2019 FALL CALCULUS 0412: SECOND MIDTERM (NOVEMBER 21 2019)

- Please answer the following questions in details, which means you need to state all theorems and all reasons you used. If you only write the the answer of the question, you will not get grade. The definitions of terminology were taught in the lectures, so you cannot ask instructor or TA about mathematical definitions while taking the midterm. When you use the L'Hospital's rule, please note "H" above "=".
- Please mark your name, student ID, and question numbers clearly on your answer sheet.

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(1) State the following theorems.
  (a) **(5 points)** Mean value theorem.
  (b) **(5 points)** Cauchy mean value theorem.

(2) Answer the following questions.
  (a) **(5 points)** We know that $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$. Find the value of $\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x$ for any $a \in \mathbb{R}$.
  (b) **(5 points)** Find the value $a \in \mathbb{R}$ such that
$$\lim_{x\to\infty}\left(\frac{x+a}{x-a}\right)^x = e.$$

(3) Find the following values.
  (a) **(5 points)** $\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+n}\right)$.
  (b) **(5 points)** $\lim_{n\to\infty}\left(\frac{1}{\sqrt{n}\sqrt{n+1}}+\frac{1}{\sqrt{n}\sqrt{n+2}}+\cdots+\frac{1}{\sqrt{n}\sqrt{n+n}}\right)$.

(4) Prove the following identities.
  (a) **(10 points)** Let f' be a continuous function, then
$$\lim_{h\to 0}\frac{f(x+h)-f(x-h)}{2h} = f'(x).$$
  (b) **(10 points)** Let f'' be a continuous function, then
$$\lim_{h\to 0}\frac{f(x+h)+f(x-h)-2f(x)}{h^2} = f''(x).$$

(5) Let f be a continuous function defined on $[a,b]$. Prove the following inequalities.
  (a) **(5 points)** $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$.
  (b) **(5 points)** $\left|\int_0^{2\pi} f(x)\cos ex\,dx\right| \le \int_0^{2\pi} |f(x)|\,dx$.

(6) **(10 points)** Let $f(x) = \sin 2x + 4\sin x - x$, where $x \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$. Find the intervals of increasing and decreasing, local maximum and local minimum, interval of concavity and inflection points.

(7) Let f be a continuous function on $[0, \pi]$. Prove the following statements.
 (a) **(5 points)** $\int_0^\pi x f(\sin x)\, dx = \frac{\pi}{2} \int_0^\pi f(\sin x)\, dx$.
 (b) **(5 points)** $\int_0^{\frac{\pi}{2}} f(\sin x)\, dx = \int_0^{\frac{\pi}{2}} f(\cos x)\, dx$.

(8) Answer the following questions
 (a) **(5 points)** Find the value of $\int_{-\pi}^{\pi} \frac{\sin x}{|x|}\, dx$.
 (b) **(5 points)** Evaluate $\int_0^1 x e^{-x^2}\, dx$.
 (c) **(10 points)** For $a, b > 0$, evaluate the area of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

(9) **(10 points)** Find the derivative of the function $f(x) = \int_{2x^2+1}^{x^3} t e^t\, dt$.

(10) **(10 points)** Let f be a continuous function satisfying

$$\int_0^x f(t)\, dt + \int_0^{x^3} e^{-t} f(t^{1/3})\, dt = x e^{2x} + \pi^x + x^\pi, \text{ for all } x > 0.$$

Find the explicit formula for $f(x)$.