2020 FALL CALCULUS 0312: FIRST MIDTERM (OCTOBER 23, 2020)

- Please answer the following questions in details, which means you need to state all theorems or results you used. L'Hospital rule is **prohibited** to use in this midterm (no grade will be given if you really used). The definitions of terminology were taught in the lectures, so you cannot ask instructor or TA about mathematical definitions while taking the midterm.
- Please mark your name, student ID, and question numbers clearly on your answer sheet.

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1. Compute the following limits:
   (a) (5 points) $\lim_{x\to 0} \dfrac{\sin 2x}{\sin 3x}$.
   (b) (5 points) $\lim_{x\to\infty} \left(\sqrt{x^2+4x+5} - \sqrt{x^2+x+1}\right)$.
2. (10 points) $\lim_{x\to-\infty} \dfrac{2x-\sqrt{4x^2+x}}{\sqrt{x^2-1}-x}$.
3. Prove or disprove that the following limits exist.
   (a) (5 points) $\lim_{x\to 0} \dfrac{\sin|x|}{x}$.
   (b) (5 points) $\lim_{x\to 0} \dfrac{\cos x - 1}{\sin(x\sin x)}$.
4. (10 points) Find $\frac{d}{dx}(\sec x)^x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$.
5. (10 points) Show that $\frac{1}{x+1} < \ln(x+1) - \ln x < \frac{1}{x}$ holds for any $x > 0$.
6. (10 points) Show that $\lim_{x\to 0}(1+x)^{1/x} = e$.[1]
7. (10 points) Prove that $e^x \geq 1+x$ for all $x \in \mathbb{R}$.
8. Prove the following statements.
   (a) (10 points) Consider the logarithmic function $\ln x$ for $x > 0$. Show that $\ln x \leq x-1$ for any $x > 0$.
   (b) (10 points) Based on (a), let $p_1, p_2, \dots, p_m > 0$ and $q_1, q_2, \dots, q_m > 0$ be arbitrary numbers such that

$$\sum_{k=1}^{m} p_k = \sum_{k=1}^{m} q_k = 1.$$

[1]Hint: Consider $g(x) = (1+x)^{1/x}$, and take $\ln g(x)$ into account.

Show that

$$\sum_{k=1}^{m} p_k \log q_k \leq \sum_{k=1}^{m} p_k \log p_k.$$

9. (20 points bonus) Let $f(x)$ be a function such that f is twice differentiable, and $f''(x)$ is continuous. Let $a \in \mathbb{R}$ be a number such that $f''(a) \neq 0$. Suppose that there exists a *constant* $\theta \in (0,1)$ such that

$$f(a+h) - f(a) = f'(a+\theta h)h,$$

for all $h > 0$. Show that $\theta = \frac{1}{2}$.

10. (10 points) Construct a concrete non-constant function $f(x)$ such that

$$f(a+h) - f(a) = f'(a + \frac{1}{2}h)h,$$

for any $a \in \mathbb{R}$ and $h > 0$.