Spectral Graph Theory and its Applications

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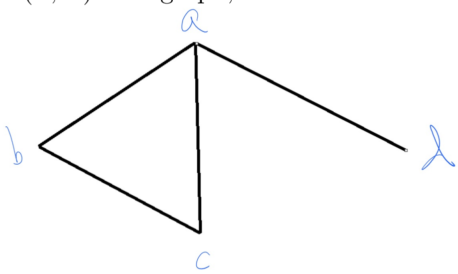
Abstract

This notes were given in a series of lectures by Prof. Fan Chung in National Taiwan University.

1 Introduction

1.1 Basic notations

Let $G = (V, E)$ be a graph, where V is a vertex set and E is an edge set.



(Graph 1)

We denote the edge set $E = \{\{a, b\}, \{b, c\}, \cdots\}$.

Definition 1.1. We say that $\{u, v\} \in E$

$$\begin{aligned} &\text{if and only if } u \sim v, \\ &\text{if and only if } u \text{ is adjacent to } v, \\ &\text{if and only if } u \text{ is a neighborhood of } v. \end{aligned}$$

Moreover, we let

$$\begin{aligned} N(v) &:= \{x \in V | x \sim v\}, \\ d_v &:= \text{degree of } v = |N(v)|. \end{aligned}$$

$(d_v)_{v\in V}$ denotes the degree of sequence.

Example 1.2. In graph 1, we say $(d_v)_{v\in V} = \{d_a, d_b, d_c, d_d\} = \{3, 2, 2, 1\}$.

Theorem 1.3. *(Erdos-Gallai, 1960) Let $(a_i)_{i=1,2,\cdots,n}$ be a degree of sequence of a graph $a_i \geq a_{i+1}$ if and only if*

$$\sum_{i=1}^{r} a_i \leq r(r-1) + \sum_{i=r+1}^{n} \min\{r, a_i\}.$$

Proof. ($\Rightarrow$) Trivial, since when the graph has r vertices, it will be less than $r(r-1)$; outside r-graph part, it will be less than the second part of the inequality.

($\Leftarrow$) Exercise ! $\square$

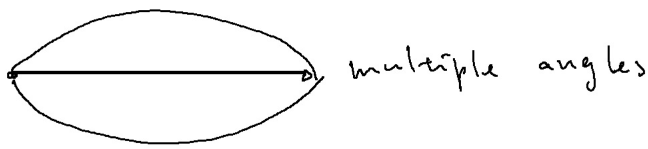
There are many points of view of the graph theory. The graph theory will involve topology, algebra, analysis and probability. It also can be applied to the computer science (algorithm, complexity), network science (big data) and optimization.

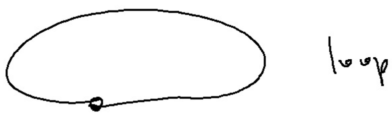
1.2 Classification

There are many types of graphs. We give brief definitions in the following.

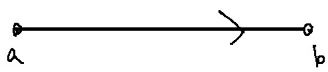
Definition 1.4. We call G to be an undirected graph if $E = \{\text{unordered pair}\}$. We call G is a simple graph if G satisfies

1.undirected,
2.no multiple angles,
3.no loops .

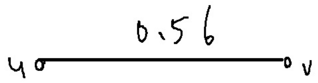




The directed graph will be denoted by



The hypergraph: $E = \{\text{subsets}\}$.
The weighted graph: $w(e) \in \mathbb{R}^+ \cup \{0\}$.



Let $G = (V, E)$, A be an adjacency matrix, which means

$$A(u,v) = \begin{cases} 1 & \text{if } u \sim v, \\ 0 & \text{otherwise.} \end{cases}$$

It is not hard to see $d_u = \sum_v A(u,v)$. Path $u = u_0, u_1, \cdots, u_t = v$, where $u_i \sim u_{i+1}$. We call it to be a simple path if all $u_i's$ are distinct. Denote $A^t(u,v) =\#$ of paths from u to v of length t.

Exercise 1.5. $\#$ of k paths in $G = (V,E) \geq n\overline{d}^{-k}$, where $n = |V|$, $\overline{d}$ =average degree= $\dfrac{\sum_u d_u}{n}$. For instance, when $k = 2$, 2-paths, by using the Cauchy-Schwarz inequality, we have

$$\sum d_i^2 \geq n(\sum \frac{d_i}{n})^2.$$

We also have

$$\# \text{ of paths of length } t \text{ in } G = \left\langle \mathbf{1}, A^t \mathbf{1} \right\rangle,$$

where $\mathbf{1} = (1, 1, \cdots, 1)$.

Exercise 1.6. Let ρ_0 =spectral radius of A (the largest eigenvalue of A). Prove that

$$\rho_0 = \lim_{k\to\infty} \left(\frac{\langle \mathbf{1}, A^k \mathbf{1} \rangle}{\langle \mathbf{1}, \mathbf{1} \rangle} \right)^{\frac{1}{k}}.$$

1.3 Random walk on G

In this section, we consider G to be undirected graphs. Let $P(u,v)$ =the probability of matrix to u from v (transition probability matrix).

$$P(u,v) = \begin{cases} \frac{1}{d_u}, & \text{if } u \sim v \\ 0, & \text{otherwise.} \end{cases} = \frac{1}{d_u} A(u,v) = (D^{-1}A)(u,v).$$

Note that $P = D^{-1}A$ with D being a diagonal degree matrix, $D(u,v) = d_v$ (not symmetric in general).

Let $f : V \to \mathbb{R}^+ \cup \{0\}$ be a probability distribution and $\sum_v f(v) = 1$.

Example 1.7. $f_0 = \chi_u$, where $\chi_u(v) = \begin{cases} 1, & \text{if } v = u \\ 0, & \text{otherwise.} \end{cases}$

There are two versions to describe the graph. In the previous example, after one step the probability becomes

$$\sum_v \chi_u(z) P(z,v) = (\chi_u, P)(v).$$

After t steps, the probability from u reaching v is $\chi_u P^t(v)$. Here we use the row vector multiplying with the matrix.

Example 1.8. Uniform distribution $\longleftrightarrow$Stationary distribution. $(\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n}) = \frac{\mathbf{1}}{n}$. Does $\lim_{t\to\infty} fP^t$ exists ? unique ? If the existence holds, then we have

$$\lim_{t\to\infty} fP^t = \Pi,$$

where Π is the stationary distribution (ergodic).

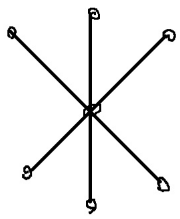
Example 1.9. G is k-regular, $d_u = k$ and $\frac{1}{n}\mathbf{1} \cdot P = \frac{1}{n}\mathbf{1}$, where P is the probability matrix.

Example 1.10. Let $G = G_1 \cup G_2$ be disjoint union, where G_1 is k_1-regular, G_2 is k_2-regular. Then the limit $\Pi = \Pi_1 \times \Pi_2$, where

$$\Pi_1 = (\frac{1}{k_1}, \cdots, \frac{1}{k_1}, 0, \cdots, 0),$$
$$\Pi_2 = (0, 0, \cdots, 0, \frac{1}{k_2}, \cdots, \frac{1}{k_2}),$$

are stationary.

Example 1.11. S_k on $k+1$ vertices.



$\chi_c P^{odd}(c) = 0$ and $\chi_c P^{even}(c) = 1$. $\frac{1}{2k}(k, 1, \cdots, 1) = \Pi$ is stationary.

Let $G = (V, E)$ be a graph without isolated vertices, $S \subset V$. We define

$$\mathrm{Vol}S = \sum_{v \in S} d_v$$

and

$$\Pi(u) = \left(\frac{d_u}{\mathrm{Vol}G}\right)_{u \in V}.$$

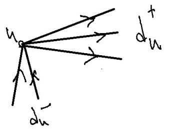
Moreover,

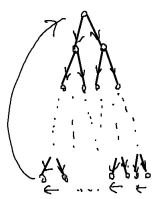
$$\begin{aligned}(\Pi P)(v) &= \sum_{u \sim v} \Pi(u) P(u, v) \\ &= \sum_{u \sim v} \frac{d_u}{\mathrm{Vol}G} \cdot \frac{1}{d_u} = \frac{d_v}{\mathrm{Vol}G} = \Pi(v).\end{aligned}$$

When the graph is directed, we can define

$$\Pi^+ = \left(\frac{d_u^+}{\mathrm{Vol}G}\right) \text{ and } \Pi^- = \left(\frac{d_u^-}{\mathrm{Vol}G}\right),$$

where $\mathrm{Vol}G = \sum_u d_u^+ = \sum_v d_v^-$.





$\Pi^+ P = \Pi^-$.

Problem 1.12. Can we find a function f such that $fP = f$?

In general, there is no close form for this f.

1.4 The note of convergence

Let M be a symmetric matrix, then all its eigenvalues are real. Let $M^t = U^*\Lambda^t U$, where Λ is the diagonal matrix. Recall that $P = D^{-1}A = D^{-\frac{1}{2}}(D^{-\frac{1}{2}}AD^{-\frac{1}{2}})D^{\frac{1}{2}}$ and let $M = D^{-\frac{1}{2}}AD^{-\frac{1}{2}}$, then $P^t = D^{-\frac{1}{2}}M^tD^{\frac{1}{2}}$. Let $\phi_i's$ are orthonormal eigenvectors associated with eigenvalues ρ_i of M, then $\phi_0 = \dfrac{\mathbf{1}D^{\frac{1}{2}}}{\sqrt{\mathrm{Vol}G}}$. We will see more details in the following lecture.

2 Spectral graph theory - An introduction

The tool is based on the generalized fast Fourier transformation (GFFT). Let M denote all positive entries matrices, A be adjacent matrices, $A(u, v) \geq 0$, $\forall u, v$. Recall that the Perron-Frobenius theorem states that there exists a unique eigenvector with positive components. All negative eigenvalues matrix can be considered as a Laplacian.

2.1 The combinatorial Laplacian

Let $G = (V, E)$ be a simple graph without isolated vertices, no loops, $|V| = n$. Denote A to be adjacency matrix, D to be diagonal matrix and $P = D^{-1}A$ to be the transition probability matrix with eigenvalues $\rho_i = 1 - \lambda_i$, where λ_i's will be introduced later.

Definition 2.1. Let $\mathcal{L}$ be (normalized) Laplacian, $\mathcal{L} = I - D^{-\frac{1}{2}}AD^{-\frac{1}{2}}$. L has eigenvalues $\lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_{n-1}$ ($|V| = n$).

$$\mathcal{L}(u, v) = \begin{cases} 1, & \text{if } u = v, \\ \dfrac{1}{\sqrt{d_u d_v}}, & \text{if } u \sim v, \\ 0, & \text{otherwise.} \end{cases}$$

$P(u, v) = \mathrm{Prob}\{\text{moving to } v \text{ from } u\}$, $fP(v) = \sum_{u \sim v} f(u)P(u, v)$, where f is a row vector. Let $B(u, e)$ be oriented all edges arbitrarily, $u \in V, e \in E$ such that

$$B(u, e) = \begin{cases} 1, & \text{if } u \text{ is the head of } e, \\ -1, & \text{if } u \text{ is the tail of } e, \\ 0, & \text{otherwise.} \end{cases}$$

Fact 2.2. $B \cdot B^* = P - A = L$: *combinatorial Laplacian.*

Fact 2.3. *We have*

$$Lf(u) = d_u f(u) - \sum_{v, v \sim u} f(v) = \sum_{v, v \sim u} (f(u) - f(v)).$$

Fact 2.4. $\mathcal{L} = D^{-\frac{1}{2}}LD^{-\frac{1}{2}} = D^{-\frac{1}{2}}B \cdot B^*D^{-\frac{1}{2}} = (D^{-\frac{1}{2}}B) \cdot (D^{-\frac{1}{2}}B)^*$.

Fact 2.5. *All eigenvalues $\lambda_i \geq 0$ and $\lambda_0 = 0$.*

A homological view: We can regard C_1 as 1-chains and C_0 as 0-chains. Then

$$B : C_0 \to C_1 \text{ and } B^* : C_1 \to C_0.$$

Let $\overrightarrow{e} = \{u, v\}$ be oriented edge, then B sends $\overrightarrow{e}$ to $u - v$ and

$$B : \sum_{\overrightarrow{e} \in \overrightarrow{E}} \alpha_{\overrightarrow{e}} \overrightarrow{e} \to \sum_{v \in V} \beta_v v.$$

Proposition 2.6. *Consider the Rayleigh quotient and let $f = gD^{-\frac{1}{2}}$,*

$$\frac{g\mathcal{L}g^*}{gg^*} = \frac{gD^{-\frac{1}{2}}LD^{-\frac{1}{2}}g^*}{gg^*} = \frac{fLf^*}{fDf^*},$$

where

$$\begin{aligned}
fLf^* &= \sum_u f(u) \sum_{v, v\sim u} (f(v) - f(u)) \\
&= \sum_u f(u) \sum_{v, v\sim u} (f(u) - f(v)) \\
&= \sum_{\{u,v\}\in E} (f(u)(f(u) - f(v)) + f(u)(f(v) - f(u)) \\
&= \sum_{\{u,v\}\in E} (f(u) - f(v))^2
\end{aligned}$$

and $fDf^* = \sum_v f(v)^2 d_v$. Therefore, the Rayleigh quotient is

$$\frac{g\mathcal{L}g^*}{gg^*} = \frac{\sum_{\{u,v\}\in E}(f(u) - f(v))^2}{\sum_{x\in V} f^2(x) d_x}.$$

For zero eigenvalue, $f(u) = f(v)$ if $u \sim v$.

Fact 2.7. $\phi_0 = f_0 D^{\frac{1}{2}} = \dfrac{\mathbf{1}D^{-\frac{1}{2}}}{\sqrt{VolG}}$*, where $VolG = \sum_v d_v$.*

Fact 2.8. *G has k connected components if and only if the zero eigenvalue has multiplicity k.*

Proof. ($\Rightarrow$) Easy. ($\Leftarrow$) Exercise. □

Fact 2.9. *$\lambda_i \leq 2$ for all i.*

Proof. By using the Rayleigh quotient, we have

$$\lambda_i \leq \sup_f \frac{\sum_{\{u,v\}\in E}(f(u) - f(v))^2}{\sum_{x\in V} f^2(x) d_x} = \sup_f \frac{\sum_{\{u,v\}\in E}(f(u) - f(v))^2}{\sum_{u\sim v}(f^2(u) + f^2(v))} \leq 2,$$

since $(a - b)^2 \leq 2(a^2 + b^2)$. □

Example 2.10. Let $A = \begin{pmatrix} 0 & & 1 \\ & \ddots & \\ 1 & & 0 \end{pmatrix}$, $L = \begin{pmatrix} 1 & & -\frac{1}{n-1} \\ & \ddots & \\ -\frac{1}{n-1} & & 1 \end{pmatrix}$. Let θ be the n-th root of $\theta^n = 1$, then

$$\begin{aligned} L \begin{pmatrix} 1 \\ \theta \\ \vdots \\ \theta^{n-1} \end{pmatrix} &= (1 - \frac{1}{n-1}(\theta + \theta^2 + \cdots + \theta^{n-1})) \begin{pmatrix} 1 \\ \theta \\ \vdots \\ \theta^{n-1} \end{pmatrix} \\ &= \begin{cases} 0, & \text{if } \theta = 1, \\ \dfrac{n}{n-1}, & \text{if } \theta \neq 1. \end{cases} \end{aligned}$$

Moreover, 0 has multiplicity 1 and $\dfrac{n}{n-1}$ has multiplicity $n-1$.

Example 2.11. More complicated, we consider matrices

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & \ddots & \ddots & \ddots & 0 \\ \vdots & 0 & \ddots & \ddots & 1 & 0 \\ 0 & 0 & \ddots & \ddots & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \text{ and } L = \begin{pmatrix} 1 & -\frac{1}{2} & 0 & \cdots & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & \ddots & 0 \\ 0 & -\frac{1}{2} & \ddots & \ddots & \ddots & 0 \\ \vdots & 0 & \ddots & \ddots & -\frac{1}{2} & 0 \\ 0 & \ddots & \ddots & -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & \cdots & 0 & -\frac{1}{2} & 1 \end{pmatrix}.$$

Similar calculation, we have

$$L \begin{pmatrix} 1 \\ \theta \\ \vdots \\ \theta^{n-1} \end{pmatrix} = (1 - \frac{\theta}{2} - \frac{\theta^{-1}}{2}) \begin{pmatrix} 1 \\ \theta \\ \vdots \\ \theta^{n-1} \end{pmatrix} = (1 - \cos\frac{2\pi j}{n}) \begin{pmatrix} 1 \\ \theta \\ \vdots \\ \theta^{n-1} \end{pmatrix},$$

where $\theta = e^{\frac{2\pi i j}{n}}$.

Example 2.12. A hypercubes Q_n has 2^n vertices, $n2^{n-1}$ edges. $V = (0,1)^n$ ={string of $0,1$ of length n }={subsets of an n-set $\{1, 2, \cdots, n\}$}. Its eigenvalues of $\mathcal{L}$ are $\dfrac{2k}{n}$, $k = 0, 1, 2 \cdots, n$ with multiplicity $\begin{pmatrix} n \\ k \end{pmatrix}$ and eigenvectors θ_s, $s \subset [n] = \{1, 2, \cdots, n\}$. $\theta_s(T) = \dfrac{(-1)^{|s|\frac{1}{n}}}{2^{\frac{n}{2}}}$ and $L\theta_s = \dfrac{2k}{n}\theta_s$, with $k = |s|$ (exercise).

2.2 More facts

Fact 2.13. *G is a vertex disjoint union of G_1 and G_2, then we have $\Lambda_G = \Lambda_{G_1} \cup \Lambda_{G_2}$, where Λ_S denotes the spectrum of S.*

Fact 2.14. *If G is connected and $G \neq K_n$, then $\lambda_1 \leq 1$.*

Proof. Let ϕ_0 be the first eigenfunction with respect to λ_0 and $f = D^{-\frac{1}{2}}g$. By using the Rayleigh quotient, we have

$$\begin{aligned}\lambda_1 &= \inf_{g\neq\phi_0} \frac{\langle g, Lg\rangle}{\langle g, g\rangle} = \inf_{\sum f(x)d_x=0} \frac{\langle f,(D-A)f\rangle}{\langle f, Df\rangle} \\ &= \inf_{\sum f(x)d_x=0} \frac{\sum_{x\sim u}(f(x)-f(u))^2}{\sum_x f^2(x)d_x} = \inf_{\sum f(x)d_x=0} R(f),\end{aligned}$$

where $R(f)$ is the standard Rayleigh quotient. Now, we choose f_1 by picking different vertices a and b with $a \nsim b$,

$$f_1(v) = \begin{cases} d_b, & \text{if } v = a,\\ d_a, & \text{if } v = b,\\ 0, & \text{otherwise.}\end{cases}$$

Then we have $\lambda_1 \le R(f) = \dfrac{d_a d_b^2 + d_b d_a^2}{d_a d_b^2 + d_b d_a^2} = 1.$ □

Fact 2.15. *Suppose G has diameter D, then $\lambda_1 \ge \dfrac{1}{D(VolG)}$, where $VolG = \sum_{v\in V} d_v$.*

Proof. Suppose that $\lambda_1 = R(f)$ for some f, f is the harmonic eigenfunction. $\sum f(x)d_x = 0$, and choose u_0 such that $|f(u_0)| = \max_u |f(u)|$, then $\exists v$ such that $f(u_0)f(v) < 0$ and $\exists$ a path $P = \{u_0, u_1, \cdots, u_t = v\}$, $t \le D$. Again,

$$\begin{aligned}\lambda_1 &= \frac{\sum_{x\sim u}(f(x)-f(u))^2}{\sum_x f^2(x)d_x} \ge \frac{(\sum_{\{x,y\}\in E(P)} |f(x)-f(y)|)^2}{(\mathrm{Vol}G)\cdot|f(u_0|^2} \\ &\ge \frac{(\sum_{x,y\in E(P)} |f(x)-f(y)|)^2}{D(\mathrm{Vol}G)|f(u_0)|^2} \ge \frac{|f(u_0)-f(u_t)|^2}{D(\mathrm{Vol}G)|f(u_0)|^2} \ge \frac{1}{D(\mathrm{Vol}G)}.\end{aligned}$$

□

Remark 2.16. $\lambda_1 \ge \dfrac{1}{n^3}$.

Exercise 2.17. Find connected graphs with λ_1 being the minimum.

3 More results for discrete Laplacian

3.1 Cartesian product

Let $G_i = (V_i, E_i)$ be graphs. $G_1 \,\square\, G_2$ has vertex set $V_1 \times V_2$.

Definition 3.1. We say $(u_1, u_2) \sim (v_1, v_2)$ if and only if $u_1 = v_1$ and $u_2 \sim v_2$ or $u_2 = v_2$ and $u_1 \sim v_1$.

By the definition, it is easy to see $Q_n = P_2 \,\square\, P_2 \,\square\, \cdots \,\square\, P_2$.

Let $G = (V, E)$ be a graph with $|V| = n$ and $\mathcal{L} = I - D^{-\frac{1}{2}}AD^{-\frac{1}{2}}$. Recall the eigenvalues are $0 = \lambda_0 \le \lambda_1 \le \cdots \le \lambda_{n-1} \le 2$, if $\lambda_0 = 0$, then the graph G has disconnected components. In further, the spectral gap $\lambda_1 - \lambda_0$ plays an important role in the spectral graph theory.

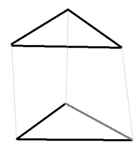
3.2 Vertex transitive and edge transitive

Let $G = (V, E)$ be a graph and σ be (graph) automorphism such that $\sigma : V \to V$ satisfying $x \sim y$ if and only if $\sigma(x) \sim \sigma(y)$.

Definition 3.2. We say that G is vertex transitive if $\forall u, v \in V$, $\exists \sigma$ automorphism such that $\sigma(u) = v$. We call G to be edge transitive if $\forall e, e' \in E$, $\exists \sigma$ automorphism such that $\sigma(e) = e'$.

Exercise 3.3. Find examples of graphs such that vertex transitive will not imply the edge transitive. Simultaneously, the edge transitive also cannot imply the vertex transitive.

Hint: We can consider the Cartesian product of two triangles. Note that in the following graph, the vertices are transitive, but the edges are not.



Theorem 3.4. *If G is edge transitive, then $\lambda_1 \geq \dfrac{1}{D^2}$.*

Theorem 3.5. *If G is vertex transitive and G has degree k, then $\lambda_1 \geq \dfrac{1}{kD^2}$.*

Definition 3.6. We define the index of G to be

$$\mathrm{Ind}G = \frac{\mathrm{Vol}G}{\min_i 2|E_i|},$$

where E_i is the i-th equivalent class of edges (the graph above also shows that these two triangular have equivalent edges).

Theorem 3.7. $\lambda_1 \geq \dfrac{1}{D^2 \, indG}$ *if G is vertex transitive.*

Proof. By using the Rayleigh quotient again,

$$\lambda_1 = \inf_f \sup_c \frac{\sum_{x\sim u}(f(x) - f(u))^2}{\sum_x (f(x) - c)^2 d_x},$$

if we choose $c = \dfrac{\sum f(x) d_x}{\sum d_x}$, then we have

$$\begin{aligned}
\lambda_1 &= \inf_f \frac{\sum_{x\sim u}(f(x) - f(u))^2}{\sum_x f^2(x) d_x - \frac{(\sum f(x) d_x)^2}{\sum d_x}} = \inf_f \frac{\sum_{x\sim u}(f(x) - f(u))^2 \mathrm{Vol}G}{\sum_{u,v}(f(u) - f(v)) d_u d_v} \\
&= \frac{n}{k} \inf_f \frac{\sum_{x\sim u}(f(x) - f(u))^2}{\sum_{u,v}(f(u) - f(v))^2}.
\end{aligned}$$

Fix a vertex x_0, $\forall y \in V$, we consider $P(x_0, y)$ of length $\leq D$ and $P(x_0) = \{P(x_0, y) | y \in V\}$. For $x \in V$, $x = \sigma(x_0)$, $P(x) = \{\sigma P(x_0, y)\}$. Now, for every edges $e \in E$, we set $N_e = \#$ of recurrence of e in $\cup P(x)$. In addition,

$$N_e \leq \frac{n^2 D}{2|E_i|} \leq \frac{n^2 D}{2 \min |E_i|} \leq \frac{n^2 D \mathrm{ind} G}{\mathrm{Vol} G} \leq \frac{n D \mathrm{ind} G}{k}.$$

Let $e = \{u, v\}$ and $f(e) = |f(u) - f(v)|$, then

$$\begin{aligned}\sum_x \sum_y (f(x) - f(y))^2 &\leq \sum_x \sum_y (\sum_{e \in P(x,y)} f(e))^2 \leq \sum_x \sum_y D \cdot \sum_{e \in P(x,y)} f^2(e) \\ &\leq \sum_{e \in E} f^2(e) D \cdot N_e \leq \sum_{e \in E} f^2(e) D \cdot \frac{n D \mathrm{ind} G}{k}.\end{aligned}$$

Therefore, we can conclude

$$\lambda_1 \geq \frac{n \sum_{e \in E} f^2(e)}{k \sum_{u,v} (f(u) - f(v))^2} \geq \frac{1}{D^2 \mathrm{ind} G}.$$

□

Let $G = (V, E)$ undirected weighted graph. Recall that we have introduced $P(x, y) = \mathrm{Prob}\{\text{moving to } y \text{ from } x\}$, $0 \leq P(x, y) \leq 1$ and $\sum_y P(x, y) = 1$. Let Π be the stationary distribution, then we have $\Pi P = \Pi$, which means $\sum_x \Pi(x) P(x, y) = \Pi P(y)$.

Definition 3.8. We call $P(x, y)$ to be the reversible random walk if

$$\Pi(x) P(x, y) = \Pi(y) P(y, x),$$

which means $P(x, y)$ is reversible.

Let $\omega_{xy} = c\Pi(x) P(x, y)$, then $\omega_{xy} = \omega_{yx}$ and

$$\Pi(x) P(x, y) = \frac{1}{c} \omega_{xy} = \frac{d_x}{\mathrm{Vol} G} \frac{\omega_{xy}}{d_x} = \Pi(y) P(x, y),$$

where $d_x = \sum_y \omega_{xy}$. The typical random walk is $P(x, y) = \dfrac{\omega_{xy}}{d_x}$, $\Pi(x) = \dfrac{d_x}{\mathrm{Vol} G}$ and $\mathrm{Vol} G = \sum_{x \in G} d_x$.

Recall that in Section 1, we have proposed the following questions: Does fP^t converge ? How long does it take ? How about the rates of convergence ? Here are partial answers.

The necessary (also sufficient) conditions for the convergence is

1. Irreducibility - This is equivalent to the *connectivity* of graphs.

2. Aperiodicity - This is equivalent to *non-bipartite* of graphs.

If $P = D^{-1} A = D^{-\frac{1}{2}} (D^{-\frac{1}{2}} A D^{-\frac{1}{2}}) D^{\frac{1}{2}} = D^{-\frac{1}{2}} M D^{\frac{1}{2}}$, we have $P^t = D^{-\frac{1}{2}} M^t D^{\frac{1}{2}}$. Let ϕ_i be orthonormal eigenvectors for M, $i = 0, 1, \cdots n-1$, if we write $f D^{-\frac{1}{2}} =$

$\sum \alpha_i \phi_i$, $\alpha_i = fD^{-\frac{1}{2}} \cdot \frac{D^{\frac{1}{2}}\mathbf{1}}{\sqrt{\mathrm{Vol}G}} = \frac{f\mathbf{1}}{\sqrt{\mathrm{Vol}G}} = \frac{1}{\sqrt{\mathrm{Vol}G}}$ and $\phi_0 = \frac{\mathbf{1}D^{\frac{1}{2}}}{\sqrt{\mathrm{Vol}G}}$. Thus, (we denote $\|\cdot\| \equiv \|\cdot\|_{L^2}$)

$$\begin{aligned}\|fP^t - \Pi\| &= \|fD^{-\frac{1}{2}}M^tD^{\frac{1}{2}} - \Pi\| = \|fD^{-\frac{1}{2}}(I-L)^tD^{\frac{1}{2}} - \Pi\| \\ &= \|\sum_{i=0}^{n-1} \alpha_i(1-\lambda_i)^t\phi_i D^{\frac{1}{2}} - \Pi\| = \|\sum_{i>0} \alpha_i(1-\lambda_i)^t\phi_i D^{\frac{1}{2}}\| \\ &\le (\max_{i>0} |1-\lambda_i|^t)C \to 0 \text{ as } t \to \infty,\end{aligned}$$

if $\overline{\lambda} := |1-\lambda_i| < 1$, for some constant $C > 0$. Moreover, we have the following main results.

1. $\lambda_1 > 0$ if and only if the graph is connected if and only if irreducibility holds.
2. $\lambda_{n-1} < 2$ if and only if the graph is non-bipartite if and only if the aperiodicity holds.

3.3 Lazy walk

We give examples of the lazy walk in the following.

Example 3.9. $Z = \frac{I+P}{2}$, $\lambda_1' = \frac{\lambda_1}{2}$, adding loops $\omega_{u,u} = d_u$.

Example 3.10. $Z = \frac{cI+P}{c+1}$, choose the best constant c (e.g. $c = \frac{\lambda_1+\lambda_{n-1}}{2} - 1$).

4 Cheeger's inequality - graph version

Recall that $\mathcal{L}$ has eigenvalues $0 = \lambda_0 \le \lambda_1 \le \cdots \le \lambda_{n-1}$, where λ_1 controls the connectivity of graphs. Moreover, λ_1 gives information about the convergence of random walks. In the spectral graph theory, we consider two important problems: Expansion property and isoperimetric property.

4.1 Isoperimetric property

In the graph theory, there are two kinds of boundaries.

Definition 4.1. We call $\delta(S)$ to be a vertex boundary if

$$\delta(S) = \{v \in V | v \notin S, v \sim u \text{ for some } u \in S\}.$$

We call $\partial(S)$ to be an edge boundary if

$$\partial(S) = \{\{u,v\} \in E | u \in S, v \notin S\}.$$

For a specific "volume" (say 100), what is the subset with volume ≤ 100 minimizing the vertex/edge boundary ? Let $G = (V,E)$ be a graph with $|V| = n$, we want to discuss the Cheeger's inequality to help us to us to understand the isoperimetric inequality.

4.2 Cheeger's inequality

Recall that $\mathrm{Vol}(S) = \sum_{v \in S} d_v$.

Definition 4.2. (Cheeger's ratio) $h(S) = \dfrac{|\partial S|}{\mathrm{Vol}(S)}$.

Definition 4.3. The Cheeger constant h_G of G is

$$h_G := \min\{h(S) | S \subset V, \ \mathrm{Vol}(S) \leq \frac{1}{2}\mathrm{Vol}(G)\}.$$

Here we only consider all graphs with finite edges and vertices. It is easy to see that for any S with $\mathrm{Vol}(S) \leq \frac{1}{2}\mathrm{Vol}(G)$, $|\partial S| \geq h_G \mathrm{Vol}(S)$.

Theorem 4.4. *(Cheeger inequality) If graph is not complete, we have*

$$h_G \geq \lambda_1 \geq \frac{h_G^2}{2} \text{ and } \sqrt{2\lambda_1} \geq h_G \geq \frac{\lambda_1}{2}.$$

The complete graphs are much easier to understand, so we only consider the incomplete graphs.

Exercise 4.5. Find graphs G, G' for which $h_{G_1} = c_1\lambda_1$ and $h_{G'} = c_2\lambda_1^2$.

Lemma 4.6. $2h_G \geq \lambda_1$.

Proof. Use the Rayleigh quotient, we have

$$\lambda_1 = \inf_{\sum f(x)d_x = 0} \frac{\sum_{x \sim y}(f(x) - f(y))^2}{\sum_x f^2(x)d_x} \leq R(g) \text{ with } \sum g(x)d_x = 0.$$

h_G is achieved by S, $h_G = \dfrac{|\partial S|}{\mathrm{Vol}(S)}$ with $\mathrm{Vol}(S) \leq \dfrac{1}{2}\mathrm{Vol}(G)$. Choose g defined by

$$g(v) = \begin{cases} \dfrac{1}{\mathrm{Vol}(S)}, & \text{if } v \in S, \\ \dfrac{1}{\mathrm{Vol}(S^c)}, & \text{if } v \notin S. \end{cases}$$

Then

$$\begin{aligned} \lambda_1 \leq R(g) &= \frac{(\frac{1}{\mathrm{Vol}S} + \frac{1}{\mathrm{Vol}S^c})|\partial S|}{(\frac{1}{\mathrm{Vol}S})^2\mathrm{Vol}S + (\frac{1}{\mathrm{Vol}S^c})^2\mathrm{Vol}S^c} \\ &= (\frac{1}{\mathrm{Vol}S} + \frac{1}{\mathrm{Vol}S^c})|\partial S| = \frac{|\partial S|\mathrm{Vol}G}{\mathrm{Vol}S \cdot \mathrm{Vol}S^c} \leq 2h_G. \end{aligned}$$

□

Theorem 4.7. $\lambda_1 \geq \dfrac{h_G^2}{2}$.

Proof. Suppose

$$\lambda_1 = R(f) = \frac{\sum_{u \sim v}(f(u) - f(v))^2}{\sum_u f^2(u)d_u}$$

for some f. Order the vertices crossing f by $f(u_1) \geq f(u_2) \geq \cdots \geq f(u_n)$. Let $C_i = \{\{v_j, v_k\} | 1 \leq j \leq i < k \leq n\}$. Define

$$\alpha := \alpha_f = \min_{1 \leq i \leq n} \frac{|C_i|}{\min\{\sum_{j \leq i} d_j, \sum_{j > i} d_j\}} \geq h_G$$

by the definition of h_G. It suffices to show that $\lambda_1 \geq \dfrac{\alpha^2}{2}$.

Now, we define $V_+ = \{v : f(v) \geq 0\}$ and $V_- = \{v : f(v) < 0\}$. WLOG, we may assume $\sum_{f(v)<0} d_v \geq \sum_{f(y) \geq 0} d_u$ (if not, changing sign). Define

$$g(x) = \begin{cases} f(x), & \text{if } x \in V_+, \\ 0, & \text{otherwise.} \end{cases}$$

Then we have

$$\lambda_1 f(u) d_u = \sum_{v, v \sim u} (f(u) - f(v)).$$

The above equality holds since if we set $\phi_1 = D^{\frac{1}{2}} f$, then $\lambda_1 \phi_1 = L\phi_1$, then

$$\begin{aligned} \lambda_1 f(x)(d_x)^{\frac{1}{2}} &= D^{-\frac{1}{2}}(D - A)D^{-\frac{1}{2}}(D^{\frac{1}{2}} f(x)) \\ &= \frac{1}{d_x^{\frac{1}{2}}} \sum_y (f(x) - f(y)) \end{aligned}$$

and

$$\lambda_1 = \inf_{g \perp \phi_0} \frac{\langle Lg, g \rangle}{\langle g, g \rangle} = \inf_{\sum f(x) d_x = 0} \frac{\langle f, Lf \rangle}{\langle f, Df \rangle}.$$

Therefore,

$$\begin{aligned} \lambda_1 &= \frac{\sum_{u \in V_+} f(u) \sum_{v \sim u} (f(u) - f(v))}{\sum_{u \in V_+} f^2(u) d_u} \geq \frac{\sum_{u \sim v} (g(u) - g(v))^2}{\sum_u g^2(u) d_u} \cdot \frac{\sum_{u \sim v} (g(u) + g(v))^2}{\sum_{u \sim v} (g(u) + g(v))^2} \\ &\geq \frac{(\sum_{u \sim v} (g^2(u) - g^2(v))^2}{2(\sum_u g^2(u) d_u)^2} \geq \frac{(\sum_i |g^2(u_i) - g^2(u_{i+1})| \cdot |C_i|)^2}{2(\sum_u g^2(u) d_u)^2} \\ &\geq \frac{(\sum (g^2(v_i) - g^2(v_{i+1})) \cdot h_G}{2(\sum_u g^2(u) d_u)^2} \geq \frac{h_G}{2} \frac{(\sum_i g(v_i)(\sum_{j \leq i} d_j - \sum_{j \leq i-1} d_j)}{(\sum_u g^2(u) d_u)^2} = \frac{h_G^2}{2}, \end{aligned}$$

where we used the summation by parts in the last inequality. ☐

Exercise 4.8. Prove that $\lambda_1 \geq 1 - \sqrt{1 - h_G^2}$ by using $(a+b)^2 = 2(a^2 + b^2) - (a-b)^2$.

Remark 4.9. Cheeger's inequality can be rewritten as

$$2h_G \geq \lambda_1 \geq \frac{\alpha^2}{2} \geq \frac{h_G^2}{2}.$$

5 Another characterization of h_G

Let $G = (V, E)$ be a graph with $|V| = n$ and $\mathcal{L} = I - D^{-\frac{1}{2}} A D^{-\frac{1}{2}}$ be harmonic. $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n-1}$, then we have $0 < \lambda_1 < \dfrac{1}{2}$. Recall that we have defined the Cheeger's constant

$$h_G = \min_{\substack{S \subset V \\ \mathrm{Vol}_{S \le \frac{1}{2}} \mathrm{Vol}_G}} \frac{e(S, S^c)}{\mathrm{Vol} S},$$

where $e(S, S^c) = |\partial S|$ in the previous section. In fact, the Cheeger's inequality is

$$2h_G \ge \lambda_1 > \frac{h_G^2}{2} \text{ and } \lambda_1 \ge \frac{\alpha^2}{2} \ge \frac{h_G^2}{2}.$$

We have the following theorem.

5.1 Another characterization of Cheeger's constant

Theorem 5.1. *The Cheeger's constant*

$$h_G = \inf_f \sup_c \frac{\sum_{x \sim y} |f(x) - f(y)|}{\sum_{x \in V} |f(x) - c| d_x} (:= h').$$

Proof. First, we claim that $h_G \le h'$. It is enough to show that "$\le$" holds for all $f : V \to \mathbb{R}$. Choose a constant c_0 to be the median of f, which means either

$$\sum_{x, f(x) < c} d_x \le \sum_{x, f(x) \ge c} d_x,$$

or

$$\sum_{x, f(x) \le c} d_x > \sum_{x, f(x) > c} d_x,$$

which was obtained by the ideas of $\mathrm{Vol} S \le \frac{1}{2} \mathrm{Vol} G$ (cut f from the middle...). Now, we put $g = f - c$ and for all $\sigma > 0$, we consider

$$\sum_{x, g(x) < \sigma} d_x \le \sum_{x, g(x) > \sigma} d_x.$$

Let

$$\widetilde{g}(\sigma) = |\{\{x, y\} \in E | g(x) \le \sigma \le g(y)\}| \ge h_G \sum_{x, g(x) < \sigma} d_x.$$

Thus, we have

$$\begin{aligned}
\sum_{x \sim y} |f(x) - f(y)| &= \int_{-\infty}^{\infty} \widetilde{g}(\sigma) d\sigma \\
&= \int_{-\infty}^{0} \left(\frac{\widetilde{g}(\sigma)}{\sum_{x, g(x) < \sigma} d_x} \sum_{x, g(x) < \sigma} d_x \right) d\sigma + \int_{-\infty}^{0} \left(\frac{\widetilde{g}(\sigma)}{\sum_{x, g(x) > \sigma} d_x} \sum_{x, g(x) > \sigma} d_x \right) d\sigma \\
&\ge h_G \left(\int_{-\infty}^{0} \left(\sum_{g(x) < \sigma} d_x \right) d\sigma + \int_{0}^{\infty} \left(\sum_{g(x) > \sigma} d_x \right) d\sigma \right. \\
&= h_G \sum_{x \in V} |g(x)| d_x = h_G \sum_{x \in V} |f(x) - c| d_x,
\end{aligned}$$

which implies $h' \ge h_G$.

For $h_G \geq h'$, we suppose that $h_G = \dfrac{|\partial X|}{\text{Vol}X}$ for some $X \subset V$, with $\text{Vol}X \leq \text{Vol}X^c$. Define

$$\varphi(v) = \begin{cases} 1, & \text{if } v \in X \\ 0, & \text{otherwise,} \end{cases}$$

then we have

$$\begin{aligned} \sup_c \frac{\sum_{x\sim y} |\varphi(x) - \varphi(y)|}{\sum_x |\varphi(x) - c| d_x} &= \sup_c \frac{2|\partial X|}{|1-c|\text{Vol}X + |1+c|\text{Vol}X^c} \\ &= \frac{2|\partial X|}{2\text{Vol}X} = h_G, \end{aligned}$$

which proves the theorem. □

Now, we give another proof for $\lambda_1 \geq \dfrac{h_G^2}{2}$ (important mathematical theorems should be proved more than one method).

5.2 A second proof of $\lambda_1 \geq \dfrac{h_G^2}{2}$

Proof. Suppose f achieves λ_1, which means $\lambda_1 = R(f)$. Then

$$\lambda_1 = R(f) = \sup_c \frac{\sum_{x\sim y}(f(x) - f(y))^2}{\sum_x (f(x) - c)^2 d_x} \geq \frac{\sum_{x\sim y}(f(x) - f(y))^2}{\sum_x (f(x) - c_0)^2 d_x},$$

where c_0 is the median of f. If we set $g(x) = f(x) - c_0$, then

$$\begin{aligned} \lambda_1 &\geq \frac{\sum_{x\sim y}(g(x) - g(y))^2}{\sum_x g^2(x) d_x} \cdot \frac{\sum_{x\sim y}(g(x) + g(y))^2}{\sum_{x\sim y}(g(x) + g(y))^2} \\ &\geq \frac{(\sum_{x\sim y} g^2(x) - g^2(y))^2}{2(\sum_x g^2(x) d_x)^2} \geq \frac{h'^2}{2}. \end{aligned}$$

Remark 5.2. Note that $L = D - A$,

$$\langle f, Lf \rangle = \sum_x f(x) \sum_y (f(x) - f(y)) = \sum_{x\sim y} (f(x) - f(y))^2,$$

since "$x \to f(x)(f(x) - f(y))$" + "$y \to f(y)(f(y) - f(x))$".

Exercise 5.3. Prove that

$$h_G \geq \inf_{f, \sum f(x) d_x = 0} \frac{\sum_{x\sim y} |f(x) - f(y)|}{\sum_x |f(x)| d_x} \geq \frac{1}{2} h_G.$$

5.3 More properties

Let $G = (V, E)$ be a graph as before. $\mathcal{L} = I - D^{-\frac{1}{2}} A D^{-\frac{1}{2}} = D^{-\frac{1}{2}}(D - A) D^{-\frac{1}{2}} = D^{-\frac{1}{2}} L D^{-\frac{1}{2}}$. For edge expansion, we have $h(S) = \dfrac{|\partial S|}{\text{Vol}G}$. $\mathcal{L}$ has eigenvalues $\lambda_0 \leq$

$\lambda_1 \leq \cdots \leq \lambda_{n-1}$ and its eigenvectors are $\varphi_0, \varphi_1, \cdots, \varphi_{n-1}$. Let $f_0, f_1, \cdots, f_{n-1}$ be combinatorial eigenvectors, where $f_i = \varphi_i D^{-\frac{1}{2}}$.

$$\lambda_1 = \inf_{g \perp \varphi_0} \frac{\langle g, \mathcal{L} g \rangle}{\langle g, g \rangle} = \inf_{\sum f(x) d_x = 0} \frac{\langle f, Lf \rangle}{\langle f, f \rangle} = \inf_{\sum f(x) d_x = 0} \frac{\sum_{x \sim y} (f(x) - f(y))^2}{\sum_x f^2(x) d_x}.$$

If $\sum f(x) d_x = 0$, we have

$$\begin{aligned}
&\langle f, Lf \rangle \geq \lambda_1 \langle f, Df \rangle, \\
&\langle f, Af \rangle \leq (1 - \lambda_1) \langle f, Df \rangle, \\
&| \langle f, Af \rangle | \leq \max_{i \neq 0} |1 - \lambda_i| \langle f, Df \rangle.
\end{aligned}$$

Let $g_G := \min\{\frac{\mathrm{Vol}\delta S}{\mathrm{Vol}S} | \mathrm{Vol}S \leq \frac{1}{2}\mathrm{Vol}G\}$, moreover, if the graph is regular, we can consider the quantity $\frac{|\delta S|}{|S|}$.

Theorem 5.4. $\frac{Vol\delta S}{VolS} \geq \frac{1}{(\sigma + 2\epsilon)^2}$, *where* $\sigma = 1 = \lambda_1$ *and* $\epsilon = \frac{Vol(S \cup \delta S)}{VolG}$.

Example 5.5. (Ramanujan k-regular) $\sigma \sim \frac{2\sqrt{k-1}}{k}$ and $\frac{\mathrm{Vol}\delta S}{\mathrm{Vol}S} \geq \frac{k}{4}$+lower order term.

Proof. Consider $f = \chi_S + \sigma \chi_{\delta S}$, where χ_A is a characteristic function on A. Let $g = f - c\chi_V$ with $c = \frac{\sum_u f(u) d_u}{\mathrm{Vol}G}$, $\sum g(x) d_x = 0$. Note that $\sum g^2(x) d_x = \sum (f(x) - c)^2 d_x = \sum f^2(x) d_x - c^2 \mathrm{Vol}G$. Then

$$\begin{aligned}
\langle f, Af \rangle &= \langle f, (D - (D - A)) f \rangle = \langle g + cI, (D - L)(g + cI) \rangle \\
&= \langle g, Ag \rangle + c^2 \mathrm{Vol}G \leq (1 - \lambda_1) \langle g, Dg \rangle + c^2 \mathrm{Vol}G \\
&\leq (1 - \lambda_1) \langle f, Df \rangle + \lambda_1 c^2 \mathrm{Vol}G \\
&\leq (\sigma + (1 - \sigma)\epsilon) \langle f, Df \rangle
\end{aligned}$$

since

$$c^2 \mathrm{Vol}G = \frac{(\sum_{u \in S \cup \delta S} f(u) d_u)^2}{\mathrm{Vol}G} \leq \frac{\sum_u f^2(u) \cdot \mathrm{Vol}(S \cup \delta S)}{\mathrm{Vol}G} \leq \epsilon \langle f, Df \rangle.$$

On the other hand,

$$\begin{aligned}
\langle f, Af \rangle &= \langle \chi_S + \chi_{\delta S}, A(\chi_S + \chi_{\delta S}) \rangle = e(S, S) + 2\sigma e(S, \delta S) + \sigma^2 e(\delta S, \delta S) \\
&\geq (1 - 2\sigma) e(S, S) + 2\sigma e(S, S \cup \delta S) \\
&= (1 - 2\sigma) e(S, S) + 2\sigma \mathrm{Vol}S \geq 2\sigma \mathrm{Vol}S,
\end{aligned}$$

where $e(S, T) = \#\{(u, v) | u \in S, v \in T, u \sim v\}$ and $e(S, S \cup \delta S) = \mathrm{Vol}S$. Thus, we have

$$(\sigma + (1 - \sigma)\epsilon) \langle f, Df \rangle \leq (\sigma + (1 - \sigma)\epsilon)(\mathrm{Vol}S + \sigma^2 \mathrm{Vol}\delta S)$$

and

$$\frac{\mathrm{Vol}\delta S}{\mathrm{Vol}S} \geq \frac{\sigma - \epsilon}{\sigma^2(\sigma + \epsilon)} \geq \frac{1}{(\sigma + 2\epsilon)^2} \text{ if } \sigma \leq \frac{1}{2}, \lambda_1 \geq \frac{1}{2}.$$

If $\lambda_1 < \frac{1}{2}$, we have

$$\frac{\mathrm{Vol}\delta S}{\mathrm{Vol}S} \geq \frac{|\partial S| + |\partial(\delta S)|}{\mathrm{Vol}S} \geq h(S) + h(S \cup \delta S).$$

□

Exercise 5.6. $h(S) \geq \lambda_1(1 - \frac{\mathrm{Vol}S}{\mathrm{Vol}G}) \geq 2\lambda_1(1-\epsilon)$.

Theorem 5.7. *We have*

$$\frac{Vol\delta X}{Vol X} \geq \frac{1-\sigma^2}{\sigma^2 + \frac{Vol X}{Vol X^c}},$$

where $\sigma = \max_{i\neq 0} |1 - \lambda_i|$.

5.4 Diameter of graphs

Consider $d(u, v)$ =length of the shortest path joining u and v, then we have $\mathrm{diam}G = \max_{u,v} d(u, v)$ and $d(X, Y) = \min\{d(u, v)|u \in X, v \in Y\}$.

Theorem 5.8. *When* $X \neq Y^c$,

$$d(X,Y) \leq \left\lceil \frac{\log \sqrt{\frac{Vol X^c \cdot Vol Y^c}{Vol X \cdot Vol Y}}}{\log \frac{\lambda_{n-1}+\lambda_1}{\lambda_{n-1}-\lambda_1}} \right\rceil.$$

Example 5.9. (Ramanujan k-regular)

$$\mathrm{diam}G \leq \frac{\log(n-1)}{\log \frac{1}{1-\lambda}} \sim \frac{\log n}{\log \sqrt{k}} = 2\frac{\log n}{\log k}.$$

Consider $X, Y = X^c \backslash \delta X$, we have

$$0 = \left\langle D^{\frac{1}{2}}\mathbf{1}_Y, (I - \mathcal{L})D^{\frac{1}{2}}\mathbf{1}_x \right\rangle \geq \frac{\mathrm{Vol}X \cdot \mathrm{Vol}Y}{\mathrm{Vol}G} - \sigma \sum_{i\neq 0} \alpha_i^2 \sum_{i\neq 0} \beta_i^2,$$

where $D^{\frac{1}{2}}\mathbf{1}_Y = \sum \alpha_i\phi_i$ and $D^{\frac{1}{2}}\mathbf{1}_x = \sum \beta_i\phi_i$ and $\alpha_0\beta_0 = \frac{\mathrm{Vol}X \cdot \mathrm{Vol}Y}{\mathrm{Vol}G}$. This will imply that $\sigma^2\mathrm{Vol}X^c \cdot \mathrm{Vol}Y^c \geq \mathrm{Vol}X \cdot \mathrm{Vol}Y$, and

$$\sigma^2(\mathrm{Vol}G - \mathrm{Vol}X)(\mathrm{Vol}X + \mathrm{Vol}\delta X) \geq \mathrm{Vol}X \cdot (\mathrm{Vol}G - \mathrm{Vol}X - \mathrm{Vol}\delta X)$$

and

$$\mathrm{Vol}\delta X(\mathrm{Vol}X + \sigma^2(1 - \mathrm{Vol}X)) = \mathrm{Vol}X(\mathrm{Vol}G - \mathrm{Vol}X)(1-\sigma^2)).$$

6 Matchings and applications

Definition 6.1. A set edges $M \subset E(G)$ is called matching if $\forall e, e' \in M$, $e \cap e' = \emptyset$.

6.1 Perfect matching

Definition 6.2. A matching is perfect if $|M| = \dfrac{|V(G)|}{2}$, i.e., covers all vertices in G. Sometimes we can think it as a permutation $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$.

For $X \subset V(G)$, $N_G(X) = \{y : y \sim x \text{ for } x \in X\}$.

Theorem 6.3. *(Hall's theorem) A bipartite graph $G = (A \cup B, E)$ with $|A| = |B|$ has a perfect matching if and only if for any nonempty $X \subset A$, $|N_G(X)| \geq |X|$ or $\dfrac{|\delta(X)|}{|X|} \geq 1$.*

Proof. If G has a perfect matching, it is easy to see that $|N_G(X)| \geq |X|$. For the other direction, it suffices to show that matching covering A. We use the induction on $|A|$. Assume that ($\Leftarrow$) holds for $1 \leq |A| < n$ and we let G have $|A| = n$.

Case 1. If $|N_G(X)| > |X|$ for all nonempty $X \subsetneq A$. Pick an edge $\{a, b\} \in E(G)$, delete a, b to obtain H. For $\emptyset \neq X \subset A \backslash \{a\}$, $|N_H(X)| \geq |N_G(X)| - 1 \geq (|X| + 1) - 1 = |X|$. So H has matched covering $A \backslash \{a\}$. Now, we add $\{a, b\}$ back to get matching covering A in G.

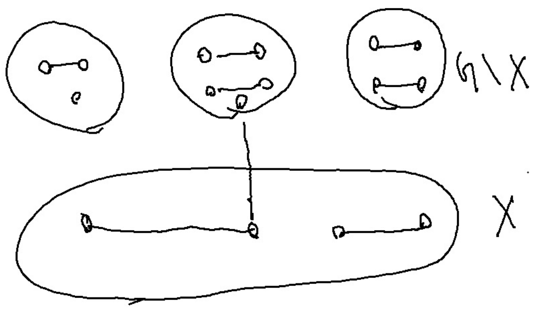
Case 2. Exists $X \subsetneq A$ with $|N(X)| = |X|$. By induction, matching covering X in H. For $Y \subset A \backslash X$, we have

$$|N_H(X)| + |N_H(Y)| \geq |N_G(X \cup Y)| \geq |X \cup Y| = |X| + |Y|.$$

Since $|N_H(X)| = |X|$, so $|N_H(Y)| \geq |Y|$, which implies matching covering $A \backslash X$ in K. $\square$

Theorem 6.4. *(Tutte's theorem) Let $q(G)$ denote the number of connected components of odd order. G has perfect matching if and only if for any $X \subset V(G)$, $q(G \backslash X) \leq |X|$, where $G \backslash X$ is a subgraph induced by $V \backslash X$.*

Proof. (Sketch) To see $\Rightarrow$, note that each connected component of odd order sends at least one edge to a vertex of X. Each vertex of X only receives at most one edge from each component.



We leave $\Leftarrow$ as an exercise. $\square$

6.2 Doubly stochastic matrices / magic squares

Definition 6.5. A square matrix is doubly stochastic if nonnegative entries with row/column sums are all 1.

Example 6.6. $P = D^{-1}A$ of random walks on regular graph.

Definition 6.7. A magic square of weight d is a square matrix, nonnegative integers with row/column sums are all d.

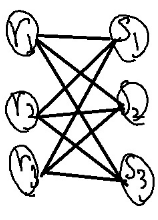
Definition 6.8. An $n \times n$ permutation matrix T_σ, defined by $T_{ij} = \begin{cases} 1, & \text{if } j = \sigma(i), \\ 0, & \text{otherwise,} \end{cases}$ where σ is a permutation of $\{1, 2, \cdots, n\}$.

Theorem 6.9. *(Birkhoff - Von Neumann) Every magic square of weight d is the sum of d permutation matrices.*

Proof. Let T be an $n \times n$ magic square of weight $d \geq 1$. Define a bipartite $G = (A \cup B, E)$, where $i \in A, j \in B$, $i \sim j$ if $T_{ij} > 0$. Consider $X \subset A$, $\sum_{i \in X} T_{ij} = |X|d$, so $\sum_{j \in N(X)} T_{ij} \geq |X|d$. But $\sum_{j \in N(X)} T_{ij} = |N(X)|d$, so there is a perfect matching by Hall's theorem, which we denote by σ. Let T_σ be the associated permutation matrix and $T - T_\sigma$ is a magic square of weight $d - 1$. Finish with the induction on d. $\square$

6.3 Applications: Two-sided matching problems

Ordinal preferences:



	1st	2nd	3rd
r_1	s_1	s_2	s_3
r_2	s_1	s_2	s_3
r_3	s_1	s_2	s_3

	1st	2nd	3rd
s_1	r_1	r_3	r_2
s_2	r_2	r_1	r_3
s_3	r_2	r_1	r_3

Find the perfect matching that is "optimal" with respect to preferences.

Definition 6.10. A matching is stable if and only if the following doesn't hold: 1. r_1 is assigned to s_1, but prefers r_j's assignment of s_j, or 2. s_j prefers r_1 over than r_j.

This is related to the “Gale-Shapley Algorithm”.

Definition 6.11. A matching r-pareto efficient if no pair r_i, r_j switch assignments and both receive prefered assignment.

Here $\begin{pmatrix} r_1 & r_2 & r_3 \\ s_1 & s_2 & s_3 \end{pmatrix}$ is only a stable matching not r-pareto efficient.