On the inviscid limit problem for viscous incompressible flows in the half plane - Approach from vorticity formulation

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Abstract

The note is mainly for personal record, if you want to read it, please be careful. This Notes was taken when Prof. Yasunori Maekawa (Tohoku University, Japan) visiting Taiwan. I followed his lecture and took the note.

Overviews:

$$\text{(NS)}\quad \begin{cases} \partial_t u + u\cdot\nabla u - \nu\Delta u + \nabla p = 0 & t>0, x\in\mathbb{R}^2_+ \\ \nabla\cdot u = 0 & t\geq 0, x\in\mathbb{R}^2_+ \\ u=0 & t\geq 0, x\in\partial\mathbb{R}^2_+ \\ u=a & t=0, x\in\mathbb{R}^2_+ \end{cases}$$

where $\mathbb{R}^2_+ = \{(x_1,x_2)|x_2>0\}$, $u=(u_1,u_2)$: velocity, $a=(a_1,a_2)$: initial velocity, p: pressure and $\nu>0$: kinematic viscosity (constant). $u=0$ on $\partial\mathbb{R}^2_+$: no-slip B.C.. $\omega = \partial_1 u_2 - \partial_2 u_1 := \mathrm{Rot}u$: vorticity. Note that $\mathrm{Rot}\Delta u$=$\Delta\mathrm{Rot}u$, $\mathrm{Rot}(u,\nabla u) = u\cdot\nabla\mathrm{Rot}u$ (since $\nabla\cdot u=0$), $\mathrm{Rot}(\nabla p)=0$. Then these imply

$$\partial_t\omega + u\cdot\nabla\omega - \nu\Delta\omega = 0, t>0, x\in\mathbb{R}^2_+.$$

- Vorticity is useful if there is no boundary (e.g. $\mathbb{R}^2$, T^2), which is well-studied.
- Behavior near the boundary? No-slip B.C. on u is a source of the vorticity on the boundary.
  Basic question: B.C. on ω? (vorticity formulation of (NS)): Topic 1.
  For typical flows, ν is very small (for water 20°C, $\nu\sim 1.0\times 10^{-6}\ m^2/s$).

Problem: Behavior of $u=u^{(\nu)}$ in the limit $\nu\to 0$. Formally we obtain the Euler equation: For $u_E=(u_{E,1},u_{E,2})$,

$$\text{(E)}\quad \begin{cases} \partial_t u_E + u_E\cdot\nabla u_E + \nabla p_E = 0 & t>0, x\in\mathbb{R}^2_+ \\ \nabla\cdot u_E = 0 & t\geq 0, x\in\mathbb{R}^2_+ \\ u_{E,2}=0 & t\geq 0, x\in\partial\mathbb{R}^2_+ \\ u_E=a & t=0, x\in\mathbb{R}^2_+ \end{cases}.$$

In general, $u_{E,1}\neq 0$ on $\partial\mathbb{R}^2_+$, while $u^{(\nu)}=0$ and $a=0$ on $\partial\mathbb{R}^2_+$.
$\Rightarrow$Boundary layer appears.

Formal estimate of the boundary layer thickness δ (Parndtl 1904).
$\partial_t u_1 + u\cdot\nabla u_1 - \nu\Delta u_1 + \partial_1 p = 0$, $\partial_1 u_1 + \partial_2 u_2 = 0$. Near the boundary, we expect $u_1, \partial_1 u_1, \partial_1^2 u_1, \partial_t u_1 \sim O(1)$, $\partial u_2 \sim O(\frac{1}{\delta})$, $\partial_2^2 u_1 \sim O(\frac{1}{\delta^2})$, $\partial_1 p \sim O(1)$. From $\nabla\cdot u = 0$, we have $\partial_2 u_2 \sim O(1) \Rightarrow u_2 \sim O(\delta)$, $u_2\partial_2 u_1 \sim O(\delta)\cdot O(\frac{1}{\delta}) = O(1)$, $u_1\partial_1 u_1 \sim O(1)$. $\Rightarrow \frac{\nu}{\delta^2} \sim O(1)$, $\therefore \delta = \nu^{1/2}$.

Near the boundary $u^{(\nu)} \sim u_p^{(\nu)} = (v_{p,1}(t, x_1, \frac{x_2}{\nu^{1/2}}), u_{p,2}(t, x_1, \frac{x_2}{\nu^{1/2}}))$, $x = (x_1, X_2)$ where $X_2 = \frac{x_2}{\nu^{1/2}}$.

$$(\mathrm{P})\quad \begin{cases} \partial_t v_{p,1} - \partial_{X_2}^2 v_{p,1} + v_p\cdot\nabla_x v_{p,1} + \partial_1\pi_p = 0,\ t>0, x\in\mathbb{R}_+^2 \\ \nabla_x\cdot v_p = 0, \partial_{X_2}\pi_p = 0, v_p|_{t=0} = 0 \\ v_p|_{X_2=0} = 0, \lim_{X_2\to 0} v_{p,1} = u_{E,1}|_{x_2=0}, \lim_{X_2\to\infty}\pi_p = p_E|_{x_2=0} \end{cases}.$$

We want to justify $u^{(\nu)} \sim \begin{cases} u_E, & \frac{x_2}{\nu^{1/2}} \gg 1 \\ u_p^{(\nu)}, & \frac{x_2}{\nu^{1/2}} \ll 1 \end{cases}$ at least locally in time. Sammartino-Caflisch (CMP, 98): Given data (initial velocity) a is analytic. This result is not applicable for, e.g., $a \in C_{0,\sigma}^\infty(\mathbb{R}_+^2)$.

Now, we justify $u^{(\nu)} \sim \begin{cases} u_E, & \frac{x_2}{\nu^{1/2}} \gg 1 \\ u_p^{(\nu)}, & \frac{x_2}{\nu^{1/2}} \ll 1 \end{cases}$ when the initial vorticity b =Rot$a$ satisfies dist($\partial\mathbb{R}_+^2$,supp$b$)>0 and Sobolev regularity: Topic 2.

1 Vorticity formulation for NS equation in $\mathbb{R}_+^2$

1.1 Derivation

Vorticity field $\omega = \partial_1 u_2 - \partial_2 u_1$ and $\nabla\cdot u = 0$ imply $-\Delta u = \nabla^\perp\omega$, where $\nabla^\perp = (\partial_2, -\partial_1)$. Biot-Savart law gives $u = \nabla^\perp(-\Delta_D)^{-1}\omega$, where $h = (-\Delta_D)^{-1} f$ is the solution of $\begin{cases} -\Delta h = f & \text{in } \mathbb{R}_+^2 \\ h = 0 & \text{on } \partial\mathbb{R}_+^2 \end{cases}$, Note that $\nabla\cdot\nabla^\perp(-\Delta_D)^{-1}\omega = 0$, Rot$\nabla^\perp(-\Delta_D)^{-1}\omega = \omega$, $\gamma_{\partial\mathbb{R}_+^2}\partial_1(-\Delta_D)^{-1}\omega = 0$ (trace).
(Derivation of vorticity B.C.)
Requirement: $\gamma_{\partial\mathbb{R}_+^2}\partial_2(-\Delta_D)^{-1}\omega = 0$.

In $\mathbb{R}_+^2$, ω satisfies $\partial_t\omega + u\cdot\nabla\omega - \nu\Delta\omega = 0$ in $\mathbb{R}_+^2$, therefore,

$$\begin{aligned} 0 &= \partial_t\gamma_{\partial\mathbb{R}_+^2}\partial_e(-\Delta_D)^{-1}\omega \\ &= \gamma_{\partial\mathbb{R}_+^2}\partial_2(-\Delta_D)^{-1}\partial_t\omega \\ &= \gamma_{\partial\mathbb{R}_+^2}\partial_2(-\Delta_D)^{-1}(\nu\Delta\omega - u\cdot\nabla\omega) \\ &= \gamma_{\partial\mathbb{R}_+^2}\partial_2(-\Delta_D)^{-1}(\nu\Delta(\omega-\omega_{har}) - u\cdot\nabla\omega) \\ &= -\nu\gamma_{\partial\mathbb{R}_+^2}(\omega-\omega_{har}) - \gamma_{\partial\mathbb{R}_+^2}\partial(-\Delta_D)^{-1}(u\cdot\nabla\omega) \\ &= -\nu\gamma_{\partial\mathbb{R}_+^2}\partial_2\omega - \nu(-\partial_1^2)^{1/2}\gamma_{\partial\mathbb{R}_+^2}\omega - \gamma_{\partial\mathbb{R}_+^2}\partial_2(-\Delta_D)^{-1}(u\cdot\nabla\omega) \end{aligned}$$

where ω_{har} satisfies $\begin{cases} \Delta\omega_{har} = 0 & \text{in } \mathbb{R}^2_+ \\ \omega_{har} = \omega & \text{on } \partial\mathbb{R}^2_+ \end{cases}$ and the last equation comes from the Dirichlet-to-Neumann map on the half-plane property. $\therefore \nu(\partial_2\omega + (-\partial_1^2)^{1/2}\omega) = -\partial(-\Delta_D)^{-1}(u\cdot\nabla\omega)$ on $\partial\mathbb{R}^2_+$.

$$\text{(V)} \begin{cases} \partial_t\omega + M(\omega,\omega) - \nu\Delta\omega = 0 & t>0, x\in\mathbb{R}^2_+ \\ \nu(\partial_2\omega + (-\partial_1^2)^{1/2}\omega) = N(\omega,\omega) & t>0, x\in\partial\mathbb{R}^2_+ \\ \omega = \text{Rot}a & t=0, x\in\mathbb{R}^2_+ \end{cases}$$

where $M(f,g) = J(f)\cdot\nabla g$, $J(f) = \nabla^\perp(-\Delta_D)^{-1}f$, $N(f,g) = -\gamma_{\partial\mathbb{R}^2_+}J_1(M(f,g))$. Then (NS)$\Leftrightarrow$(V) + $\gamma_{\partial\mathbb{R}^2_+}a = 0$ (Note that we have used the fractional Laplacian, so the nonlocal property automatically holds).

1.2 Solution formula for linearization

$$\text{(LV)} \begin{cases} \partial_t\omega - \nu\Delta\omega = f & t>0, x\in\mathbb{R}^2_+ \\ \nu(\partial_2\omega + (-\partial_1^2)^{1/2}\omega = g & t>0, x\in\partial\mathbb{R}^2_+ \\ \omega = b & y=0, x\in\mathbb{R}^2_+ \end{cases}$$

and we have$\widehat{(-\partial_1^2)^{1/2}\omega}(\xi) = |\xi_1|\widehat{\omega}(\xi,\cdot)$ and $b=0$ for simplicity, where $\widetilde{\omega}(s,\xi_1,x_2) = \frac{1}{(2\pi)^{1/2}}\int_0^\infty\int_{\mathbb{R}}\omega(t,x)e^{-ix_1\xi_1 - ts}dx_1dt$. Then we have

$$\begin{cases} \partial_2^2\widetilde{\omega} - (\frac{s}{\nu} + \xi^2)\widetilde{\omega} = -\frac{1}{\nu}\widetilde{f} & x_2 > 0 \\ \partial_2\widetilde{\omega} + |\xi_1|\widetilde{\omega} = \frac{\widetilde{g}}{\nu} & x_2 = 0, \widetilde{\omega}\to 0\ (x_2\to\infty). \end{cases}$$

$G(t,x) = \frac{1}{4\pi t}e^{-|x|^2/4t}$, $E(x) = -\frac{1}{2\pi}\log|x|$, $(h_1\star h_2)$ is the standard convolution, and $(h_1 * h_2) = \int_{\mathbb{R}^2_+} h_x(x-y^*)h_2(y)dy$, where $y^* = (y_1,-y_2)$. $h*(gH^1_{\partial\mathbb{R}^2_+})(x) = \int_{\mathbb{R}} h(x_1-y_1,x_2)g(y_1)dy_1$, $\Gamma(t,x) = 2(\partial_1^2 + (-\partial_1^2)^{1/2}\partial_2)(E*G(t))(x)$, $e^{t\Delta_N}f = G(t)*f + G(t)\star f$, $\Gamma(0)\star f = \lim_{t\to 0}\Gamma(t)\star f = 2(\partial_1^2 + (-\partial_1^2)^{1/2}\partial_2)E\star f$.

Theorem 1.1. *The solution formula for (LV) is given by*

$$\begin{aligned} \omega(t) &= e^{\nu tB}b - \Gamma(0)\star b + \int_0^t e^{\nu(t-s)B}(f(s) - g(s)H^1_{\partial\mathbb{R}^2_+})ds \\ &\quad - \int_0^t \Gamma(0)\star(f(s) - g(s)H^1_{\partial\mathbb{R}^2_+})ds \end{aligned}$$

where $e^{\nu tB} := e^{\nu t\Delta_N} + \Gamma(\nu t)\star$.

Refer to "Solution formula for the Stokes equation: Solonnikov (AMS Transl 68(, Ukai (CPAM 87). "

Proposition 1.2. *If* $g = \gamma_{\partial\mathbb{R}^2_+}J_1(f) = \gamma_{\partial\mathbb{R}^2_+}\partial_2(-\Delta_D)^{-1}f$, *then* $\Gamma(0)\star(f - gH^1_{\partial\mathbb{R}^2_+}) = 0$ *in* $\mathbb{R}^2_+$. *In particular, if* $\gamma_{\partial\mathbb{R}^2_+}J_1(b) = 0$, *then* $\Gamma(0)\star b = 0$ *in* $\mathbb{R}^2_+$.

(Note: $\nabla\cdot a = 0$, b=Rot$a$, $\gamma_{\partial\mathbb{R}^2_+}a = 0 \Rightarrow \gamma_{\partial\mathbb{R}^2_+}J(b) = 0$.

Proof. (Prop 1.2)

$$\begin{aligned} E*(f-gH^1_{\partial\mathbb{R}^2_+}) &= \int_{\mathbb{R}^2_+} E(x-y^*)f(y)dy + \int_{\partial\mathbb{R}^2_+} E(x-y^*)\partial_{11}(-\Delta_D)^{-1} f d\nu^1_y \\ &= \int_{\mathbb{R}^2_+} \nabla_y E(x-y^*)\cdot\nabla_y(-\Delta_D)^{-1} f dy \\ &= -\int_{\mathbb{R}^2_+} \Delta_y E(x-y^*)f(y)dy = 0 \end{aligned}$$

if $x\in\mathbb{R}^2_+$. □

Proposition 1.3. *$\Gamma(\nu t)*b-\Gamma(0)*b = -\nu\int_0^t \Im G(\nu s)ds*b$ in $\mathbb{R}^2_+$ and $\Im = 2(\partial_1^2+(-\partial_1^2)^{1/2}\partial_2)$.*

Proof. RHS$=\int_0^t \Im(-\Delta_{\mathbb{R}^2})^{-1}\partial_\xi(G(\nu s))ds*b = \Im E*G(\nu t)*b-\Im E*b$. □

1.3 Properties of $\{e^{tB}\}_{t\geq 0}$

Set $\dot W^{1.q}_{0,\sigma}(\mathbb{R}^2_+) = \overline{C^\infty_{0,\sigma}(\mathbb{R}^2_+)}^{\|\nabla a\|_{L^q}}$, $1<q<\infty$. $X_q=\{\mathrm{Rot}a\in L^q(\mathbb{R}^2_+)| a\in \dot W^{1,q}_{0,\sigma}(\mathbb{R}^2_+)\}$.

Theorem 1.4. *$\{e^{tB}\}_{t>0}$ defines a C_0-analytic semigroup in X_q. Moreover, its generator B_q is given by*

$$D(B_q)=\{f\in X_q\cap W^{2,q}(\mathbb{R}^2_+)|\gamma_{\partial\mathbb{R}^2_+}(\partial_2 f+(-\partial_1^2)^{1/2}f)=0\},$$

and $B_q f=\Delta f$ when $f\in D(B_q)$. Moreover, $\|\nabla^2 f\|_{L^q}\leq C\|B_q f\|_{L^q}$, $f\in D(B_q)$.

2 Inviscid limit problem

2.1 Result

$$\text{(NS)}\quad \begin{cases} \partial_t u^{(\nu)} + u^{(\nu)}\cdot\nabla u^{(\nu)} - \nu\Delta u^{(\nu)} + \nabla p^{(\nu)} = 0 & t\in(0,T), x\in\mathbb{R}^2_+ \\ \nabla\cdot u = 0 & t\in(0,T), x\in\mathbb{R}^2_+ \\ u=0 & t\in(0,T), x\in\partial\mathbb{R}^2_+ \\ u=a & t=0, x\in\mathbb{R}^2_+ \end{cases}$$

v_p: velocity of the Prandtl flow, $\widetilde v_p=(\widetilde v_{p,1},\widetilde v_{p,2})$, $\widetilde v_{p,1}=v_{p,1}-\gamma_{\partial\mathbb{R}^2_+}u_{E.1}$,$\widetilde v_{p.2}=\int_{x_2}^\infty \partial_1\widetilde v_{p,1}dY_2$.
(Velocity of the modified Prandtl flow)
$\widetilde u^{(\nu)}_{p,1}(t,x)=\widetilde v_{p,1}(t,x_1,\frac{x_2}{\nu^{1/2}})$, $\widetilde u^{(\nu)}_{p,2}(t,x)=\nu^{1/2}\widetilde v_{p,2}(t,x_1,\frac{x_2}{\nu^{1/2}})$.

Theorem 2.1. *(M., to appear in CPAM)*
Let $a\in\dot W^{1,p}_{0,\sigma}(\mathbb{R}^2_+)\cap W^{4.2}(\mathbb{R}^2_+)$ for some $1<p<2$ and $b=\partial_1 a_2-\partial_2 a_1\in W^{4,1}(\mathbb{R}^2_+)\cap W^{4,2,}(\mathbb{R}^2_+)$. Assume that $d_0=$dist($\partial\mathbb{R}^2_+$,suppb)$>$0 . Then $\exists T,C>0$ such that

$$\sup_{0<t<T}\|u^{(\nu)}(t)-u_E(t)-\widetilde u^{(\nu)}_p(t)\|_{L^\infty}\leq C\nu^{1/2}$$

for sufficiently small $\nu>0$. The time T is estimated from below as $T\geq c\min\{d_0,1\}$ with $c>0$ depending only on $\|b\|_{W^{4,1}\cap W^{,4,2}}$.

(Away the boundary, (NS) is estimated by Euler equation,; Near the boundary, (NS) is estimated by Prandtl equation)
We assume $0 < d_0 \ll 1$, for simplicity.
Note: $\omega_E = \mathrm{Rot} u_E \in C^1([0,T] \times \overline{\mathbb{R}^2_+}) \cap L^\infty(0,T; W^{4,1} \cap W^{4,2})$ $\forall T > 0$ satisfies $\partial_t \omega_E + u_E \cdot \nabla \omega_E = 0$.
In particular, $\cup_{0<t<T_0} \mathrm{supp}\omega_{E(t)} \subset \{x \in \mathbb{R}^2_+ | x_2 \geq \frac{d_0}{2}\}$ for $T_0 = c_0 d_0$ with some $C_0 > 0$ depending only on $\|b\|_{W^{4,1} \cap W^{4,2}}$.

- Key observations for the proof of theorem 2.1:
  (1) Analyticity of the data near the boundary (from the Prandtl)$\Rightarrow$ Local solvability of the Prandtl equation (cf. Sammartino-Caflisch (CMP '98), Lombardo-Cannone-Sammartino (SIMA JMA '03), Kukavica-Vicol (CMS '13).
  (2) Exponential smallness in ν^{-1} of the vorticity in the region between the boundary layer and suppω_E $\Rightarrow$ Small direct interaction between the vorticity of the outer flow and the vorticity created on the boundary.

We make the ansatz $\omega^{(\nu)} = \omega_E + R_{\frac{1}{\nu}} \omega_p + R_{\frac{1}{\nu}} \omega^{(\nu)}_{IP} + \omega^{(\nu)}_{IE}$, the last two are remainder terms, $(R_{\frac{1}{\nu}} f)(t,x) = \frac{1}{\nu^{1/2}} f(t, x_2, \frac{x_2}{\nu^{1/2}})$.

$$(V_E) \begin{cases} \partial_t \omega_E + M(\omega_E, \omega_E) = 0 \\ w_E|_{t=0} = \mathrm{Rot} a \end{cases}$$

where $M(f,g) = J(f) \cdot \nabla g$, $J(f) = \nabla^\perp (-\Delta_D)^{-1} f$.

$$(V_P) \begin{cases} \partial_t \omega_p - \partial^2_{X_2} \omega_p = -v_p \cdot \nabla_X \omega_p & t > 0, x \in \mathbb{R}^2_+ \\ \partial_t \omega_p = -\int_0^\infty v_p \cdot \nabla_X \omega_p dY_2 + N(\omega_E, \omega_E) & \text{on the boundary} \\ \omega_p|_{t=0} = 0 \end{cases}$$

2.2 Function spaces

- $\mu, \rho, \theta \geq 0$: parameters.
- ξ_1: wave number in x_1 direction.
- $X_2 = \nu^{-1/2} x_2$.
- $(\alpha)_+ = \max(\alpha, 0)$, $\alpha \in \mathbb{R}$.

$\varphi^{(\mu,\rho)}_{p,\nu} = \exp\left(\frac{(\mu - \nu^{1/2} X_2)_+ |\xi_1|}{4} + \rho X_2^2 \right)$,
$\varphi^{(\mu,\rho)}_{E,\nu}(\xi_1, X_2) = \exp\left(\frac{(\mu - x_2)_+}{4} |\xi_1| + \frac{\theta}{\nu} (\frac{d_0}{8} - x_2)^2_+ \right)$,
$\hat{f}(\xi_1, x_2) = \mathcal{F}_{x_1 \to \xi_1}[f(\cdot, x_2](\xi_1)$, $\|\hat{f}\|_{L^p_{\xi_1} L^q_{x_2}} = (\int_{\mathbb{R}} (\int_0^\infty |\hat{f}(\xi_1, x_2)|^q dx_2)^{p/q} d\xi_1)^{1/p}$,
$\|f\|_{X^{(\mu,p)}_{IP_{\nu,0}}} = \sum_{k=0,1} \|\varphi^{(\mu,\rho)}_{p,\nu} X_2^{k/2} \hat{f}\|_{L^2_{\xi_1} L^{1+k}_{x_2}}$, $\|f\|_{X^{(\mu,p)}_{IP_{\nu,j}}}$, $j = 1,2$: additional Sobolev regularity. $\|f\|_{X^{(\mu,p)}_{p,2}} := \|f\|_{X^{(\mu,p)}_{IP_0,2}}$ $(\nu = 0)$, $\|f\|_{X^{(\mu\,\theta)}_{IE_\nu,0}} = \|\varphi^{(\mu,\theta)}_{E,\nu} \hat{f}\|_{L^2_{\xi_1}, L^2_{x_2}} + \|\varphi^{(0,\theta)}_{E,\nu} f\|_{L^1_x}$, $\|f\|_{X^{(\mu,\theta)}_{IE_\nu,j}}$, $j = 1,2$: additional Sobolev regularity.

Theorem 2.2. *$\exists C, T, \mu, \rho, \theta > 0$ such that*

$$\sup_{0<t<T} \|\omega_p(t)\|_{X_p^{(\mu \frac{p}{t})}} \leq C,$$

$$\sup_{0<t<T} (\|\omega_{IP}^{(\nu)}(t)\|_{X_{IP_{\nu,1}}^{(\mu, \frac{\rho}{t})}} + \|\omega_{IE}^{(\nu)}(t)\|_{X_{IE_{\nu,1}}^{(\mu, \frac{\theta}{t})}}) \leq C\nu^{1/2}.$$

$u^{(\nu)} = J(\omega^{(\nu)})$.

2.3 Biot-Savart law

Lemma 2.3. *We have*

$$\begin{aligned}\mathcal{F}(\partial_1(-\Delta_D)^{-1}f)(\xi_1, x_2) &= \frac{1}{2}\frac{i\xi_1}{|\xi_1|}\{\int_0^\infty e^{-|\xi_1|(x_2-z_2)}(1-e^{-2|\xi_1|z_2})\hat{f}(\xi_1, z_2)dz_2 \\ &\quad + \int_0^\infty e^{-|\xi_1|(z_2-x_2)}(1-e^{-2|\xi_1|x_2})\hat{f}(\xi_1, z_2)dz_2\},\end{aligned}$$

$$\begin{aligned}\mathcal{F}(\partial_2(-\Delta_D)^{-1}f)(\xi_1, x_2) &= \frac{1}{2}\{-\int_0^{x_2} e^{-|\xi_1|(x_2-z_2)}(1-e^{-2|\xi_1|z_2})\hat{f}(\xi_1, z_2)dz_2 \\ &\quad + \int_{x_2}^\infty e^{-|\xi_1|(z_2-x_2)}(1+e^{-2|\xi_1|x_2})\hat{f}(\xi_1, z_2)dz_2\}.\end{aligned}$$

Note: If $\text{supp} f \subset \{x \in \mathbb{R}^2 | x_2 \geq m > 0\}$, then for $x_2 < m$, we have

$$\begin{aligned}|\mathcal{F}[\partial_j(-\Delta_d)^{-1}f](\xi_1, x_2)| &\leq c\int_m^\infty e^{-|\xi_1|(z_2-x_2)}|\hat{f}(\xi_1, z_2)|dz_2 \\ &\leq e^{-|\xi_1|(m-x_2)}\int_0^{|\infty} |\hat{f}(\xi_1, z_2)|dz_2.\end{aligned}$$

- Key ingredient of the proof:
  (1) Solution formula for linearized (vorticity) problem. Weighted norm estimates for linear and bilinear maps.
  (2) Pointwise estimate for the Green function of $\partial_t - \nu\Delta + u \cdot \nabla$ under the Neumann boundary condition.
  (3) Abstract Cauchy-Kowalevsky theorem.

- Key propositions

Proposition 2.4. *Let $0<s<t$, $\mu > 0$, $0 < \rho, \theta \ll 1$. Then*

$$\|R_\nu e^{\nu(t-s)\Delta_N} R_{\frac{1}{\nu}} f\|_{X_{IP_{\nu,o}}^{(\mu, \frac{p}{t})}} \precsim \|f\|_{X_{IP_{\nu,0}}^{(\mu, \frac{p}{s})}},$$

$$\|e^{\nu(t-s)\Delta_N} f\|_{X_{IE_{\nu,0}}^{(\mu, \frac{\theta}{t})}} \precsim \|f\|_{X_{IE_{\nu,o}}^{(\mu, \frac{\theta}{s})}}.$$

Proof. Set $g(t, X_2) = \frac{1}{(4\pi t)^{1/2}} e^{-x_2^2/4t}$, $g_N(t, X_2, Y_2) = g(t, X_2 - Y_2) + g(t, X_2 + Y_2)$. Then

$$\begin{aligned}\mathcal{F}[R_\nu e^{\nu(t-s)\Delta_N} R_{\frac{1}{\nu}} f](\xi_1, X_2) &= e^{-\nu(t-s)\xi_1^2}\int_0^\infty g_N(t-s, X_2, Y_2)\hat{f}(\xi_1, Y_2)dY_2 \\ &= e^{-\nu(t-s)\xi_1^2}\int_0^\infty g_N(t-s, X_2, Y_2)e^{-\frac{1}{4}(\mu-\nu^{1./2}Y_2)_+} \\ &\quad \times e^{|\xi_1|}\varphi_{p,\nu}^{(\mu,0)}\hat{f}dY_2.\end{aligned}$$

Therefore,

$$
\begin{aligned}
|\varphi_{p,\nu}^{(\mu,\frac{p}{t})}\mathcal{F}[R_\nu e^{\nu(t-s)\Delta_N}R_{\frac{1}{\nu}}f](\xi_1,X_2) &\precsim \frac{e^{\frac{p}{t}X_2^2}}{(t-s)^{1/2}}\int_0^\infty e^{-\nu(t-s)\xi_1^2-\frac{|X_2-Y_2|^2}{4(t-s)}} \\
&\times e^{\frac{(\mu-\nu^{1/2}X_2)_+-(\mu-\nu^{1/2}Y_2)_+}{4}|\xi_1|}|\varphi_{p,\nu}^{(\mu,0)}\hat{f}|dY_2.
\end{aligned}
$$

By

$$
\begin{aligned}
\mu-\nu^{1/2}X_2)_+|\xi_1| &\leq (\mu-\nu^{1/2}Y_2)_+|\xi_1|+\nu^{1/2}|X_2-Y_2||\xi_1| \\
&\leq (\mu-\nu^{1/2}Y_2)_+|\xi_1|+\nu(t-s)\xi_1^2+\frac{|X_2-Y_2|^2}{4(t-s)} \\
&\precsim \frac{e^{\frac{p}{t}X_2^2}}{(t-s)^{1/2}}\int_0^\infty e^{-\frac{3}{4}\nu(t-s)\xi_1^2}e^{-\frac{|X_2-Y_2|^2}{8(t-s)}}|\varphi_{p,\nu}\hat{f}|dY_2,
\end{aligned}
$$

$\therefore$Young-type inequality with weights gives

$$
\|\varphi_{p,\nu}^{(\mu,\frac{p}{t})}\mathcal{F}[R_\nu e^{\nu(t-s)\Delta_N}R_{\frac{1}{\nu}}f]\|_{L^1_{X_2}}\precsim\|\varphi_{p,\nu}^{(\nu,\frac{p}{s})}\hat{f}(\xi_1)\|_{L^1_{X_2}}
$$

and

$$
\|\varphi_{p,\nu}^{(\mu,\frac{p}{t})}\mathcal{F}[R_\nu e^{\nu(t-s)\Delta_N}R_{\frac{1}{\nu}}f]\|_{L^2_{\xi_1}L^1_{X_2}}\precsim\|\varphi_{p,\nu}^{(\nu,\frac{p}{s})}\hat{f}(\xi_1)\|_{L^2_{\xi_1}L^1_{X_2}}.
$$

□

$\omega^{(\nu)}=\omega_E+R_{\frac{1}{\nu}}\omega_p+R_{\frac{1}{\nu}}\omega_{IP}^{(\nu)}+\omega_{IE}^{(\nu)}$, the last term is the remainder of the outer flow.

$$
(V_{IE_\nu})\begin{cases}\partial_t\omega_{IE}-\nu\Delta\omega_{IE}+M(\omega^{(\nu)},\omega_{IE})=H^{(\nu)} \\ \partial_2\omega_{IE}=0 & \text{on the boundary} \\ \omega_{IE}|_{t=0}=0\end{cases}
$$

and $M(\omega^{(\nu)},\omega_{IE})=u^{(\nu)}\cdot\nabla\omega_{IE}$, $\nu(\partial_2\omega^{(\nu)}+(-\partial_1^2)^{1/2}\omega^{(\nu)})$ =Nonlinear $\Rightarrow\partial_2\omega_{IE}=0$.

$$
\begin{aligned}
H^{(\nu)} &= -M(R_{\frac{1}{\nu}}\omega_{IP}^{(\nu)}+\omega_{IE}^{(\nu)},\omega_E)+\nu\Delta\omega_E \\
&= -J(R_{\frac{1}{\nu}}\omega_{IP}^{(\nu)}+\omega_{IE}^{(\nu)})\cdot\nabla\omega_E+\nu\Delta\omega_E,
\end{aligned}
$$

$\therefore\cup_{0<t<T}\mathrm{supp}H^{(\nu)}(t)\subset\{x\in\mathbb{R}^2|x_2\geq\frac{d_0}{2}\}$.

Proposition 2.5. $\omega_{IE}^{(\nu)}=\int_0^t\int_{\mathbb{R}^2_+}P_{u^{(\nu)}}^{(\nu)}(t,x;s,y)H^{(\nu)}(s,y)dyds$, $0<P_{u^{(\nu)}}^{(\nu)}(t,x;s,y)\leq\frac{1}{2\pi\nu(t-s)}\exp\left(-\frac{(|x-y|-\int_s^t\|u^{(\nu)}(t)\|_{L^\infty})_+^2}{4\nu(t-s)}\right)$.

(From Carlen-Loss '95 Duke)

Exponential decay in ν^{-1} of ω_{IE} near boundary is observed as follows:

$$
\begin{aligned}
|\omega_{IE}^{(\nu)}(t,x)| &\precsim \int_0^t\int_{\frac{d_0}{2}}^\infty\frac{1}{\nu(t-s)}\exp\left(-\frac{(|x-y|-\int_s^t\|u^{(\nu)}(t)\|_{L^\infty})_+^2}{4\nu(t-s)}\right) \\
&\times|H^{(\nu)}|dyds.
\end{aligned}
$$

$\therefore$ If $x_2 \leq \dfrac{d_0}{4}$ and $0 < t < T = O(d_0) \ll 1$, $\sup_{0<t<T} \|u^{(\nu)}(t)\|_{L^\infty} \leq C$, then

$$|\omega_{IE}(t,x)| \precsim d_0^{-2} e^{-\frac{d_0^2}{8\nu t}} \int_0^t \|H^{(\nu)}\|_{L^1} \, ds.$$

For the third part, Cauchy-Kowalevsky theorem. (Nirenberg '72, Nishida '77 ...)