The Dirichlet-to-Neumann map and electric impedance tomography (EIT)

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Abstract

The note is mainly for personal record, if you want to read it, please be careful. This note is given Prof. *Jenn-Nan Wang*'s lecture during the summer course in NCTS, 2015. We would like to introduce the notation of the Dirichlet-to-Neumann map (DtN map) and discuss its basic properties. The aim here is to study the inverse problem of determining the parameters of the equation by the DtN map. For the conductivity equation, this is known as the electrical impedance tomography or Calderon's problem. I plan to discuss fundamental questions of the problem: uniqueness, stability, and reconstructions. The course will be self- contained, but basic knowledge on PDE and Sobolev spaces is helpful.

1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain. Assume that $\partial\Omega$ is smooth. We take u: electric potential on Ω. By Ohm's law, J is a current, then

$$J = -\frac{\nabla u}{R} = -\gamma(x)\nabla u,$$

R: resistance in Ω and $\gamma(x) = \dfrac{1}{R(x)}$: conductivity. Here $\gamma(x)$ is a scalar function (isotropic). When we have an anisotropic conductivity, then

$$J_\alpha = -a^{\alpha\beta}\partial_\beta u.$$

If there is no sink or no source in Ω, then for $D \Subset \Omega$,

$$\begin{aligned} 0 &= \int_{\partial D} J \cdot \nu dS = -\int_{\partial D} \gamma(x)\nabla u \cdot \nu dS \\ &= -\int_D \nabla \cdot (\gamma(x)\nabla u) dx, \end{aligned}$$

true for all $D \Subset \Omega$. Then

$$\nabla \cdot (\gamma(x)\nabla u) = 0 \text{ in } \Omega.$$

1.1 Forward problem

Assume that $\gamma(x|$ is known in Ω, $\gamma(x) \geq \delta > 0$ a.e., $\gamma(x| \in L^\infty(\Omega)$. Consider the boundary value problem

$$\begin{cases} \nabla \cdot (\gamma \nabla u) = 0 & \text{in } \Omega, \\ u = f \in H^{1/2}(\partial\Omega) & \text{on } \partial\Omega. \end{cases} \tag{1}$$

The problem (1) is well-posed, $\exists! u \in H^1(\Omega)$ and $\|u\|_{H^1(\Omega)} \leq C\|f\|_{H^{1/2}(\partial\Omega)}$. This can be proved by the Lax-Milgram theorem. Given any $f \in H^{1/2}(\partial\Omega)$, $\exists! u_f \in H^1(\Omega)$, one can determine the Neumann data and given by $\gamma(x)\dfrac{\partial u_f}{\partial \nu}|_{\partial\Omega}$. Hence, we can define

$$\Lambda_\gamma : f \to \gamma(x)\frac{\partial u_f}{\partial \nu}|_{\partial\Omega} : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega).$$

Call Λ_γ to be the Dirichlet-to-Neumann map (voltage-to-current).

Now if we have anisotropic medium, then the electric potential u satisfies

$$\partial_\alpha(a^{\alpha\beta}(x)\partial_\beta U) = 0 \text{ in } \Omega. \tag{2}$$

Assume that $(a^{\alpha\beta}) \in L^\infty(\Omega)$ is elliptic. $\exists! u_f \in H^1(\Omega)$ solving (2) and satisfying $u_f|_{\partial\Omega} = f \in H^{1/2}(\partial\Omega)$. The Dirichlet-to-Neumann map $\Lambda_a : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ is given by

$$\Lambda_a f = a^{\alpha\beta}\partial_\beta u \nu_\alpha,$$

$\nu = (\nu_1, \cdots, \nu_n)$ (also call $\Lambda_a f$ to be the conormal derivative). We can also study the elasticity system

$$\sum_{jik=1}^{N} \partial_j(C^{ijkl}\partial_k u^l) : \text{ elliptic system.}$$

Elasticity: $C^{ijkl} = C^{klij} = C^{jikl} = C^{ijlk}$. Ellipticity: $C^{ijkl}\xi_j\xi_k\eta_i\eta_l \geq c|\xi|^2|\eta|^2$, for all $\xi, \eta \in \mathbb{R}^n$. For the elasticity system, we can specify the displacement $u|_{\partial\Omega}$,

$$\begin{cases} \sum_{ijkl=1}^{N} \partial_j(C^{ijkl}\partial_k u^l) = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases}$$

DN-map $\Lambda_C : f \to C^{ijkl}\partial_k u^l \nu_j|_{\partial\Omega}$ (displacement to traction).

1.2 Inverse problem

Assume that $\gamma(x)$ is NOT known. But Λ_γ is given (i.e. you can take all possible measurements on $\partial\Omega$). The question now is to determine $\gamma(x)$ from Λ_γ (also assume $\gamma(x) \geq \delta > 0$ and $\gamma \in L^\infty(\Omega)$).

Known: $\exists \Lambda : \gamma \to \Lambda_\gamma \in$ map from $H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$. Can we inverse Λ and find Λ^{-1}? Note that Λ is not linear, i.e., $\Lambda_{\gamma_1+\gamma_2} \neq \Lambda_{\gamma_1} + \Lambda_{\gamma_2}$. We are interested in the well-posedness of the inverse problem.

1. Uniqueness (or the injectivity of Λ^{-1}). If $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$, then $\gamma_1 = \gamma_2$.

2. Stability. $\|\gamma_1 - \gamma_2\|_{L^\infty(\Omega)} \le \omega(\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{\frac{1}{2},-\frac{1}{2}})$, $\omega(t)$ is modulus of continuity.

3. Reconstruction. Reconstruct $\gamma(x)$ from Λ_γ.

4. **Existence. Characterize the DN map: Find conditions on $T : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ bounded, linear.**

To determine $\gamma(x)$ from Λ_γ was proposed by Calderón ('60).

2 Calderón's approach

Let us consider

$$\begin{cases} \nabla \cdot (\gamma(x)\nabla u) = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega, \end{cases}$$

u is the minimizer of $\int_\Omega \gamma(x)|\nabla u|^2 dx$ over $u \in H^1(\Omega)$ with $u|_{\partial\Omega} = f$. Define

$$\begin{aligned} Q_\gamma(f) &= \int_\Omega \gamma(x)|\nabla u|^2 dx \text{ (power)} \\ &= \int_{\partial\Omega} f \Lambda_\gamma f dS. \end{aligned}$$

Knowing Q_γ is equivalent knowing Λ_γ. We can find a bilinear form $B_\gamma(\cdot,\cdot) : H^{1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)$ from the quadratic form (by polarization) $B_\gamma(f,f) = Q_\gamma(f)$ and $B_\gamma(f,g) \in \mathbb{R}$. B_γ is symmetric, i.e., $B_\gamma(f,g) = B_\gamma(g,f)$. This implies that Λ_γ is symmetric,

$$\langle \Lambda_\gamma f, g \rangle = \langle f, \Lambda_\gamma g \rangle .$$

In general, Λ_γ is self-adjoint, i.e., $\Lambda_\gamma = \Lambda_\gamma^*$.

2.1 Linearization

Let $\gamma = 1 + \epsilon h$, $\epsilon \ll 1$, $h \in L^\infty(\Omega)$. Consider $\nabla \cdot (\gamma(x)\nabla v) = 0$ in Ω and we write $v = u + \epsilon\widetilde{u}$, where u satisfies $\Delta u = 0$. Look at

$$\begin{aligned} \lim_{\epsilon\to 0} \frac{B_\gamma(f,g) - B_1(f,g)}{\epsilon} &= \int_\Omega h(x)\nabla u_f \cdot \nabla u_g \\ &= dB_1(f,g)(h) = 0, \end{aligned}$$

where $\Delta u_f = \Delta u_g = 0$ in Ω, $u_f = f$ and $u_g = g$ on $\partial\Omega$. $dB_1(f,g)(h) = 0$ means that

$$\int_\Omega h\nabla u \cdot \nabla v dx = 0, \ \forall \Delta u = \Delta v = 0. \tag{3}$$

Does this imply that $h \equiv 0$? If we can show that

$$\{\nabla u \cdot \nabla v | u, v \text{ are harmonic}\}$$

is dense in $L^2(\Omega)$, then we are done.

Consider $\zeta \in \mathbb{C}^N$ $(N \geq 2)$ and put the ansatz $u = e^{\zeta \cdot x}$, $\Delta u = (\zeta \cdot \zeta) e^{\zeta \cdot x} = 0$, which implies $\zeta \cdot \zeta = 0$. If we write $\zeta = \mathrm{Re}\zeta + i\mathrm{Im}\zeta$, then

$$\zeta \cdot \zeta = |\mathrm{Re}\zeta|^2 + 2i\mathrm{Re}\zeta \cdot \mathrm{Im}\zeta - |\mathrm{Im}\zeta|^2 = 0,$$

if and only if

$$|\mathrm{Re}\zeta| = |\mathrm{Im}\zeta|, \ \mathrm{Re}\zeta \cdot \mathrm{Im}\zeta = 0.$$

Let $\zeta_1 = \frac{1}{2}(\xi - ik)$, $\zeta_2 = \frac{1}{2}(-\xi - ik)$, $|\xi| = |k|$, $\xi \cdot k = 0$, $\xi, k \in \mathbb{R}^N$ $(\zeta_1 \cdot \zeta_1 = \zeta_2 \cdot \zeta_2 = 0)$. Substituting $u = e^{\zeta_1 \cdot x}$ and $v = e^{\zeta_2 \cdot x}$ into (3), implies

$$\begin{aligned} 0 &= \int_\Omega h \nabla(e^{\zeta_1 \cdot x}) \cdot \nabla(e^{\zeta_2 \cdot x}) dx \\ &= (\zeta_1 \cdot \zeta_2) \int_\Omega h e^{(\zeta_1 + \zeta_2) \cdot x} = \int_\Omega h(x) e^{-ik \cdot x} dx \\ &= \int_{\mathbb{R}^N} (\chi_\Omega h) e^{-ik \cdot x} dx = (\widehat{\chi_\Omega h})(k), \end{aligned}$$

for all $k \neq 0$. Thus, $\chi_\Omega h \equiv 0$, or $h = 0$. The linearized DN map is locally injective. But we can not conclude that Λ is locally injective, i.e., $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$ and γ_1, γ_2 are close to 1, can we imply $\gamma_1 = \gamma_2$? The answer is NO and the reason is NO implicit function theorem for Λ.

3 Proof of global uniqueness

This result was refered to Sylvester-Uhlmann 1987. Let $\gamma_1(x), \gamma_2(x) \in C^\infty(\overline{\Omega})$ and $\Lambda_{\gamma_1}, \Lambda_{\gamma_2}$ be two corresponding DN maps. Assume that $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$, then $\gamma_1 \equiv \gamma_2$. The conductivity equation $\nabla \cdot (\gamma \nabla u) = 0$ in Ω, Λ_γ is the DN map with respect to this equation.

Schrodinger equation

$$\begin{cases} \Delta u - qu = 0 & \text{in } \Omega, \\ u = f \in H^{1/2}(\partial\Omega) & \text{on } \partial\Omega, \end{cases}$$

where $q(x) \in L^\infty(\Omega)$. Assume 0 is not a Dirichlet eigenvalue of $(\Delta - q)$. Define the DN map $\Lambda_q : f \to \frac{\partial u}{\partial \nu}|_{\partial\Omega}$. The inverse problem is given Λ_q to determine q. Let $v = \sqrt{\gamma} u$, i.e., $u = \gamma^{-\frac{1}{2}} u$, then

$$0 = \gamma^{-\frac{1}{2}} \nabla \cdot (\gamma \nabla (\gamma^{-\frac{1}{2}} v)) = (\Delta - q) v,$$

where $q = \frac{\Delta\sqrt{\gamma}}{\sqrt{\gamma}}$. Now if we can show that $\Lambda_{q_1} = \Lambda_{q_2}$ implying $q_1 = q_2$, then

$$q = q_1 = q_2 = \frac{\Delta\sqrt{\gamma_1}}{\sqrt{\gamma}_1} = \frac{\Delta\sqrt{\gamma_2}}{\sqrt{\gamma_2}},$$

which means

$$\begin{cases} \Delta\sqrt{\gamma_1} - q\sqrt{\gamma_1} = 0, \\ \Delta\sqrt{\gamma_2} - q\sqrt{\gamma_2} = 0. \end{cases}$$

If $\gamma_1=\gamma_2=\gamma$ on $\partial\Omega$, then $\sqrt{\gamma}$ can be determined by solving

$$\begin{cases}\Delta\sqrt{\gamma}-q\sqrt{\gamma}=0 & \text{in } \Omega,\\ \sqrt{\gamma}|_{\partial\Omega} \text{ is given,}\end{cases}$$

then $\sqrt{\gamma_1}=\sqrt{\gamma_2}$ of course in Ω.

3.1 Relation between Λ_γ and Λ_q with $q=\dfrac{\Delta\sqrt{\gamma}}{\sqrt{\gamma}}$

Assume that u solves $\nabla\cdot(\gamma\nabla u)=0$ in Ω with $u|_{\partial\Omega}=f$.

$$\begin{aligned}\Lambda_q(\sqrt{\gamma}f)&=\frac{\partial(\sqrt{\gamma}u)}{\partial\nu}|_{\partial\Omega}=\frac{1}{2}\gamma^{-\frac{1}{2}}\frac{\partial\gamma}{\partial\nu}|_{\partial\Omega}f+\sqrt{\gamma}\frac{\partial u}{\partial\nu}|_{\partial\Omega}\\&=\frac{1}{2}\gamma^{-\frac{1}{2}}\frac{\partial\gamma}{\partial\nu}|_{\partial\Omega}f+\gamma^{-\frac{1}{2}}\gamma\frac{\partial u}{\partial\nu}|_{\partial\Omega}\\&=\frac{1}{2}\gamma^{-\frac{1}{2}}\frac{\partial\gamma}{\partial\nu}|_{\partial\Omega}f+\gamma^{-\frac{1}{2}}\Lambda_\gamma f,\end{aligned}$$

i.e., $\forall g\in H^{\frac{1}{2}}(\partial\Omega)$,

$$\Lambda_q(g)=\frac{1}{2}\gamma^{-1}\frac{\partial\gamma}{\partial\nu}|_{\partial\Omega}g+\gamma^{-\frac{1}{2}}\Lambda_\gamma(\gamma^{-\frac{1}{2}}g)$$

or equivalently,

$$\Lambda_q(\cdot)=\frac{1}{2}\gamma^{-1}\frac{\partial\gamma}{\partial\nu}|_{\partial\Omega}\cdot+\gamma^{-\frac{1}{2}}\Lambda_\gamma(\gamma^{-\frac{1}{2}}\cdot).$$

Theorem 3.1. *(Kohn-Vogelius, CPAM '84) Assume that $\gamma_1,\gamma_2\in C^\infty(\overline{\Omega})$. If $\Lambda_{\gamma_1}=\Lambda_{\gamma_2}$, then for all $\ell\in\mathbb{N}\cup\{0\}$,*

$$(\frac{\partial}{\partial\nu})^\ell\gamma_1(x)=(\frac{\partial}{\partial\nu})^\ell\gamma_2(x),\ \forall x\in\partial\Omega.$$

Corollary 3.2. $\Lambda_{\gamma_1}=\Lambda_{\gamma_2}$ *implies* $\Lambda_{q_1}=\Lambda_{q_2}$ *with* $q_1=\dfrac{\Delta\sqrt{\gamma_1}}{\sqrt{\gamma}_1}$, $q_2=\dfrac{\Delta\sqrt{\gamma_2}}{\sqrt{\gamma_2}}$.

Problem 3.3. $\Lambda_{q_1}=\Lambda_{q_2}$ implies $q_1=q_2$, $q_1,q_2\in L^\infty(\Omega)$. How do we solve it ?

Step 1. If $\Lambda_{q_1}=\Lambda_{q_2}$, then

$$0=\langle\Lambda_{q_1}f,g\rangle-\langle f,\Lambda_{q_2}g\rangle=\int_\Omega(q_2-q_1)u_1u_2dx,$$

where $\Delta u_1-q_1u_1=0=\Delta u_2-q_2u_2$, $u_1|_{\partial\Omega}=f$, $u_2|_{\partial\Omega}=g$, i.e.,

$$\int_\Omega(q_1-q_2)u_1u_2dx=0.$$

Can we show that $q_1-q_2=0$?

Need to find special solutions of $(\Delta-q)u=0$ in Ω.

3.2 Complex geometrical optics (CGO) solutions

Our aim is to find $u = e^{\zeta \cdot x}(1+r)$, where $\zeta \cdot \zeta = 0$, $\zeta \in \mathbb{C}^N$ and

$$\|r\|_{L^2(\Omega)} \leq \frac{C}{|\zeta|},$$

where $C = C(\|q\|, N, \Omega)$.

Suppose that CGO solutions exist. Let $\xi, k, \eta \in \mathbb{R}^N$, $N \geq 3$ and $\zeta_1 = \frac{\xi}{2} - i(\frac{k}{2} + \frac{\eta}{2})$, $|\xi|^2 = |k|^2 + |\eta|^2$, $\xi \cdot k = \xi \cdot \eta = k \cdot \eta = 0$ imply $\zeta_1 \cdot \zeta_1 = 0$. Choose $\zeta_2 = -\frac{\xi}{2} - i(\frac{k}{2} - \frac{\eta}{2})$, similarly, $\zeta_2 \cdot \zeta_2 = 0$. $\zeta_1 + \zeta_2 = -ik$. Let

$$u_1 = e^{\zeta_1 \cdot x}(1+r_1) \text{ and } u_2 = e^{\zeta_2 \cdot x}(1+r_2).$$

Thus,

$$\begin{aligned} 0 &= \int_\Omega (q_1 - q_2) u_1 u_2 dx \\ &= \int_\Omega (q_1 - q_2) e^{(\zeta_1 + \zeta_2)\cdot x}(1 + r_1 + r_2 + r_1 r_2), \end{aligned}$$

or

$$\int_\Omega (q_2 - q_1) e^{-ik\cdot x} = \int_\Omega (q_1 - q_2) e^{-ik\cdot x}(r_1 + r_2 + r_1 r_2). \tag{4}$$

Fix the frequency k, let $|\eta| \to \infty$, then $|\zeta_1|, |\zeta_2| \to \infty$ as well. Therefore the right hand side of (4) will tend to zero since the decaying properties of r_ℓ for $\ell = 1, 2$.

Remark 3.4. This approach does NOT work for $N = 2$ (there is no freedom to fix k and change η).

3.3 Construction of CGO

Ansatz: $u = e^{\zeta \cdot x} w$ and

$$e^{-\zeta \cdot x}(\Delta - q)(e^{\zeta \cdot x} w) = (\Delta + e\zeta \cdot \nabla - q) w = 0.$$

If $w = 1 + r$, then r satisfies

$$(\Delta + 2\zeta \cdot \nabla - q) r = q.$$

There are infinitely many solutions solving the above equation, the difficulty is

$$\|r\|_{L^2(\Omega)} \leq \frac{C}{|\zeta|}.$$

First of all, we assume $q = 0$ and consider

$$(\Delta + 2\zeta \cdot \nabla) r = f \text{ in } \Omega.$$

Goal: $\|r\|_{L^2(\Omega)} \leq C \frac{\|f\|_{L^2(\Omega)}}{|\zeta|}$.

The symbol of $\Delta_\zeta := \Delta + 2\zeta \cdot \nabla$ is

$$-|\xi|^2 + 2i\zeta \cdot \xi = -|\xi|^2 + 2i(\mathrm{Re}\zeta + i\mathrm{Im}\zeta) \cdot \xi,$$

this symbol vanishes for some $\xi \neq 0$. The operator Δ_ζ is NOT elliptic if we consider ζ as another variable.

Theorem 3.5. *Suppose $\zeta \cdot \zeta = 0$, $|\zeta| \geq k > 0$, $f \in L^2_{1+\delta}$ with $-1 < \delta < 0$. Then there exists a unique $w \in L^2_\delta$ solving $\Delta_\zeta w = f$ in $\mathbb{R}^N$ and*

$$\|w\|_{L^2_\delta} \leq \frac{C(\delta, N, k)}{|\zeta|} \|f\|_{L^2_{1+\delta}},$$

where $\|f\|^2_{L^2_\delta} = \int_{\mathbb{R}^N} (1 + |x|^2)^\delta |f|^2 dx$.

Corollary 3.6. *Let $-1 < \delta < 0$, there exists $\epsilon = \epsilon(\delta)$ and $C = C(\delta)$ such that for every $q \in L^2_{1+\delta} \cap L^\infty_{\frac{1}{2}}$ and every $\zeta \in \mathbb{C}^N$ ($\zeta \cdot \zeta = 0$) and*

$$\frac{\|(1 + |x|^2)^{\frac{1}{2}} q\|_{L^\infty}}{|\zeta|} \leq \epsilon, \tag{5}$$

there exists a unique solution u to

$$(\Delta - q)u = 0 \text{ in } \mathbb{R}^N$$

of the form $u = e^{\zeta \cdot x}(1 + r(x, \zeta))$ and

$$\|r\|_{L^2_\delta} \leq \frac{C}{|\zeta|} \|f\|_{L^2_{\delta+1}}.$$

Proof. (Sketch) Since $\Delta_\zeta w = f$, from Theorem 3.5, we have

$$\|\Delta_\zeta^{-1} f\|_{L^2_\delta} \leq \frac{C}{|\zeta|} \|f\|_{L^2_{1+\delta}},$$

regard $\Delta_\zeta^{-1} : L^2_{1+\delta} \to L^2_\delta$ and $\|\Delta_\zeta^{-1}\| \leq \dfrac{C}{|\zeta|}$. Take the conjugate operator

$$e^{-\zeta \cdot x}(\Delta - q)(e^{\zeta \cdot x}(1 + r)) = (\Delta_\zeta - q)(1 + r) = 0,$$

or $(\Delta_\zeta - q)r = q$ and $(I - \Delta_\zeta^{-1} q)r = \Delta_\zeta^{-1} q$, The inverse $(I - \Delta_\zeta^{-1} q)^{-1}$ exists provided $|\zeta| \gg 1$ (see Neumann series, and $q \in L^\infty_{1/2}$ is guaranteed for the existence of (5)). □

Remark 3.7. The proof only works for isotropic conductivity, $\gamma(x)$ is a scalar function.

Theorem 3.8. *(P. Hahner) Let $\zeta \cdot \zeta = 0$, $\zeta \in \mathbb{C}^N$ and Ω is open bounded domain in $\mathbb{R}^N$, $N \geq 2$. Then $\exists C = C(\Omega, N) > 0$ such that $\exists \phi$ solving*

$$(D^2 + 2\zeta \cdot D)\phi = f \text{ in } \Omega, \tag{6}$$

and satisfying

$$\|\phi\|_{L^2(\Omega)} \leq \frac{C}{|\zeta|} \|f\|_{L^2(\Omega)}, \ \|\nabla \phi\|_{L^2(\Omega)} \leq C\|f\|_{L^2(\Omega)},$$

where $D = \dfrac{1}{i}\nabla$ ($D^2 = D \cdot D = -\Delta$).

Remark 3.9. To construct $u = e^{\zeta\cdot x}(1+\phi)$, $\zeta\cdot\zeta = 0$ satisfying $(D^2+q)u = 0$ in Ω, $\|\phi\|_{L^2(\Omega)} \le \dfrac{C}{|\zeta|}$. We need to prove the existence of

$$e^{-\zeta\cdot x}(D^2+q)(e^{\zeta\cdot x}(1+\phi)) = 0 \text{ in } \Omega,$$

if and only if

$$(D^2 + 2\zeta\cdot D + q)\phi = -q \text{ in } \Omega.$$

Proof. WLGO, we may assume that $\Omega \subset Q := [-\pi,\pi]^N$. $\zeta\cdot\zeta = 0$ iff $|\mathrm{Re}\zeta| = |\mathrm{Im}\zeta|$, $\mathrm{Re}\zeta\cdot\mathrm{Im}\zeta = 0$. By rotation, we can also assume that $\zeta = s(e_1 + ie_2)$, where e_1, e_2 are standard orthonormal basis in $\mathbb{R}^N$ and $s = \dfrac{|\zeta|}{\sqrt{2}}$. $D^2 + 2\zeta\cdot D = D^2 + 2s(D_1 + iD_2)$. Let us consider a set of functions $\{e^{i(k+\frac{e_2}{2})\cdot x}\}$, $k = (k_1, k_2, \cdots, k_N) \in \mathbb{Z}^N$. Define the inner product

$$(u,v) = \frac{1}{(2\pi)^N}\int_Q u\bar{v}dx,\ \ u,v\in L^2(Q).$$

We can show that $\{w_k := e^{i(k+\frac{e_2}{2})\cdot x}\}_{k\in\mathbb{Z}^N}$ is a complete orthonormal basis in $L^2(Q)$. Let us write $\phi = \sum\phi_k w_k$, $f = \sum f_k w_k$, and $\|f\|_{L^2}^2 = \sum|f_k|^2$.

Observation: $D(e^{i(k+\frac{e_2}{2})\cdot x}\cdot) = e^{i(k+\frac{e_2}{2})\cdot x}(D + (k+\dfrac{e_2}{2}))\cdot$. In order to satisfy (6), ϕ_k satisfies

$$p_k\phi_k := [|k_2 + \frac{e_2}{2}|^2 + 2s(k_1 + i(k_2 + \frac{1}{2}))]\phi_k = f_k,\ \ k\in\mathbb{Z}^N.$$

Note that p_k is never 0 and $\phi_k = \dfrac{f_k}{p_k}$ and $\phi = \sum\phi_k w_k$. Now, we estimate ϕ_k,

$$|\phi_k| = \frac{|f_k|}{||k_2 + \frac{e_2}{2}|^2 + 2s(k_1 + i(k_2+\frac{1}{2}))|} \le \frac{|f_k|}{s}.$$

So $\sum|\phi_k|^2 \le \dfrac{1}{s^2}\sum|f_k|^2$, i.e., $\|\phi\|_{L^2(Q)} \le \dfrac{C}{s}\|f\|_{L^2(Q)}$.

Now, we note that $D\phi = \sum_k \phi_k(k+\dfrac{e_2}{2})w_k$. Therefore,

$$\phi_k(k+\frac{e_2}{2}) = \frac{(k+\frac{e_2}{2})f_k}{|k_2+\frac{e_2}{2}|^2 + 2s(k_1 + i(k_2+\frac{1}{2}))}.$$

We consider two cases: One is $|k+\dfrac{e_2}{2}| < 4s$, then

$$|\phi_k(k+\frac{e_2}{2})| \le \frac{4s}{2s|k_2+\frac{1}{2}|}|f_k| \le 4|f_k|.$$

The other is $|k+\dfrac{e_2}{2}| \ge 4s$, then

$$\begin{aligned}||k+\frac{e_2}{2}|^2 + 2sk_1| &\ge |k+\frac{e_2}{2}|^2 - 2s|k+\frac{e_2}{2}| \\ &\ge ||k+\frac{e_2}{2}|^2 - \frac{1}{2}|k+\frac{e_2}{2}|^2 \\ &= \frac{1}{2}|k+\frac{e_2}{2}|^2.\end{aligned}$$

So

$$\begin{aligned}|\phi_k(k+\frac{e_2}{2})| &\le \frac{|k+\frac{e_2}{2}||f_k|}{\frac{1}{2}|k+\frac{e_2}{2}|^2} = \frac{|f_k|}{\frac{1}{2}|k+\frac{e_2}{2}|}\\ &\le \frac{|f_k|}{\frac{1}{2}4s} = \frac{|f_k|}{2s}.\end{aligned}$$

Thus,

$$\|\nabla\phi\|_{L^2(Q)} \le C\|f\|_{L^2(Q)}.$$

□

4 Another construction (based on the Carleman estimates)

Construct solutions $u = e^{\zeta\cdot x}v$ satisfying $(D^2+q)u = 0$ in Ω. Write

$$\begin{aligned}\zeta\cdot x &= \frac{1}{h}(\alpha+i\beta)\cdot x\\ &= \frac{1}{h}(\varphi(x)+i\psi(x)),\end{aligned}$$

where $|\alpha| = |\beta| = 1$, $\alpha\perp\beta$, and $\varphi(x) = \alpha\cdot x$, $\psi(x) = \beta\cdot x$. Now, we need to solve

$$e^{-\frac{\varphi+i\psi}{h}}(D^2+q)(e^{\frac{\varphi+i\psi}{h}}(1+r)) = 0,$$

or

$$e^{-\frac{\varphi+i\psi}{h}}(D^2+q)(e^{\frac{\varphi+i\psi}{h}}r) = f\ (=q) \in L^2(\Omega).$$

Want

$$\begin{aligned}\|r\|_{L^2(\Omega)} &= \|e^{i\frac{\psi}{h}}r\|_{L^2(\Omega)}\\ &\le Ch\|e^{-\frac{\varphi+i\psi}{h}}(D^2+q)(e^{\frac{\varphi+i\psi}{h}}r)\|_{L^2(\Omega)}\end{aligned}$$

if and only if

$$\|\widetilde{r}\|_{L^2(\Omega)} \le Ch\|e^{-\frac{\varphi}{h}}(D^2+q)e^{\frac{\varphi}{h}}\widetilde{r}\|_{L^2(\Omega)}.$$

Theorem 4.1. *(Carleman estimate) Let $q\in L^\infty(\Omega)$, $\varphi(x) = \alpha\cdot x$ with $|\alpha| = 1$. Then there exists $C = C(\Omega, N, \|q\|_{L^\infty}) > 0$ and $h_0 = h_0(\Omega, N, \|q\|_{L^\infty}) > 0$ such that $0 < h \le h_0$, we have*

$$\|v\|_{L^2(\Omega)} \le Ch\|e^{\frac{\varphi}{h}}(D^2+q)(e^{-\frac{\varphi}{h}}v)\|_{L^2(\Omega)}, \tag{7}$$

$\forall v \in C_c^\infty(\Omega)$.

Proof. It suffices to show

$$\|v\|_{L^2(\Omega)} \le ch\|e^{\frac{\varphi}{h}}D^2(e^{-\frac{\varphi}{h}}v)\|_{L^2(\Omega)}.$$

If this true, then (7) will be trivial since

$$\begin{aligned}\|v\|_{L^2(\Omega)} &\le ch\|e^{\frac{\varphi}{h}}(D^2+q-q)(e^{-\frac{\varphi}{h}}v)\|_{L^2(\Omega)}\\ &\le ch\|e^{\frac{\varphi}{h}}D^2(e^{-\frac{\varphi}{h}}v)\|_{L^2(\Omega)} + Ch\|q\|_\infty\|v\|_{L^2(\Omega)},\end{aligned}$$

by choosing h small enough and absorbing to the left hand side, then we are done.

Let $P_0 = h^2 D^2 = (hD)^2$ and

$$\begin{aligned}
P_{0,\varphi} &= e^{\frac{\varphi}{h}} P_0(e^{-\frac{\varphi}{h}}) = \sum_j (hD_j + i\partial_{x_j}\varphi)^2 \\
&= \sum_j (hD_j + i\alpha_j)^2 = (hD)^2 - 1 + 2i\alpha \cdot (hD) \\
&= A + iB
\end{aligned}$$

with $A = (hD)^2 - 1$, $B = 2i\alpha \cdot (hD)$ and noting that $A = A^*$, $B = B^*$, $[A, B] = 0$ (constant coefficients). Our estimate is equivalent to

$$Ch\|v\|_{L^2(\Omega)} \le C\|P_{0,\varphi}v\|_{L^2(\Omega)}.$$

Compute

$$\begin{aligned}
\|P_{0,\varphi}v\|^2_{L^2(\Omega)} &= (P_{0,\varphi}v, P_{0,\varphi}v) \\
&= ((A + iB)v, (A + iB)v) \\
&= \|Av\|^2_{L^2} + \|Bv\|^2_{L^2} - i([A, B]v, v) \\
&= \|Av\|^2_{L^2} + \|Bv\|^2_{L^2} \ge \|Bv\|^2_{L^2} \\
&\ge C'h^2\|v\|^2_{L^2},
\end{aligned}$$

where the last inequality is given by the Poincare inequality. □

Theorem 4.2. *Let $\varphi(x) = \alpha \cdot x$, with $|\alpha| = 1$ and $q \in L^\infty(\Omega)$. $\exists C > 0, h_0 > 0$ such that for any $f \in L^2(\Omega)$, there exists a solution $v \in L^2(\Omega)$ solving*

$$e^{\frac{\varphi}{h}}(D^2 + q)(e^{-\frac{\varphi}{h}}v) = f \text{ in } \Omega$$

and

$$\|v\|_{L^2(\Omega)} \le Ch\|f\|_{L^2(\Omega)}.$$

Proof. Let $P_\phi = e^{\frac{\varphi}{h}}(h^2D^2 + h^2q)e^{-\frac{\varphi}{h}} = P_{0,\varphi} + h^2 q$ and $P^*_{0,\varphi} = P_{0,-\varphi} + h^2\overline{q}$. We have the Carleman estimate for P^*_φ, i.e.,

$$\|v\|_{L^2(\Omega)} \le \frac{C}{h}\|P^*_\varphi v\|_{L^2(\Omega)},$$

for all $v \in C^\infty_c(\Omega)$. Prove the existence by the Hahn-Banach theorem.

Define $\mathcal{D} = P^*_\varphi C^\infty_c(\Omega)$ be a subspace of $L^2(\Omega)$. A linear functional $L : \mathcal{D} \to \mathbb{C}$ by

$$L(P^*_\varphi v) = (v, f) \ \forall v \in C^\infty_c(\Omega).$$

Note that L is well-defined, i.e., $P^*_\varphi v_1 = P^*_\varphi v_2$ implies $v_1 = v_2$ by using the Carleman estimate we proved before.

$$\begin{aligned}
|L(P^*_\varphi v)| &= |(v, f)| \le \|v\|_{L^2}\|f\|_{L^2} \\
&\le \frac{C}{h}\|f\|_{L^2}\|P^*_\varphi v\|_{L^2}.
\end{aligned}$$

Therefore, L is a bounded linear functional on $\mathcal{D}$ with the norm bounded by $\frac{C}{h}\|f\|_{L^2}$. By the Hahn-Banach theorem, $\exists$ a bounded linear functional $\widetilde{L}$: $L^2(\Omega) \to \mathbb{C}$ and $\widetilde{L}|_{\mathcal{D}} = L$ and $\|\widetilde{L}\| \leq \frac{C}{h}\|f\|_{L^2}$. By the Riesz representation theorem, there is a unique $\widetilde{u} \in L^2$ such that

$$\widetilde{L}(w) = (w, \widetilde{u}),\ w \in L^2(\Omega)$$

and $\|\widetilde{u}\|_{L^2(\Omega)} \leq \frac{C}{h}\|f\|_{L^2}$.

Then for any $v \in C_c^\infty(\Omega)$, by the weak derivative

$$\begin{aligned}(v, P_\varphi \widetilde{u}) &= (P_\varphi^* v, \widetilde{u}) = \widetilde{L}(P_\varphi^* v)\\ &= L(P_\varphi^* v) = (v, f),\end{aligned}$$

i.e., $P_\varphi \widetilde{u} = f$ and $\|\widetilde{u}\|_{L^2(\Omega)} \leq \frac{C}{h}\|f\|_{L^2}$, which implies

$$\|h^2\widetilde{u}\|_{L^2(\Omega)} \leq Ch\|f\|_{L^2(\Omega)}.$$

□

Theorem 4.3. *For $\rho = \varphi + i\psi$ ($\nabla\rho\cdot\nabla\rho = 0$), $\varphi = \alpha\cdot x$. We can find a solution $u = e^{-\frac{\rho}{h}}(a + r)$ solving $(D^2 + q)u = 0$ in Ω and $L\hat{}$*

$$\|r\|_{L^2(\Omega)} \leq Ch\|q\|_{L^2(\Omega)}.$$

Proof. Want $e^{\frac{\rho}{h}}(D^2+q)(e^{-\frac{\rho}{h}}(a+r)) = 0$ or $e^{\frac{\varphi}{h}}(D^2+q)(e^{-\frac{\varphi}{h}}(e^{-\frac{i\psi}{h}}r)) = -e^{-\frac{i\psi}{h}}(D^2+q)(e^{-\frac{\rho}{h}}a)$. RHS$= -e^{-\frac{i\psi}{h}}(-h^{02}\nabla\rho\cdot\nabla\rho + h^{-1}[2\nabla\rho\cdot\nabla + \Delta\rho] + (D^2+q))a$, want

$$\nabla\rho\cdot\nabla\rho = 0 \text{ and } (2\nabla\rho\cdot\nabla + \Delta\rho)a = 0 \text{ (transport equation).}$$

Note that $\nabla\rho\cdot\nabla\rho = 0$ gives $|\nabla\varphi| = |\nabla\psi|$ and $\nabla\varphi\cdot\nabla\psi = 0$. Since $\varphi(x) = \alpha\cdot x$, we choose $\psi(x) = \beta\cdot x$ with $\alpha \perp \beta$, $|\beta| = 1$. By the choice of φ, ψ, the transport equation holds automatically. We simply take $a = 1$ and $\|f\|_{L^2(\Omega)} = \|q\|_{L^2(\Omega)}$. □

5 Inverse scattering problem

Let us consider the acoustic equation in $\mathbb{R}^3$

$$(\Delta + k^2 n(x))u = 0 \text{ in } \mathbb{R}^3,$$

where k is the wave number and $n(x)$ is refractive index. Assume that supp$(m) \subset B$ (some ball) with $m = n - 1$, i.e., $(\Delta + k^2 m + k^2)u = 0$ in $\mathbb{R}^3$ (perturbed) and $(\Delta + k^2)v = 0$ in $\mathbb{R}^3$ (unperturbed, Helmholtz equation).

Take u^{inc} satisfying $(\Delta + k^2)u^{inc} = 0$, for example, $u^{inc}(x) = e^{ikx\cdot d}$, with $d \in \mathbb{S}^2$ (d is the direction of the plane waves). Let $u = u^{inc} + u^s$ solving

$$\begin{cases}(\Delta + k^2 n(x))u = 0 \text{ in } \mathbb{R}^3,\\ u^s \text{ satisfies } \lim_{r\to\infty} r(\frac{\partial u^s}{\partial r} - iku^s) = 0,\end{cases}$$

where $r = |x|$.

5.1 Lippmann-Schwinger equation

$u = u^{inc} - k^2 \int_{\mathbb{R}^3} \Phi(x,y)m(y)u(y)dy$, $\Phi(x,y) = \dfrac{1}{4\pi}\dfrac{e^{ik|x-y|}}{|x-y|}$ is the outgoing fundamental solution of $\Delta + k^2$. When $|x| \gg 1$, $|x-y| = |x| - \widehat{x}\cdot y + O(\dfrac{1}{|x|})$, with $\widehat{x} = \dfrac{x}{|x|}$, then

$$u = u^{inc} + \frac{e^{ik(x)}}{|x|}u_\infty(\widehat{x}) + O(\frac{1}{|x|^2}),$$

where $u_\infty(\widehat{x}) = -\dfrac{k^2}{2\pi}\int_{\mathbb{R}^3} e^{-ik\widehat{x}\cdot y}m(y)u(y)dy$ is called the far-field vector.

In most cases, we use the plane wave as the incident field, which means $u^{inc} = e^{ikx\cdot d}$. We write

$$u_\infty(\widehat{x}) = a(d, \widehat{x}, k): \text{ scattering amplitude.}$$

Inverse problem: Determine $n(x)$ (or $m(x)$) from $a(d,\widehat{x},k)$, $\forall d, \widehat{x} \in \mathbb{S}^2$, for one fixed $k > 0$.
Focus on the uniqueness theorem: $a_1(d,\widehat{x},k) = a_2(d,\widehat{x},k)$, $\forall d,\widehat{x} \in \mathbb{S}^2$, we want to show that $n_1(x) = n_2(x)$.

Remark 5.1. DN-map: Near field data. Scattering amplitude: Far-field data.

Aim: Reduce the problem to the inverse boundary value problem for $(\Delta + k^2 n(x))u = 0$ in B.

5.2 Rellich's lemma

What happens if $a(d,\widehat{x},k) = 0$?

Lemma 5.2. *(Rellich's lemma) If v solves $(\Delta + k^2)v = 0$ in B^c (the exterior domain) and satisfies*

$$\lim_{r\to\infty}\int_{|x|=r}|v|^2 dS = 0, \tag{8}$$

then $v \equiv 0$ in B^c.

Remark 5.3. There is no need to assume that k^2 is not an eigenvalue of $-\Delta$. Moreover, if we want to obtain the uniqueness from the DN-map, then we need to add such assumption.

Corollary 5.4. *If $a_1(d,\widehat{x},k) = a_2(d,\widehat{x},k)$, then*

$$u_1^s(x) = u_2^s(x) \text{ in } B^c.$$

Proof. $(\Delta + k^2)(u_1^s - u_2^s) = 0$ in B^c and $u_1^s - u_2^s = O(\dfrac{1}{|x|^2})$. Then $u_1^s - u_2^s$ satisfies the condition of the Rellich lemma. □

Proof. (Proof of the Rellich lemma, a sketch) Using the spherical harmonic $Y_n^m(\widehat{x})$. Any solution v of $(\Delta + k^2)v = 0$ in B^c is written by

$$v = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} a_n^m(r)Y_n^m(\widehat{x}).$$

Parseval's formula says

$$\int_{|x|=r} |v|^2 = r^2 \sum_{n=0}^{\infty} \sum_{m=-n}^{n} |a_n^m(r)|^2.$$

(8) means that

$$\lim_{r\to\infty} r^2 |a_n^m(r)|^2 = 0.$$

On the other hand, $a_n^m(r)$ satisfies

$$\frac{d^2 a_n^m}{dr^2} + \frac{2}{r}\frac{da_n^m}{dr} + (k^2 - \frac{n(n+1)}{r^2})a_n^m = 0.$$

The representation formula for $a_n^m(r)$ is

$$a_n^m(r) = \alpha_n^m h_n^{(1)}(rk) + \beta_n^m h_n^{(2)}(rk),$$

where $h_n^{(\ell)}$ are spherical Hankel's functions, $\ell = 1, 2$. By the asymptotic behaviors of $h_n^{(\ell)}(rk)$, $\ell = 1, 2$, we can show that $\alpha_n^m = \beta_n^m = 0$. □

So far, we have shown that $a_1(d, \widehat{x}, k) = a_2(d, \widehat{x}, k)$ implying that $u_1^s = u_2^s$ in B^c. This implies $\Lambda_1(u^s) = \Lambda_2(u^s)$ for all scattered solutions u^s, where Λ_1, Λ_2 are DN-maps for $(\Delta + k^2 n_j(x))u = 0$ in B (assume that k^2 is not an eigenvalue).

Lemma 5.5. *Span*$\{u^s(\cdot, d) : d \in \mathbb{S}^2\}$ *is dense in all solutions of* $(\Delta + k^2 n(x))u = 0$ *in* $B' \supset B$ *in the* L^2*-sense.*

Proof. See Isakov's book, inverse problems for PDEs. □

Remark 5.6. By the interior estimates for the elliptic equations, $\{u^s\}$ will be dense in the $H^1(B)$ and *Span*$\{u^s(\cdot, d)|_{\partial B}\}$ is dense in $H^{1/2}(\partial B)$. Using this lemma, $\Lambda_1 = \Lambda_2$.

Theorem 5.7. $a_1(d, \widehat{x}, k) = a_2(d, \widehat{x}, k)$ $\forall d, \widehat{x} \in \mathbb{S}^2$ *implies* $n_1(x) = n_2(x)$.

Proof. $\Lambda_1 = \Lambda_2$ implies $kn_1(x) = kn_2(x)$. □

Remark 5.8. Conversely, if $\Lambda_1 = \Lambda_2$ on ∂B, then $a_1(d, \widehat{x}, k) = a_2(d, \widehat{x}, k)$. In fact, $u_1^s(x) = u_2^s(x)$ in B^c.

Proof. Let us consider w^* satisfying

$$\begin{cases} (\Delta + k^2 n_2)w^* = 0, & \text{in } B, \\ w^* = u_1^s & \text{on } \partial B. \end{cases}$$

Define $w = \begin{cases} w^*, & x \in B \\ u_1^s, & x \in B^c \end{cases}$. Since $\Lambda_2(w^*|_{\partial B}) = \Lambda_1(w^*|_{\partial B}) = \Lambda_1(u_1^s)$, then $w \in H^2_{loc}(\mathbb{R}^3)$. Solving $(\Delta + k^2 n_2)w = 0$ with Sommerfeld radiation condition. By the uniqueness of the scattered solution, $u_2^s = w = u_1^s$ in B^c. To show the uniqueness mentioned before, we need to use the Rellich's lemma and the unique continuation property ($n \in L^\infty$ is a safe case). □

6 Stability estimates

Let $\nabla \cdot (\gamma_j(x)\nabla u_j) = 0$ in Ω and the DN-maps Λ_{γ_j}. We want to derive the following estimate

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\Omega)} \leq \omega(\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_*),$$

where $\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_* = \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{1/2}(\Omega)\to H^{-1/2}(\partial\Omega)}$ and ω is a modulus of continuity.

$$\omega(t) \leq \frac{1}{|\log t|^\tau},\ \tau \in (0,1).$$

This log type stability estimate is optimal. To derive this estimate, we need smoothness assumptions on γ_j. We cannot just assume $\gamma_j \in L^\infty(\Omega)$.

Example 6.1. (Alessandrini) Consider $\gamma_1 = 1$, $\gamma_2 = \begin{cases} 1+\gamma, & 0 < |x| < r_0, \\ 1, & r_0 < |x| < 1, \end{cases}$ $0 < r_0 < 1$ and $\nabla \cdot (\gamma_j \nabla u) = 0$ in $D \subset \mathbb{R}^2$, where D is a unit disc. Given $f \in H^{1/2}(\partial\Omega)$. In terms of the Fourier series, $f = \sum_{k=-\infty}^{\infty} \widehat{f}(k)e^{ik\theta}$. Then

$$\Lambda_{\gamma_1} f(e^{i\theta}) = \sum_{k=-\infty}^{\infty} |k|\widehat{f}(k)e^{ik\theta},$$
$$\Lambda_{\gamma_2} f(e^{i\theta}) = \sum_{k=-\infty}^{\infty} |k|\frac{2 + r(1 + r_0^{2|k|})}{2 + r(1 - r_0^{2|k|})}\widehat{f}(k)e^{ik\theta}.$$

Then

$$\begin{aligned}
&\|(\Lambda_{\gamma_1} - \Lambda_{\gamma_2})f\|_{H^{-1/2}(\partial\Omega)} \\
=&\sum_{k=-\infty}^{\infty} (1+|k|^2)^{-\frac{1}{2}}|k|^2\Big(1 - \frac{2 + r(1 + r_0^{2|k|})}{2 + r(1 - r_0^{2|k|})}\Big)(1+|k|^2)^{\frac{1}{2}}|\widehat{f}(k)|^2 \\
\leq&(\gamma r_0^2)^2\|f\|^2_{H^{1/2}(\partial\Omega)}.
\end{aligned}$$

Thus,

$$\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_* \leq \gamma r_0^2 \text{ and } \lim_{r_0\to 0}\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_* = 0,$$

but

$$\|\gamma_1 - \gamma_2\|_{L^\infty(D)} = \gamma > 0.$$

Theorem 6.2. *Let Ω be a bounded open domain with smooth boundary in $\mathbb{R}^N$, $N \geq 3$. Assume that $\gamma_1(x), \gamma_2(x) \in H^{s+2}(\Omega)$, $s > \frac{N}{2}$ ($\gamma_1, \gamma_2 \in C^2(\overline{\Omega})$). Denote $\Lambda_{\gamma_1}, \Lambda_{\gamma_2}$ the associated DN maps and $\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_*$ is a before. Furthermore, we assume $\exists E > 0$ s.t. $\frac{1}{E} \leq \gamma_j(x) \leq E$ and $\|\gamma_j\|_{H^{s+2}(\Omega)} \leq E$. Then $\exists C = C(\Omega, N, E, s)$ and $\tau = \tau(N, s) \in (0,1)$ such that*

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\Omega)} \leq \omega(\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_*),$$

where $\omega(t)$ is a modulus of continuity satisfies $\omega(t) \leq \frac{C}{|\log t|^\tau}$, for $0 < t < \frac{1}{e}$.

Remark 6.3. The log type stability estimate for Calderón's problem is "optimal" for general conductivity. It means that one cannot derive a Hölder type stability.

6.1 Steps of proof

Step 1. Recall that $q = \dfrac{\Delta\sqrt{\gamma}}{\sqrt{\gamma}}$ and

$$\Lambda_q(f) = \frac{1}{2}\gamma^{-1}\frac{\partial\gamma}{\partial\nu}|_{\partial\Omega}f + \gamma^{-\frac{1}{2}}\Lambda_\gamma(\gamma^{-\frac{1}{2}}f) \tag{9}$$

. Needs the stability estimates of $(\gamma_1 - \gamma_2)_{\partial\Omega}$ by $\Lambda_{\gamma_1} - \Lambda_{\gamma_2}$.

Lemma 6.4. *$\|\gamma_1 - \gamma_2\|_{L^\infty(\partial\Omega)} \leq C\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_*$ and $\|\dfrac{\partial\gamma_1}{\partial\nu} - \dfrac{\partial\gamma_2}{\partial\nu}\|_{L^\infty(\partial\Omega)} \leq C\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_*^\sigma$, where $\sigma \in (0,1)$ and C are given as before theorem.*

Using Lemma 6.4 and (9), we can obtain

$$\|\Lambda_{q_1} - \Lambda_{q_2}\|_* \leq C(\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_* + \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|^{\widetilde{\sigma}}),\ \widetilde{\sigma} \in (0,1).$$

Step 2. Using Green's formula and the symmetric property of Λ_q, we have the Alessandrini's identity

$$\int_\Omega (q_1 - q_2)u_1u_2dx = \langle(\Lambda_{q_1} - \Lambda_{q_2})u_1, u_2\rangle_{H^{-\frac{1}{2}}(\partial\Omega)\times H^{\frac{1}{2}}(\partial\Omega)}. \tag{10}$$

Choose $\{\eta_1, \eta_2, \xi\}$ forming orthogonal vectors, $|\eta_1| = |\eta_2| = 1$, $\xi \in \mathbb{R}^N$,

$$\zeta_1 = -\frac{s}{\sqrt{2}}(\sqrt{1 - \frac{|\xi|^2}{2s^2}}\eta_1 + \frac{1}{\sqrt{2}s}\xi + i\eta_2),\ (\zeta_1\cdot\zeta_1 = 0,\ |\zeta_1| = s),$$

$$\zeta_2 = \frac{s}{\sqrt{2}}(\sqrt{1 - \frac{|\xi|^2}{2s^2}}\eta_1 - \frac{1}{\sqrt{2}s}\xi + i\eta_2),\ (\zeta_2\cdot\zeta_2 = 0,\ |\zeta_2| = s).$$

Take $u_1 = e^{i\zeta_1\cdot x}(1+\rho_1)$ and $u_2 = e^{i\zeta_2\cdot x}(1+\rho_2)$ with $\|\rho_1\|_{L^2(\Omega)} \leq \dfrac{C}{|\zeta_1|} = \dfrac{C}{s}$ and $\|\rho\|_{L^2(\Omega)} \leq \dfrac{C}{s}$.

In order to estimate $\|u_1\|_{H^{\frac{1}{2}}(\partial\Omega)}$ and $\|u_2\|_{H^{\frac{1}{2}}(\partial\Omega)}$. It suffices to estimate $\|u_j\|_{H^1(\Omega)}$ and use the standard trace theorem.

$$\begin{aligned}\|u_1\|_{H^1(\Omega)} &\leq \|e^{i\zeta\cdot x}(1+\rho_1)\|_{L^2(\Omega)} + \|\nabla(e^{i\zeta\cdot x}(1+\rho_1))\|_{L^2(\Omega)}\\ &\leq Cse^{sa},\ \text{where } a = \sup_{x\in\Omega}|x|.\end{aligned}$$

By the trace theorem, we have

$$\|u_j\|_{H^{\frac{1}{2}}(\partial\Omega)} \leq Cse^{sa} \leq Ce^{sa},$$

provided that s is large.

Substituting u_1, u_2 into (10), we obtain

$$\begin{aligned}&|\int_\Omega (q_1 - q_2)e^{-i\xi\cdot x}dx|\\ \leq&\|\Lambda_{q_1} - \Lambda_{q_2}\|_*\|u_1\|_{H^{\frac{1}{2}}}\|u_2\|_{H^{\frac{1}{2}}} + |\int_\Omega (q_1 - q_2)e^{-i\xi\cdot x}(\rho_1 + \rho_2 + \rho_1\rho_2)|\\ \leq&Ce^{sa}\|\Lambda_{q_1} - \Lambda_{q_2}\|_* + \frac{C}{s}.\end{aligned}$$

Let $\widetilde{q_j} = \chi_\Omega q_j$, then we actually get

$$|(\widehat{\widetilde{q_1} - \widetilde{q_2}})(\xi)| \leq Ce^{sa}\|\Lambda_{q_1} - \Lambda_{q_2}\|_* + \frac{C}{s},\ s \geq C'(\Omega, E, N, s).$$

Now we want to estimate $\|q_1 - q_2\|_{H^{-1}(\Omega)}$.

$$\begin{aligned}
\|q_1 - q_2\|^2_{H^{-1}(\Omega)} \leq \|\widetilde{q_1} - \widetilde{q_2}\|^2_{H^{-1}(\mathbb{R}^N)} &= \int_{\mathbb{R}^N} \frac{|(\widehat{\widetilde{q_1} - \widetilde{q_2}})(\xi)|^2}{(1+|\xi|^2)} d\xi \\
&= (\int_{|\xi|<R} + \int_{|\xi|>R}) \frac{|(\widehat{\widetilde{q_1} - \widetilde{q_2}})(\xi)|^2}{(1+|\xi|^2)} d\xi \\
&\leq Ce^{cs}R^N\|\Lambda_{q_1} - \Lambda_{q_2}\|^2_* + \frac{CR^N}{s^2} + \frac{C}{R^2},
\end{aligned}$$

where the last term cones from the apriori assumption on q_j. Note that $\|q_j\|_{L^\infty(\Omega)} \leq M = M(E)$. Choose R such that $\frac{CR^N}{s^2} = \frac{C}{R^2}$ or $R = Cs^{\frac{2}{N+2}}$. We now have

$$\|q_1 - q_2\|^2_{H^{-1}(\Omega)} \leq Ce^{Cs}\|\Lambda_{q_1} - \Lambda_{q_2}\|^2_* + Cs^{-\frac{4}{N+2}},\ s \geq C"(\Omega, E, N, s).$$

We now take $s = \frac{1}{C}|\log\|\Lambda_{q_1} - \Lambda_{q_2}\|_*|$. Need $\|\Lambda_{q_1} - \Lambda_{q_2}\|_* < \epsilon < 1$ such that $s = \frac{1}{c}|\log\|\Lambda_{q_1} - \Lambda_{q_2}\|_*| \geq C"$. So we have

$$\|q_1 - q_2\|^2_{H^{-1}(\Omega)} \leq C(\|\Lambda_{q_1} - \Lambda_{q_2}\|_* + C|\log\|\Lambda_{q_1} - \Lambda_{q_2}\|_*|^{-\frac{4}{N+2}}.$$

Take $\|\Lambda_{q_1} - \Lambda_{q_2}\|_*$ even smaller to absorb the first term. In other words, we have shown the following theorem.

Theorem 6.5. *$q_j \in L^\infty(\Omega)$, $\|q_j\| \leq E$ $(N \geq 3)$.*

$$\|q_1 - q_2\|_{H^{-1}(\Omega)} \leq \omega(\|\Lambda_{q_1} - \Lambda_{q_2}\|_*),$$

where $\omega(t) \leq C|\log t|^{-\frac{2}{N+2}}$, $0 < t < e^{-1}$.

Remark 6.6. In the proof, we only consider $\|\Lambda_{q_1} - \Lambda_{q_2}\|_*$ is small, i.e., $\|\Lambda_{q_1} - \Lambda_{q_2}\|_* < \epsilon = \epsilon(E, N, \Omega)$. For t not small, the estimate is obvious ! Moreover, theorem holds for any $q_j \in L^\infty(\Omega)$.

Step 3. Compute $\Delta(\log\sqrt{\gamma_j}) = \nabla\cdot\nabla(\log\sqrt{\gamma_j}) = \nabla\cdot(\frac{\nabla\sqrt{\gamma_j}}{\sqrt{\gamma_j}}) = \frac{\Delta\sqrt{\gamma_j}}{\sqrt{\gamma_j}} - |\nabla(\log\sqrt{\gamma_j})|^2$. Let $w = \log\sqrt{\gamma_1} - \log\sqrt{\gamma_2} = \log\frac{\sqrt{\gamma_1}}{\sqrt{\gamma_2}}$. $\nabla\cdot((\sqrt{\gamma_1\gamma_2})\nabla w) = \sqrt{\gamma_1\gamma_2}(q_1 - q_2)$ with $w|_{\partial\Omega} = (\log\sqrt{\gamma_1} - \log\sqrt{\gamma_2})|_{\partial\Omega}$, by the elliptic estimate, one can get estimate on w.

Finally, to get estimate for $\gamma_1 - \gamma_2$, we compute

$$\begin{aligned}
\log\gamma_1 - \log\gamma_2 &= \int_0^1 \frac{d}{dt}\log((1-t)\gamma_2 + t\gamma_1)dt \\
&= (\int_0^1 \frac{d}{dt}\frac{1}{(1-t)\gamma_1 + \gamma_2}dt)(\gamma_1 - \gamma_2).
\end{aligned}$$

7 Reconstruction

Reconstruct $\gamma(x)$ by Λ_γ (Nachman's reconstruction, for $N \geq 3$).

Step 1. Boundary reconstruction, one can reconstruct $\gamma(x)$, $\dfrac{\partial\gamma}{\partial\nu}(x)$, $\forall x \in \Omega$. We can do this by the full symbol of Λ_γ: first order pseudodifferential operator.
Step 2. We can determine Λ_q by Λ_γ.
Step 3. Determine q from Λ_q.
Step 4. Solve

$$\begin{cases} \Delta\sqrt{\gamma} - q\sqrt{\gamma} = 0 & \text{in } \Omega, \\ \gamma|_{\partial\Omega} \text{ is given.} \end{cases}$$

For Step 3, we consider $\Delta u - qu = 0$ in $\Omega \to \Lambda_q$ and $\Delta v = 0$ in $\Omega \to \Lambda_0$, they are well-defined. Then

$$\begin{aligned} \int_\Omega quvdx &= \int_\Omega (v\Delta u - u\Delta v) = \int_{\partial\Omega} (v\Lambda_q u - u\Lambda_0 v)dS \\ &= \int_{\partial\Omega} (\Lambda_q - \Lambda_0)(v|_{\partial\Omega})(u|_{\partial\Omega})dS. \end{aligned}$$

Let $u = e^{\zeta_1 \cdot x}(1+\rho)$ and $v = e^{\zeta_2 \cdot x}$ with $\zeta_1 \cdot \zeta_1 = \zeta_2 \cdot \zeta_2 = 0$, $\zeta_1 + \zeta_2 = -i\xi$. Hence, we obtain

$$\int_\Omega q e^{\zeta_1 \cdot x}(1+\rho)e^{\zeta_2 \cdot x} = \int_{\partial\Omega} (\Lambda_q - \Lambda_0)(e^{\zeta_2 \cdot x}|_{\partial\Omega})(u|_{\partial\Omega})dS.$$

Let $|\zeta_1| \to \infty$, then

$$\int_\Omega q e^{-i\xi \cdot x} = \lim_{|\zeta_1| \to \infty} \int_{\partial\Omega} (\Lambda_q - \Lambda_0)(e^{\zeta_2 \cdot x}|_{\partial\Omega})(u|_{\partial\Omega})dS.$$

We need to know the boundary values of CGO solutions, which is contributed by Nachman's Annals paper in 1988. $u|_{\partial\Omega}$ satisfies an integral equation on $\partial\Omega$.