2019 FALL CALCULUS 0412: THIRD MIDTERM (DECEMBER 19 2019)

- Please answer the following questions in details, which means you need to state all theorems and all reasons you used. If you only write the the answer of the question, you will not get grade. The definitions of terminology were taught in the lectures, so you cannot ask instructor or TA about mathematical definitions while taking the midterm.
- Please mark your name, student ID, and question numbers clearly on your answer sheet.

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(1) (5 points) Evaluate $\int_e^{e^2} \ln x \, dx$.

(2) (5 points) Prove or disprove that $\int_{-\infty}^{\infty} f(x)\, dx = \lim_{t\to\infty} \int_{-t}^{t} f(x)\, dx$.

(3) (10 points) For $m, n \in \mathbb{N}$, prove that $\int_{-\pi}^{\pi} \sin nx \sin mx \, dx = \begin{cases} 0, & \text{if } m \neq n \\ \pi, & \text{if } m = n \end{cases}$.

(4) Evaluate
   (a) (2 points) $\frac{d}{dx} \tan x$.
   (b) (2 points) $\frac{d}{dx} \sec x$.
   (c) (6 points) $\int \sec x \, dx$.

(5) (10 points) Use the method of cylindrical shells to find the volume generated by rotating the function $y = e^{-x^2}$ about y-axis in the region $y = 0$, $x = 0$ and $x = 1$.

(6) (10 points) Find the exact length of the curve $y = \frac{1}{4}x^2 - \frac{1}{2}\ln x$, $1 \leq x \leq 2$.

(7) For $a > b > 0$, let $E := \left\{(x, y) : \ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\right\}$ be an ellipse in the plane. What are
   (a) (10 points) the volume of the revolution solid of E rotating about the x-axis.
   (b) (10 points) the surface area of the revolution solid of E rotating about the x-axis.

(8) (10 points) Evaluate $\int \cos\theta \cos^5(\sin\theta)\, d\theta$.

(9) For $n \in \mathbb{N}$, let $\Gamma(n) = \int_0^{\infty} x^{n-1} e^{-x}\, dx$. Show that
   (a) (5 points) $\Gamma(1) = 0! = 1$.
   (b) (10 points) $\Gamma(n+1) = n\Gamma(n)$.
   (b) (5 points) $\Gamma(n+1) = n!$.

(10) (20 points) For $a > 0$, find the value of $\int_0^{\infty} e^{-sx} \sin ax \, dx$, where $s > a$.