2019 FALL CALCULUS 0412: FIRST MIDTERM (OCTOBER 24 2019)

- Please answer the following questions in details, which means you need to state all theorems or results you used. L'Hospital rule is prohibited from using while you compute the limits (no grade will be given if you used). The definitions of terminology were taught in the lectures, so you cannot ask instructor or TA about mathematical definitions while taking the midterm.
- Please mark your name, student ID, and question numbers clearly on your answer sheet.

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1. Let $f(x)$ be a function defined on $\mathbb{R}$, $a \in \mathbb{R}$ and $L \in \mathbb{R}$. Please state the following questions by using ϵ-δ.
   (1) **(5 points)** What is the definition of $\lim_{x\to a} f(x) = L$?
   (2) **(5 points)** What is the definition of $f(x)$ is continuous at $x = a$?
2. Determine whether the statement is true or false. If it is true, explain; if it is false, explain.
   (1) **(3 points)** $\lim_{x\to 4}\left(\frac{2x}{x-4} - \frac{8}{x-4}\right) = \lim_{x\to 4}\frac{2x}{x-4} - \lim_{x\to 4}\frac{8}{x-4}$.
   (2) **(3 points)** $\lim_{x\to 3}\frac{x^2-9}{x-3} = \lim_{x\to 3}(x+3)$.
   (3) **(4 points)** The equation $x^{10} - 10x^2 + 5 = 0$ has a root in $(0, 2)$.
3. (1) **(5 points)** What does the Squeeze Theorem say?
   (2) **(5 points)** Find the value of $\lim_{x\to 0}\frac{\sin x}{x}$.
4. **(10 point)** Find the value $\lim_{x\to\infty} x^{\frac{3}{2}}\left(\sqrt{x+2} - 2\sqrt{x+1} + \sqrt{x}\right)$.
5. Let $f(x)$ be a function defined on $\mathbb{R}$.
   (1) **(5 points)** State the definition that $f(x)$ is differentiable at $x = a$.
   (2) **(5 points)** Is the function
   $$f(x) = \begin{cases} x\sin\frac{1}{x}, & \text{when } x \neq 0, \\ 0, & \text{when } x = 0, \end{cases}$$
   differentiable at $x = 0$ or not? Explain the reason.
6. Find the derivative of the following functions.
   (1) **(5 points)** $f(x) = \log_2(3^x + x^4 + 5^6)$.
   (2) **(5 points)** $f(x) = \sin^{-1}\left(\cos^2(\tan x^3)\right)$.
7. Calculate y':

(1) **(3 points)** $y = \frac{\tan x}{1+\cos x}$.
(2) **(3 points)** $y = \ln\left|\frac{x^2-4}{2x+5}\right|$.
(3) **(4 points)** $xe^y = y - 1$.

8. Find equations of the tangent line to the curve at the given point.
   (1) **(5 points)** $x^2 + 4xy + y^2 = 13$ at $(x, y) = (2, 1)$.
   (2) **(5 points)** $y = (2 + x)e^{-x}$ at $(x, y) = (0, 2)$.

9. **(10 points)** Let f be a differentiable function defined on $\mathbb{R}$, and f satisfies $f' = f$ with $f(0) = 1$. Prove that f is a monotone increasing function.

10. **(10 points)** Construct a continuous function f, but not differentiable.

11. **(20 points bonus)** Define the function f by

$$f(x) = \begin{cases} 0, & \text{when } x \in \mathbb{Q}^c, \\ \frac{1}{p}, & \text{when } x = \frac{q}{p} \in \mathbb{Q}, \end{cases}$$

where $\mathbb{Q}^c := \mathbb{R} \setminus \mathbb{Q}$ is the set of irrational numbers, and $p, q \in \mathbb{Z}$ with g.c.d.$(p, q) = 1$ (g.c.d. stands for the greatest common division). Prove that $f(x)$ is continuous on the set $\mathbb{Q}^c$.