**Method of Moving Plane**

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# 1 Motivations.

**Theorem 1.** *Let* $\Omega = \{x \in \mathbb{R}^n : |x| < R\}$ *be the ball in* $\mathbb{R}^n$. *Let* $u$ *be a positive* $C^2$ *solution of the Dirichlet problem* $\Delta u + f(u) = 0$ *in* $\Omega$ *with* $u$*=0 on* $|x| = R$*, where* $f$ *is a* $C^1$ *function. Then* $u$ *is radially symmetric and* $\dfrac{\partial u}{\partial r}$*<0, for* $r \in (0, R)$

**Theorem 2.** *Let* $u$*>0 be a* $C^2$*solution of the Dirichlet problem* $\Delta u + f(u) = 0$ *in a ring domain* $R' \le |x| < R$ *with* $u$*=0 on |x|=R. Then* $\dfrac{\partial u}{\partial r}$*<0 for* $\dfrac{R'+R}{2} \le |x| < R$.

How do we prove these two theorems? If we have the property that the solution of the PDE to be radially symmetric, then the PDE becomes to an ODE, we can use the theory in ODE to reduce this problem. For example, if we don't know the existence of some PDE, but we know the solution of the ODE is unique, then we find the solution of PDE is also unique.

# 2 Tools.

Let $L$ be the uniformly elliptic operator which is defined as $Lu = a_{ij}(x)D_{ij}u + b_i(x)D_iu + c(x)u$, where $a_{ij}\xi_i\xi_j \ge c_0|\xi|^2$ for some $c_0 > 0$

**Lemma 3.** (*Maximum Principle*)

*Suppose* $u \le 0$ *satisfies* $Lu \ge 0$. *If* $u$ *vanishes at some point in* $\Omega$*, then* $u \equiv 0$*.*

**Lemma 4.** (*Hopf boundary lemma*)

*Suppose there is a ball $B$ in $\Omega$ with a point $P \in \partial\Omega$ on its boundary and suppose $u$ is continuous in $\Omega \cup P$ and $u(P)=0$. Then if $u \neq 0$ in $B$ we have for an outward directional derivative at $P$, $\frac{\partial u}{\partial \nu}(P) > 0$ , in the sense that if $Q$ approaches $P$ in $B$ along a radius then $\lim_{P \longrightarrow Q} \frac{u(P) - u(Q)}{|P - Q|} > 0$.*

## 3 Main Results.

Consider a solution $u \in C^2(\Omega)$ of $\Delta u + b_1(x)u_{x_1} + f(u) = 0$ in $\Omega$ with $b_1(x) \in C(\overline{\Omega})$ and $f \in C^1$, where $\Omega$ is a bounded smooth domain; for $x \in \partial\Omega$, $\nu(x)$ is the exterior unit normal. Before proving our main results, we give some assumptions as follows.

**Assumptions.**

1. Let $\gamma$ be a unit vector in $R^n$ and and $\lambda$ be a real number. Define $T_\lambda = \{x|\gamma \cdot x = \lambda\}$, and let $\Sigma_\lambda$ be the hybersurface such that $\{x|\gamma \cdot x > \lambda\}$. WLOG, we let $\gamma = (1, 0, \ldots, 0)$ be the unit vector. For convenience, we denote the hypersurface $\Sigma^r_\lambda$ to be the reflective hypersurface of $\Sigma_\lambda$ with respect to the plane $T_\lambda$.

2. We assume that $\max_{x \in \overline{\Omega}} x_1 = \lambda_0$ . Let $T_{\lambda_1}$ be the plane such that it contains a maximal cap of $\Sigma_{\lambda_1}$with $\Sigma_{\lambda_1}$ contained in $\Sigma^r_{\lambda_1}$. By the definition of 1 & 2, it is easy to see that $\lambda_1 < \lambda_0$.

3. Note the point $x \in \Sigma_\lambda$, and $x^\lambda$ be the reflection of $x$ with respect to the plane $T_\lambda$. That is, $x^\lambda \in \Sigma^r_\lambda$.

4. Without any confusion, we abbreviate $\Sigma_{\lambda_1}$ to be $\Sigma$, and $\Sigma^r_{\lambda_1}$ to be $\Sigma^r$.

5. We let $x \in \Sigma_\lambda$, then we note the reflection point with respect to the plane $T_\lambda$ of $x$ is $x^\lambda$. That is, if $x = (x_1, x_2, \ldots, x_n)$, then $x^\lambda = (2\lambda - x_1, x_2, \ldots, x_n)$.

**Conditions.** We only concern the solution u satisfying $(*)$ $u > 0$ in $\Omega$, $u \in C^2(\Omega \cap \{x_1 > \lambda_1\})$, and $u = 0$ on $\partial\Omega \cap \{x_1 > \lambda_1\}$. We have a method which is called moving plane. We use the plane $T_\lambda$ and let $\lambda$ be a large number such that $T_\lambda$ is disjonted from the domain $\Omega$, then we decrease $\lambda$ to touch the domain.

**Theorem 5.** (*Main theorem*)

*Let u as above, satisfying the condition* $(*)$ *and assume* $b_1(x) \geq 0$ *in* $\Sigma \cup \Sigma^r$. *For any* $\lambda \in (\lambda_1, \lambda_0)$ *we have* $u_{x_1}(x) < 0$ *and* $u(x) < u(x^\lambda)$ *for* $x \in \Sigma_\lambda$. *Thus* $u_{x_1} < 0$ *in* $\Sigma$. *Moreover, if* $u_{x_1} = 0$ *at some point in* $\Omega$ *on the plane* $T_{\lambda_1}$ *then necessarily u is symmetric in the plane* $T_{\lambda_1}$, $\Omega = \Sigma \cup \Sigma^r \cup (T_{\lambda_1} \cap \Omega)$ *and* $b_1(x) = 0$.

Before proving Theorem6, we use the main theorem to prove theorem1 and theorem2.

*Proof.* (Theorem 1)

Applying Theorem 6, we see that $u_{x_1} < 0$ if $x_1 > 0$ for any choice of our $x_1$ axis. It follows that $u_{x_1} > 0$ if $x_1 < 0$. Hence $u_{x_1} = 0$ on $x_1 = 0$. By the last assertion of Theorem6, we infer that $u$ is symmetric in $x_1$. Since the direction of the $x_1$ axis is arbitrary it follows that $u$ is radially symmetric and $u_r < 0$ for $0 < r < R$.

□

*Proof.* (Theorem 2)

To see the reference [1].

□

Now, we need two lemmas to prove theorem6.

**Lemma 6.** *Let $x_0$ belong to $\partial\Omega$ with $\nu_1(x_0) > 0$, where $\nu(x_0) = (\nu_1(x_0), \ldots, \nu_n(x_0))$ is the outer normal at $x_0$. For some $\epsilon > 0$ assume $u$ is a $C^2$ function in $\overline{\Omega}_\epsilon$ where $\Omega_\epsilon = \Omega \cap \{|x - x_0| < \epsilon\}$, $u > 0$ in $\Omega$ and $u = 0$ on $\partial\Omega \cap \{|x - x_0| < \epsilon\}$.*
*Then $\exists \delta > 0$ such that $u_{x_1} < 0$ in $\Omega \cap \{|x - x_0| < \delta\}$.*

*Proof.* Since $u > 0$ in $\Omega$, then $u_\nu \leq 0$ on $\partial\Omega \cap \{|x - x_0| < \epsilon\} = S$. Since $u = 0$ on $S$, then $u$ is parallel to $\nu$ on $S$. By the assumption of the lemma, we have $u_{x_1} \leq 0$ on $S$ for $\nu_1 > 0$. If the lemma were false there would be a sequence of points $x^j \to x_0$, with $u_1(x^j) \geq 0$. For $j$ large the interval in the $x_1$ direction going from $x^j$ to $\partial\Omega$ hits $S$ at a point where $u_1 \leq 0$. Since $u_1(x^j) \geq 0$ and $u_1(x_0) \leq 0$ on $S$, then we have $u_{11}(x_0) \leq 0$. On the other hand, By mean value theorem, we have $u(x^j) = u(x_0) + u_{x_1}(x_0)(x_j^1 - x_0^1) + \dfrac{1}{2} u_{x_1x_1}(y_j)(x_j^1 - x_0^1)^2$, where $x_0 = (x_0^1, \ldots, x_0^n)$ and $x_j = (x_j^1, \ldots x_j^n)$. Since $u(x^j) > 0$ and $u(x_0) = 0$, $u_{x_1}(x_0) = 0$, then $u_{11}(y_j) > 0$. Hence we can imply that $u_{11}(x_0) \geq 0$. By above two facts, we have $u_1(x_0) = 0$ and $u_{11}(x_0) = 0$.

Case1. Suppose $f(0) \geq 0$. Then in $\Omega_\epsilon$, $u$ satisfies $\Delta u + b_1 u_{x_1} + f(u) - f(0) \leq 0$ or, by the mean value theorem, for some funtion $c_1(x)$, $\Delta u + b_1 u_{x_1} + c_1(x) u \leq 0$. Apply the Hopf boundary lemma to the function $-u$ we find $u_\nu(x_0) < 0$, and so $u_1(x_0) < 0$, it is a contradiction.

Case2. Suppose $f(0) < 0$. By the equation $\Delta u + b_1(x) u_{x_1} + f(u) = 0$, we see that at $x_0$, by $u_{x_1} = 0$ we have $\nabla u = 0$ and $\Delta u = -f(0)$. But then it follows that $u_{x_i x_j} = -f(0)\nu_i \nu_j$ at $x_0$ (note that the identity is not so trivial to see!). In particular $u_{11}(x_0) < 0$, it is also a contradiction.

$\square$

**Lemma 7.** *Assume that for some $\lambda \in [\lambda_1, \lambda_0)$ we have $u_1(x_0) \leq 0$ and $u(x) \leq u(x^\lambda)$ but $u(x) \neq u(x^\lambda)$ in $\Sigma_\lambda$.Then $u(x) < u(x^\lambda)$ in $\Sigma_\lambda$ and $u_1(x) < 0$ on $\Omega \cap T_\lambda$.*

*Proof.* In $\Sigma^r_\lambda$= the reflection of $\Sigma_\lambda$ with respect to the pland $T_\lambda$, consider $v(x) = u(x^\lambda)$; note $x^\lambda \in \Sigma_\lambda$. In $\Sigma^r_\lambda$, v satisfies the equation $\Delta v - b_1(x^\lambda)v_1 + f(v) = 0$ and $v_1 \geq 0$. If we subtract the eqution $\Delta u + b_1(x)u_{x_1} + f(u) = 0$, we find $\Delta(v-u) + b_1(x)(v-u)_1 + f(v) - f(u) = (b_1(x^\lambda) + b_1(x))v_1 \geq 0$ in $\Sigma^r_\lambda$, since $v_1 \geq 0$ and $b_1 \geq 0$. Using the mean value theorem in integral form we see that in $\Sigma^r_\lambda$ $w(x) \equiv v(x) - u(x) \leq 0$, $w(x) \neq 0$, and $\Delta w + b_1(x)w + c(x)w \geq 0$ for some function $c(x)$. Since $w = 0$ on $T_\lambda \cap \Omega$ it follows form the maximum principle and the Hopf boundary lemma that $w < 0$ in $\Sigma^r_\lambda$ and $w_1 > 0$ on $T_\lambda$. But on $T_\lambda$, $w_1 = v_1 - u_1 = -2u_1$, and the lemma is proved.

□

Now, we use lemma7&8 in order to prove theorem6.

*Proof.* (Theorem6)

It follows from lemma7 that for $\lambda$ close to $\lambda_0$, for $\lambda < \lambda_0$, the condition $u_{x_1}(x) < 0$ and $u(x) < u(x^\lambda)$ for $x \in \Sigma_\lambda$ holds. Decrease $\lambda$ until a critical value $\mu \geq \lambda_1$ is reached, beyond which it no longer holds. Then the condition $u_{x_1}(x) < 0$ and $u(x) < u(x^\lambda)$ for $x \in \Sigma_\lambda$ holds for $\lambda > \mu$, while for $\lambda = \mu$ we have by continuity, $u_1(x) \leq 0$ and $u(x) \leq u(x^\mu)$ for $x \in \Sigma_\mu$.

We will show that the critical value of $\mu$ is $\lambda_1$. Suppose $\mu > \lambda_1$. For any point $x_0 \in \partial(\Sigma_\mu \cap \Omega) \setminus T_\mu$ we have $x_0^\mu \in \Omega$. Since $0 = u(x_0) < u(x_0^\mu)$, we see that $u(x) \neq u(x^\mu)$ for $x \in \Sigma_\mu$. We apply the lemma8 and conclude that $u(x) < u(x^\mu)$ in $\Sigma_\mu$ and $u_1 < 0$ on $\Omega \cap T_\mu$. Thus $u_{x_1}(x) < 0$ and $u(x) < u(x^\lambda)$ for $x \in \Sigma_\lambda$ holds for $\lambda = \mu$.Since $u_1 < 0$ on $\Omega \cap T_\mu$, we see with the aid of lemma7 that for some $\epsilon > 0$ , $u_1 < 0$ in $\Omega \cap \{x_1 > \mu - \epsilon\}(**)$ (we can see it from the critical plane $T_\lambda$ to anothe critical plane $T_\mu$).

From our definition of $\mu$ we must then have the following situation. For $j = 1, 2, ...$ there is a sequence $\lambda^j$, $\lambda_1 < \lambda^j \nearrow \mu$, and a point $x_j$in $\Sigma_{\lambda^j}$ such that $u(x_j) \geq u(x_j^{\lambda^j})$, we want to get a contradiction. A subsequence which we still call $x_j$ will converge to some point $x$ in $\overline{\Sigma_\mu}$; then $x_j^{\lambda^j} \to x^\mu$ and $u(x) \geq u(x^\mu)$. Since $u_{x_1}(x) < 0$ and $u(x) < u(x^\lambda)$ for $x \in \Sigma_\lambda$ holds for $\lambda = \mu$ we must have $x \in \partial\Sigma_\mu$. If $x$ is not on the plane $T_\mu$ then $x^\mu$ lies in $\Omega$ and consequently $0 = u(x) < u(x^\mu)$ which is impossible. Therefore $x \in T_\mu$ and $x^\mu = x$. On the other hand, for $j$ sufficiently large, the straight segment joining $x_j$ to $x_j^{\lambda^j}$ belongs to $\Omega$and by the mean value theorem it contains a point $y_j$ such that $u_1(y_j) \geq 0$. Since $y_j \to x$ we obtain a contradiction to $(**)$. Thus we have proved that $\mu = \lambda_1$ and that $u_{x_1}(x) < 0$ and $u(x) < u(x^\lambda)$ for $x \in \Sigma_\lambda$ holds for $\lambda > \lambda_1$. By continuity, $u_1(x) \leq 0$ and $u(x) \leq u(x^{\lambda_1})$ in $\Sigma$.

To complete the proof of the theorem suppose $u_1 = 0$ at some point in $\Omega$ on $T_{\lambda_1}$. By lemma6, we infer that $u(x) \equiv u(x^{\lambda_1})$ in $\Sigma$. Since $u(x) = 0$ if $x \in \partial\Omega$ and $x_1 \geq \lambda_1$ it follows that $u(x^{\lambda_1}) = 0$ at the reflected point and thus the condition $\Omega = \Sigma \cup \Sigma^r \cup (T_{\lambda_1} \cap \Omega)$ holds. Finally, suppose $b_1 > 0$ at some point $x \in \Omega$ (we may take $x \notin T_{\lambda_1}$). Then from the equation $\Delta u + b_1(x)u_{x_1} + f(u) = 0$ in $\Omega$ and the symmetry of the solution in the plane $T_\lambda$ we see that $b_1(x)u_1(x) = b_1(x^{\lambda_1})u_1(x^{\lambda_1})$. If $x \in \Sigma$, the left-hand side is negative while the right-hand side is nonnegative : it is impossible; similarly if $x \in \Sigma^r$.

□

# 4 More Applications.

We can apply the moving plane method to the following equations.

1. Liouville's equation $\Delta u + e^u = 0$. with the condtion $\int e^u < \infty$, if u is a positive solution of this equation, then u is radially symmetric with respect to some point $p_0 \in \mathbb{R}$.

2. Yamabe problem $\Delta u + u^{\frac{n+2}{n-2}} = 0$.

## 5 Some References.

1. Gidas,B., W.-M. Ni AND L.Nirenberg, Symmetry and related properties via the maximum principle, Comm. Math. Phys., 68 (1979), 209-243.

2. L.C. Evans, Partial Differential Equations(Chapter 9).

3. Q.Han, F.H.Lin, Elliptic Partial Differential Equations(Chapter2).