Liapunov-Schmidt reduction

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1 Tools.

Definition 1. Let X and Y be Banach spaces. A bounded linear operator $L : X \to Y$ is called $Fredholm$ if the following two conditions hold.
(a) Ker$L$ is a finite-dimensional subspace of X .
(b) Range$L$ is a closed subspace of Y of finite codimension.

Definition 2. If L is Fredholm, the *index* of L is the integer $i(L) = dim(kerL)-codim(rangeL)$.

Proposition 3. *If $L : X \to Y$ is Fredholm, then there exists closed subspaces M and N of X and Y, respectively, such that*
(a) $X = kerL \oplus M$ and (b) $Y = N \oplus rangeL$.

Remark 4. In the following discussion, we assume L is $Fredholm$ and its index is zero. By the definition of the index, it is easy to see that $dim(kerL) = dimN$, where N is the subspace of Y introduced in the proposition3. If $kerL = \{0\}$, then L is onto and hence, by the closed graph theorem, L is invertible. Thus, we have the following implication for Fredholm operators of index zero: If $kerL = \{0\}$, then L is invertible.

2 Liapunov-Schmidt reduction.

Let $\Phi : X \times R^{k+1} \to Y$, $\Phi(0, 0) = 0$ be a smooth mapping between Banach spaces. We want to use the Liapunov-Schmidt reduction to solve the equation $\Phi(u, \alpha) = 0$ for u as a function of α near $(0, 0)$. Let L be the differential of Φ at the origin; in symbols

$Lu = \lim_{h \to 0} \frac{\Phi(u,0) - \Phi(0,0)}{h}$. (note that the linear operator L is the $Frechet$ derivative of Φ)

We assume that L is the Fredholm operator with index zero.

We have the L-S reduction in the following several steps :

1. Decompose X and Y:
   (a) $X = kerL \oplus M$.
   (b) $Y = N \oplus rangeL$.
   ***Reason**: The hypothesis that L is Fredholm guarantees that the above splittings are possible. Moreover, $kerL$ and N are finite-dimensional.

2. Split the equation $\Phi(u, \alpha) = 0$ into an equivalent pair of the equations:
   (a) $E\Phi(u, \alpha) = 0$
   (b) $(I - E)(u, \alpha) = 0$
   where $E : Y \to rangeL$ is the projection associated to the splitting in 2(b).
   ***Reason**: This is primarilyh notational and requires no comment.

3. Use the equation 1(a) $X = kerL \oplus M$: to write $u = v + w$, where $v \in kerL$ and $w \in M$. Apply the implicit function theorem to solve $E\Phi(u, \alpha) = 0$ for w as a function of v and α. This leads to a function $W : kerL \times R^{k+1} \to N$ such that $E\Phi(v + W(v, \alpha), \alpha) = 0$.
   ***Reason**: We want to show that the implicit function theorem is applicable. We extract a map $F : kerL \times M \times R^{k+1} \to rangeL$ from 2(a); i.e., $F(v, w, \alpha) = E\Phi(v + w, \alpha)$. This differential of F with respect to w at the origin is $EL = L$. Now we argue that $L : M \to rangeL$ is invertible.(For simplicity, I don't want to say too much details in this notes.)

4. Define $\phi : kerL \times R^{k+1} \to N$ by $\phi(v, \alpha) = (I - E)\Phi(v + W(v, \alpha), \alpha)$.
   ***Reason**: This is primarily notational and requires no comment.

5. Choose a basis $v_1, v_2, ...v_n$ for $kerL$ and a basis $v_1^*, v_2^*, ...v_n^*$ for $(rangeL)^\perp$. Define $g : R^n \times R^{k+1} \to R^n$ by $g_i(x, \alpha) =< v_i^*, \phi(x_1v_1 + \cdots + x_nv_n, \alpha) >$.
   ***Reason**: In the writing $(rangeL)^\perp$ we are using (for the first time) the fact that Y is equipped with the inner product in the L^2-sense (i.e. we write $< u, v >= \int_\Omega u(x)v(x)dx$). Since L is Fredholm with index zero, $dimkerL = dim(rangeL)^\perp$ and both dimensions are finite. Thus the bases for $kerL$ and $(rangeL)^\perp$ contain the same number of vectors.

We summarize the outcome of the Liapunov-Schmidt reduction.

Proposition 5. *If the linearization of $\Phi(u, \alpha) = 0$ is a Fredholm operator of index zero, then solutions of this equations are (locally) in one-to-one correspondence with solutions of the finite system $g_i(x, \alpha) = 0$, $i = 1, 2, \cdots n$. where g_i is defined by $g_i(x, \alpha) =< v_i^*, \phi(x_1 v_1 + \cdots + x_n v_n, \alpha) >$.*