Radon Transform
An Introduction

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The Radon transform is widely applicable to tomography, the creation of an image from the scattering data associated to cross-sectional scans of an object. If a function f represents an unknown density, then the Radon transform represents the scattering data obtained as the output of a tomographic scan. Hence the inverse of the Radon transform can be used to reconstruct the original density from the scattering data, and thus it forms the mathematical underpinning for tomographic reconstruction. The Radon transform data is often called a sinogram because the Radon transform of a Dirac delta function is a distribution supported on the graph of a sine wave. Consequently the Radon transform of a number of small objects appears graphically as a number of blurred sine waves with different amplitudes and phases. The Radon transform is useful in computed axial tomography (CAT scan), electron microscopy of macromolecular assemblies like viruses and protein complexes, reflection seismology and in the solution of hyperbolic partial differential equations. [wikipedia]

Definition 1. Given a function $f(x)$, Radon transform produces the value of $\int_L f(x)dx$ along all the lines L. Define $Rf(L) = \int_L f(x)dx$. If we rewrite the lines L in coordinates, that is, $L = \{x|x \cdot \omega = s\}$, then we have $Rf(\omega, s) = \int_{x \cdot \omega = s} f(x)dx$, where $(\omega, s) \in S^1 \times \mathbb{R}$

The Radon transform has the following properties:

1. **Evenness**
   $Rf(\omega, s) = Rf(-\omega, -s)$, we can identify the two points (ω, s) to

$(-\omega, -s)$.
$g = Rf$ is defined on the Mobius strip: $S^1 \times \mathbb{R}/\sim$

2. **Shift invariance**
   Let $(T_a f)(x) = f(x+a)$, and $(t_p g)(\omega, s) = g(\omega, s+p)$
   Then $R(T_a f)(\omega, s) = t_{a\cdot\omega} Rf = Rf(\omega, s + a\cdot\omega)$.

3. **Rotation invariace**
   A is a rotation operator in $\mathbb{R}^2$, then $R(f(Ax))(\omega, s) = Rf(A\omega, s)$.

Corollary 2. *Fourier transform methods might be useful.*

Definition 3. We call Q is a ridge function if Q can be written as the following form: $Q(x) = q(x\cdot\omega)$.

Proposition 4. *Q is a ridge function, i.e. $Q(x) = q(x\cdot\omega)$. Then we have the following identity: $\int_{\mathbb{R}^2} f(x)Q(x)dx = \int_{\mathbb{R}} Rf(\omega, s)q(s)ds$.*

Proof. For $Q(x) = e^{ix\cdot\xi} = e^{i\sigma\omega\cdot x}$, where $\sigma = |x|$, and $\omega = (cos\theta, sin\theta)$, then we can get the above identity. □

Theorem 5. *(**Projection-Slice formula** or **Fourier-Slice formula**) Let $\hat{f}$ be the Fourier transform of f, and $\hat{g}(\omega, \sigma)$ be the Fourier transform of g w.r.t. the parameter s. Then we have $\hat{f}(\sigma\omega) = \frac{1}{\sqrt{2\pi}}\hat{g}(\omega, \sigma)$, where $g = Rf$ is the Radon transform of f.*

Dual Radon Transform

Definition 6. $R : f \to Rf(\omega, s)$ is a Radon transform, we call $R^{\#} : g(\omega, s) \to R^{\#}g(x)$ is a dual Radon transform, such that

$$\int Rf(\omega, s)g(\omega, s)d\omega ds = \int f(x)R^{\#}g(x)dx.$$

Lemma 7. *We have the formula $R^{\#}g(x) = \int_{S^1} g(\omega, x\cdot\omega)d\omega$, where S^1 is the unit circle in $\mathbb{R}^1$.*

Remark 8. We call $R^{\#}$ is the backprojection operator.

The line with normal coordinates $(\omega, x\cdot\omega)$ are passing through x. This explains the name backprojection operator.

Theorem 9. *(L^2 – continuity)*

Let Ω be the unit disc on the plane, f supported inside, then we have the estimate $||Rf||_{L^2(C,\frac{1}{\sqrt{1-s^2}})} \leq k||f||_{L^2}$.

where $||Rf|||^2_{L^2(C,\frac{1}{\sqrt{1-s^2}})} = \int_C |Rf(\omega,s)|^2 \frac{1}{\sqrt{1-s^2}} d\omega ds$, *k is a constant and C is a cylinder with* $C = S^1 \times \mathbb{R}$.

Remark 10. In $\mathbb{R}^n (n > 2)$, one can take $d-$dimesional Radon transform fo any $1 \leq d < n$: $f(x) \to \int_H f(x)dx$, where H is a $d-$dimensional subsapce.

Example 11. In $\mathbb{R}^3$:

1. $d = 2$, Radon transform $\int_{x \cdot \omega = s} f(x)dx$
2. $d = 1$, X-ray transform $\int_{-\infty}^{\infty} f(x + t\omega)dt$.

Fourier inversion

Projection-slice formula gives an inversion procedure:

$f(x) \Longrightarrow Rf(\omega,s) \Longrightarrow \widehat{Rf}(\omega,\sigma) = \sqrt{2\pi}\widehat{f}(\sigma\omega) \underset{IFT}{\Longrightarrow} f(x)$, where IFT denotes the inverse Fourier transform.

Fourier inversion in polar coordinates

Let's elaborate on Fourier inversion, denoting $g = Rf$:

$f(x) = \frac{1}{2\pi} \int e^{ix\cdot\xi} \widehat{f}(\xi) d\xi = \frac{1}{2\pi} \int_{|\omega|=1} \int_0^{\infty} e^{i\sigma\omega\cdot x} \widehat{f}(\sigma\omega)|\sigma| d\sigma d\omega = \frac{1}{4\pi} \int_{|\omega|=1} \int_{-\infty}^{\infty} \cdots = \frac{1}{4\pi} \int_{|\omega|=1} d\omega (\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\sigma\omega\cdot x} \widehat{g}(\omega,\sigma)|\sigma| d\sigma)$, here we need to use the projection-slice formula to change the function f into g. Moreover, by using Hilbert transform, we have that $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\sigma\omega\cdot x} \hat{g}(\omega,\sigma)|\sigma| d\sigma = (H\frac{dg}{ds})(\omega, x\cdot\omega)$, where H is the Hilbert transform and $(Hg)(s) = \frac{1}{\pi} v.p. \int_{-\infty}^{\infty} \frac{g(\tau)}{s-\tau} d\tau$, then we can write the above identities as $f(x) = \frac{1}{4\pi} R^{\#}(H\frac{d}{ds}g)(x) = \frac{1}{4\pi} R^{\#} H \frac{d}{ds} g(x)$. Note that $\widehat{H\frac{dg}{ds}}(\omega,\sigma) = |\sigma|\widehat{g}(\omega,\sigma)$.

Theorem 12. *(FBP inversion)*

$$f(x) = \frac{1}{4\pi} R^{\#} H \frac{d}{ds} Rf, \textit{ where } H\frac{d}{ds} \textit{ is a filtration, } R^{\#} \textit{ is a backprojection.}$$

Remark 13. Radon transform is not invertible. It is just left invertible. Left inverse operators are not unique.

Range

Q: Is a given function $g(\omega, s)$ Radon transform for some f ? (appropriate function classes assumed)

Ex: If $g(\omega, s) \in C^\infty(S^1 \times [-1, 1])$, can we find some f such that $g = Rf$?

A: Let $g = Rf$.

1. $g(\omega, s) = g(-\omega, -s)$
2. consider kth **moment** $G_k(\omega) = \int_{\mathbb{R}} s^k g(\omega, s) ds$ - function on the unit sphere.
   $G_k(\omega) = \int_{-\infty}^{\infty} s^k ds \int_{x\cdot\omega=s} f(x)dx = \int_{\mathbb{R}^2} (x \cdot \omega)^k f(x)dx$ homogeneous polynomial of degree k in ω.

Theorem 14. *A function $g \in S(C)$, where $S(C)$ is the Schwartz space over $C = S^1 \times \mathbb{R}$. $g = Rf$ for some $f \in S(\mathbb{R}^2)$ if and only if it satisfies the following* ***range condtions****:*

1. **Evenness**. $g(\omega, s) = g(-\omega, -s)$
2. **Moment condtions**. For any $k = 0, 1, 2, \ldots,$ $G_k(\omega)$ extends to a homogeneous polynomial of degree k of $\omega \in \mathbb{R}^2$.

Remark 15. $g(\omega, s) \to \widehat{g}(\omega, \sigma) \to \widehat{f}(\sigma\omega) = c\widehat{g}(\omega, \sigma) \to f(x)$, since polar coordinate is not a nice coordinate. Note that $\sigma\omega = (-\sigma)(-\omega)$, if we want the identity $\widehat{f}(\sigma\omega) = c\widehat{g}(\omega, \sigma)$ is well-defined, we need the evenness condition.

Range revisited in Fourier domain Consider 1D FT $\widehat{g}(\omega, \sigma)$ of $g = Rf$. Get 2D FT of f along radial lines:

$\widehat{f}(\sigma\omega) = c\widehat{g}(\omega, \sigma)$. Notice the role of evenness!

Function $h(\sigma\omega) = \widehat{g}(\omega, \sigma)$ is smooth everywhere, possibly except the origin.

Stability of inversion

Stability: Small variations in data lead to small changes in the result.

Theorem 16. *(Stability estimate for Radon transform in 2D)*

D is a disk in $\mathbb{R}^2$, $f \in H_0^s(D)$, (i.e. $\int |\widehat{f}(\xi)|^2(1+|\xi|^2)^s < \infty$), $C_1||Rf||_{H^{s+1/2}} \leq ||f||_{H^s} \leq C_2||Rf||_{H^{s+1/2}}$.

Conclusion:

X-ray transform smoothes functions by 1/2 derivatives. Same with the dual. Thus, two of them add a derivative, which must be removed during inversion. (remember the filter $H\dfrac{d}{ds}$?)

More generally, d-plane transform adds $\frac{2}{d}$ derivatives, thus in FBP d derivatives need to be removed by filtration.

Instability of inverting smoothing operators

Rule of thumb:

If an operator A increases smoothness of functions, its inversion is unstable. More smoothing leads to more instability. The extreme cases are when A turns non-smooth functions into C^∞ or even worse, analytic functions.

Incomplete data

What if the values of $Rf(\omega, s)$ are known for a set of points (ω, s) only?

1. Is the reconstruction possible (uniqueness) ?

2. Reconstruction procedures.

3. Stability.

4. Any resulting image deterioration ?

Limited angle

Theorem 17. *(Data known for an open set of* ω*s and all* s *)*

Any compactly supported function is uniquely reconstructed.

Remark 18. Reconstruction is unstable.