

*with $\sigma > 0$.*

*Proof.* Compactness and Lipschitz estimates for Dirichlet correctors

$$\begin{cases} L_\epsilon(\Phi_{\epsilon,j}) = 0 & \text{in } \Omega, \\ \Phi_{\epsilon,j} = x_j & \text{on } \partial\Omega. \end{cases}$$

Show that $\|\nabla\Phi_{\epsilon,j}\|_{L^2(\Omega)} \leq C$. □

**Theorem 7.4.** *(Neumann problem, Kenig-Lin-Shen, 2013) All assumptions are the same in Theorem 7.4. Let*

$$\begin{cases} L_\epsilon(u_\epsilon) = F & \text{in } \Omega, \\ \frac{\partial u_\epsilon}{\partial \nu_\epsilon} = g & \text{on } \partial\Omega, \end{cases}$$

*then*

$$\|\nabla u_\epsilon\|_{L^\infty(\Omega)} \leq C\{\|F\|_{L^p(\Omega)} + \|g\|_{C^\sigma(\partial\Omega)}\},$$

*where $p > d$, $\sigma > 0$.*

*Proof.* Compactness and Lipschitz estimates for the Neumann correctors. Let

$$\begin{cases} L_\epsilon(\Psi_{\epsilon,j}) = 0 & \text{in } \Omega, \\ \frac{\partial \Psi_{\epsilon,j}}{\partial \nu_\epsilon} = \dfrac{\partial}{\partial \nu_0}(x_j) & \text{on } \partial\Omega, \end{cases}$$

where $\dfrac{\partial u}{\partial \nu_0} = n_i\overline{a_{ij}}\dfrac{\partial u}{\partial x_j}$. Show that $\|\nabla\Psi_{\epsilon,j}\| \leq C$. □

## 8 Real variable method and $W^{1,p}$ estimates

For the simplest case,

$$\begin{cases} -\Delta u = \text{div} f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We have $\|\nabla u\|_{L^2(\Omega)} \le C\|f\|_{L^2(\Omega)}$ and $\|u\|_{H_0^1(\Omega)} \le C\|f\|_{L^2(\Omega)}$.
**Question**: Does $\|\nabla u\|_{L^p(\Omega)} \le C\|f\|_{L^p(\Omega)}$ hold for $1 < p < \infty$?

Similarly, for the homogenization problem, our main question is if

$$\begin{cases} L_\epsilon(u_\epsilon) = \text{div} f & \text{in } \Omega, \\ u_\epsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

we have $\|\nabla u_\epsilon\|_{L^2(\Omega)} \le C\|f\|_{L^2(\Omega)}$, does $\|\nabla u_\epsilon\|_{L^p(\Omega)} \le C\|f\|_{L^p(\Omega)}$ hold for $1 < p < \infty$?

**Theorem 8.1.** *(Interior $W^{1,p}$ estimate) Suppose that $A$ is elliptic, periodic and continuous (VMO will be fine). Let $L_\epsilon(u_\epsilon) = divf$ in $B(x_0, 2R)$. Then for $2 < p < \infty$, we have*

$$\left(\fint_{B(x_0,R)} |\nabla u_\epsilon|^p\right)^{1/p} \le C_p\left\{\left(\fint_{B(x_0,2R)} |\nabla u_\epsilon|^2\right)^{1/2} + \left(\fint_{B(x_0,2R)} |f|^p\right)^{1/p}\right\},$$

*where $C_p$ depends only on $d, \mu$ and $\omega(t) = \sup\{|A(y) - A(z)|; |y - z| \le t\}$.*

**Lemma 8.2.** *Let $L_1(u) = 0$ in $B(x_0, 2r)$ for $0 < r < 1$. Then*

$$\left(\fint_{B(x_0,r)} |\nabla u|^p\right)^{1/p} \le C_p\left(\fint_{B(x_0,2r)} |\nabla u|^2\right)^{1/2}.$$

*Proof.* This is a local $W^{1,p}$ estimate. □

**Lemma 8.3.** *Suppose $A$ is elliptic, periodic and continuous. Let $L_\epsilon(u_\epsilon) = 0$ in $B(x_0, 2R)$. Then for $2 < p < \infty$, we have*

$$\left(\fint_{B(x_0,R)} |\nabla u_\epsilon|^p\right)^{1/p} \le C_p\left(\fint_{B(x_0,2R)} |\nabla u_\epsilon|^2\right)^{1/2}.$$

*Proof.* By translation and dilation, we may assume $x_0 = 1$, $R = 1$. If $\epsilon \ge \frac{1}{4}$, this follows from Lemma 8.2, since $|A(\frac{x}{\epsilon}) - A(\frac{y}{\epsilon})| \le \omega(|\frac{x}{\epsilon} - \frac{y}{\epsilon}|) \le \omega(4|x-y|) \le \omega(4t)$, when $|x - y| \le t$.

Let $0 < \epsilon < \frac{1}{4}$, consider $w(x) = u_\epsilon(\epsilon x)$, then $L_1(w) = 0$. By Lemma 8.2, $(\fint_{B(0,1)} |\nabla w|^p)^{1/p} \le C_p(\fint_{B(0,2)} |w|^2)^{1/2}$, which implies

$$\left(\fint_{B(0,\epsilon)} |\nabla u_\epsilon|^p\right)^{1/p} \le C_p\left(\fint_{B(0,2\epsilon)} |\nabla u_\epsilon|^2\right)^{1/2} \le C_p\left(\fint_{B(0,1)} |\nabla u_\epsilon|^2\right)^{1/2}.$$

By translation,

$$\left(\fint_{B(x_0,\epsilon)} |\nabla u_\epsilon|^p\right)^{1/p} \le C_p\left(\fint_{B(x_0,\epsilon)} |\nabla u_\epsilon|^2\right)^{1/2} \le C_p\left(\fint_{B(0,2)} |\nabla u_\epsilon|^2\right)^{1/2},$$

for all $x_0 \in B(0,1)$. Therefore, we have

$$\int_{B(x_0,\epsilon)} |\nabla u_\epsilon|^p \le C\epsilon^d (\fint_{B(0,2)} |\nabla u_\epsilon|^2)^{p/2},$$

and

$$\int_{B(0,1)} |\nabla u_\epsilon|^p \le C(\fint_{B(0,2)} |\nabla u_\epsilon|^2)^{p/2}.$$

□

The ideas of real variable method come from the Calderon-Zygmund decomposition.

**Theorem 8.4.** *(A real variable theorem) Let $B_0$ be ball in $\mathbb{R}^d$ and $F \in L^2(4B_0)$. Let $q > 2$ and $f \in L^p(4B)$ for $2 < p < q$. Suppose that for each ball $B \subset 2B_0$ with $|B| \le c_1|B_0|$, there exist two functions $F_B$ and $R_B$ on $2B$ such that*

1. $|F| \le |F_B| + |R_B|$ *on* $2B$,
2. $(\fint_{2B} |R_B|^q)^{1/q} \le N_1\{(\fint_{4B} |F|^2)^{1/2} + \sup_{B\subset B'\subset 4B_0} (\fint_{B} |f|^2)^{1/2}\}$,
3. $(\fint_{2B} |F_B|^2)^{1/2} \le N_2 \sup_{B\subset B'\subset 4B_0} (\fint_{B'} |f|^2)^{1/2}$.

*Then*

$$(\fint_{B_0} |F|^p)^{1/p} \le C\{(\fint_{4B_0} |F|^2)^{1/2} + (\fint_{4B_0} |f|^p)^{1/p}\},$$

*where $C$ depends on $N_1, N_2, C_1, p, q, d$.*

*Proof.* (Proof of interior $W^{1,p}$) Let $F = |\nabla u_\epsilon|$, $f = f$. Show

$$(\fint_{B(x_0,R)} |F|^p)^{1/p} \le C\{(\fint_{B(x_0,2R)} |F|^2)^{1/2} + (\fint_{B(x_0,2R)} |f|^P)^{1/p}\},$$

For each $B$ with $4B \subset B(x_0, 2R)$, we write $u_\epsilon = v_\epsilon + w_\epsilon$,where

$$\begin{cases} L_\epsilon(v_\epsilon) = \text{div} f & \text{in } \Omega = 4B, \\ v_\epsilon = 0 & \text{on } \partial\Omega = \partial(4B). \end{cases}$$

Let $F_B = |\nabla v_\epsilon|$, $R_B = |\nabla w_\epsilon|$. Clearly, $|F| \le F_B + R_B$ in $2B$ and by the energy estimate, we have

$$(\fint_{4B} |F_B|^2)^{1/2} = (\fint_{4B} |\nabla v_\epsilon|^2)^{1/2} \le C(\fint_{4B} |f|^2)^{1/2}.$$

Note that $L_\epsilon(w_\epsilon) = 0$ in $4B$ and take $q = p + 1$. By Lemma 8.3, we have $(\fint_{2B} |\nabla w_\epsilon|^q)^{1/q} \le C(\fint_{4B} |\nabla w_\epsilon|^2)^{1/2}$. This implies that

$$\begin{aligned}(\fint_{2B} |R_B|^q)^{1/q} &\le C(\fint_{4B} |R_B|^2)^{1/2} \le C(\fint_{4B} |\nabla v_\epsilon|^2)^{1/2} + C(\fint_{4B} |\nabla u_\epsilon|^2)^{1/2} \\ &\le C(\fint_{4B} |f|^2)^{1/2} + C(\fint_{4B} |F|^2)^{1/2}.\end{aligned}$$

Take $B' = B$ and apply the real variable theorem, we can obtain the $W^{1,p}$ estimate. □

# 9 Singular integrals

Finally, we discuss the singular integrals, which will be useful to the homogenization theorem.

Let $Tf(x) = \text{p.v.} \int_{\mathbb{R}^d} K(x,y)f(y)dy$, where $K(x,y)$ is the Calderon-Zygmund kernel satisfying

$$\begin{cases} |K(x,y)| \le \dfrac{C}{|x-y|^d}, \\ |K(x,y) - K(x,z)| \le \dfrac{C|x-z|^\sigma}{|x-y|^{d+\sigma}}, \quad |x-z| < \dfrac{1}{2}|x-y|, \\ |K(y,x) - K(z,x)| \le \dfrac{C|x-z|^\sigma}{|x-y|^{d+\sigma}}, \quad |x-z| < \dfrac{1}{2}|x-y|. \end{cases}$$

$T$ is called a Calderon-Zygmund operator if $T$ is bounded on $L^2(\mathbb{R}^d)$ and $T$ is associated with CZ kernel ($K(x,y) = C_d \dfrac{x_j - y_j}{|x-y|^{d+1}}$).

**Lemma 9.1.** *(Calderon-Zygmund) All CZ operators are bounded on $L^p(\mathbb{R}^d)$ for $1 < p < \infty$, and is of weak type $(1,1)$.*

Use the Marcinkiewitz interpolation, we have $1 < p < \infty$ and duality argument, we have $2 < p < \infty$.

**Theorem 9.2.** *Let $T$ be a bounded sublinear operator on $L^2(\mathbb{R}^d)$ and $q > 2$. Suppose that*

$$(\fint_B |T(g)|^q)^{1/q} \le N\{(\fint_{2B} |Tg|^2)^{1/2} + \sup_{B \subset B'} (\fint_{B'} |g|^2)^{1/2}\}$$

*for all $B \subset \mathbb{R}^d$ and for all $g \in C_0^\infty(\mathbb{R}^d)$ with supp$(g) \subset \mathbb{R}^d \backslash 4B$. Then*

$$\|T(f)\|_{L^p(\mathbb{R}^d)} \le C\|f\|_{L^p(\mathbb{R}^d)}, \ \forall 2 < p < q.$$

*Remark* 9.3. In the previous theorem, we did not use any conditions on CZ kernel.

# 10 Exercises

Consider the second order elliptic operator

$$L_\epsilon = -\nabla \cdot (A(\frac{x}{\epsilon})\nabla),$$

where $\epsilon > 0$. In the following problems we always assume that the coefficients matrix

$$A = A(y) = (a_{ij}(y))_{d \times d}$$

is real, bounded measurable, and satisfies the ellipticity condition

$$\frac{1}{\mu}|\xi|^2 \le a_{ij}(y)\xi_i\xi_j \le \mu|\xi|^2 \text{ for } y \in \mathbb{R}^d \text{ and } \xi \in \mathbb{R}^d,$$

where $\mu > 0$, and periodicity condition

$$A(y+z) = A(y) \text{ for } y \in \mathbb{R}^d \text{ and } z \in \mathbb{Z}^d.$$

The summation convention that the repeated indices are summed is used.

1. Define the weak solution of the Dirichlet problem

$$L_\epsilon(u_\epsilon) = F \text{ in } \Omega \text{ and } u_\epsilon = f \text{ on } \partial\Omega,$$

where $\Omega$ is a bounded Lipschitz domain, $F \in H^{-1}(\Omega)$ and $f \in H^{1/2}(\partial\Omega)$. Prove that the existence and uniqueness of the weak solutions.

2. Define the weak solutions of the Neumann problem

$$L_\epsilon(u_\epsilon) = F \text{ in } \Omega \text{ and } \frac{\partial u_\epsilon}{\partial \nu_\epsilon} = g on \partial\Omega,$$

where $\Omega$ is a bounded Lipschitz domain, $F \in H_0^{-1}(\Omega)$ and $g \in H^{-1/2}(\partial\Omega)$. Prove the existence and uniqueness of the weak solutions.

3. Show that if $L_\epsilon(u_\epsilon) = \text{div} f$ in $B(x_0, 2r)$, then

$$\int_{B(x_0 r)} |\nabla u_\epsilon|^2 dx \leq C\{\frac{1}{r^2}\int_{B(x_0,2r)} |u_\epsilon|^2 dx + \int_{B(x_0,2r)} |f|^2 dx\}.$$

4. Show that there exists a 1-periodic function $u$ such that $u \in H^1_{loc}(\mathbb{R}^d)$ and

$$L_1(u) = -L_1(x_j) \text{ in } \mathbb{R}^d.$$

Such $u$, unique up to constants, are called the correctors and denote by $\chi_j$. The homogenized matrix $\overline{A} = (\overline{a_{ij}})$ is defined by

$$\overline{a_{ij}} = \int_{\mathbb{T}^d} \{a_{ij} + a_{ik}\frac{\partial \chi_j}{\partial x_k}\} dy.$$

5. Find the correctors $\overline{A}$, defined in Problem 4, in the case $d = 1$ (Signolo example).

6. Finish the proofs of Lemma 3.3 and Lemma 3.4 (the Div-Curl lemma).

7. Let

$$S_\epsilon(u) = \fint_{B(x,\epsilon)} u(y) dy.$$

Prove that

$$\|gS_\epsilon(u)\|_{L^p(\mathbb{R}^d)} \leq \sup_{x\in\mathbb{R}^d} (\fint_{B(x,\epsilon)} |g|^p)^{1/p} \|u\|_{L^p(\mathbb{R}^d)}$$

for any $1 \leq p < \infty$.

8. Let $u \in H^1(\mathbb{R}^d)$. Show that

$$\|S_\epsilon(u) - u\|_{L^2(\mathbb{R}^d)} \leq \epsilon \|\nabla u\|_{L^2(\mathbb{R}^d)}.$$

9. Suppose that $u_\epsilon \in H^1_0(\Omega)$ and $-\nabla \cdot (A(\frac{x}{\epsilon})\nabla u_\epsilon) = F$ in $\Omega$. Show that $\epsilon \to 0$,

$$\int_\Omega A(\frac{x}{\epsilon})\nabla u_\epsilon \cdot (\nabla u_\epsilon)\varphi dx \to \int_\Omega \overline{A}\nabla u_0 \cdot (\nabla u_0)\varphi dx$$

for any $\varphi \in C_0^\infty(\Omega)$, where $u_0$ is the solution of the homogenized problem.

10. Let $u_\epsilon$ be the weak solution of

$$\begin{cases} (\partial_t + L_\epsilon)u_\epsilon = F & \text{in } \Omega \times (0,T), \\ u_\epsilon = 0 & \text{on } \partial\Omega \times (0,T), \\ u_\epsilon = f & \text{on } \Omega \times \{t = 0\}, \end{cases}$$

where $F \in L^2(\Omega\times(0,T))$, $f \in L^2(\Omega)$ and $\Omega$ is a bounded Lipschitz domain. Show that $u_\epsilon \to u_0$ strongly in $L^2(\Omega \times (0,T))$ and $A(\frac{x}{\epsilon})\nabla u_\epsilon \rightharpoonup \overline{A}\nabla u_0$ weakly in $L^2(\Omega \times (0,T))$, where $u_0$ is the solution of

$$\begin{cases} (\partial_t + L_0)u_0 = F & \text{in } \Omega \times (0,T), \\ u_0 = 0 & \text{on } \partial\Omega \times (0,T), \\ u_0 = f & \text{on } \Omega \times \{t = 0\}, \end{cases}$$

and $L_0 = -\nabla \cdot (\overline{A}\nabla)$.