2020 FALL REAL ANALYSIS (I): FINAL EXAM (DECEMBER 24, 2020)

- Please mark your name, student ID, and question numbers clearly on your answer sheet.
- State all reasons, lemmas, theorems clearly, while you are using during answering the questions.
- You can use all results coming from *advanced calculus* without any proofs.
- The total score is 120 points. 20 points are your bonus.

1. (a) (5 points) State and prove the Young's inequality.
   (b) (10 points) State and prove the Hölder's inequality (for the version $f \in L^p$ and $g \in L^q$ with suitable exponents (p, q)).

2. (20 points) Prove the integral version of Minkowski's inequality for $1 \leq p \leq \infty$, and measurable function $f(x, y)$:

$$\left[\int \left(\int |f(x,y)|\, dx\right)^p dy\right]^{1/p} \leq \int \left(\int |f(x,y)|^p\, dy\right)^{1/p} dx.$$

3. (15 points) Let $f(x, y)$, $0 \leq x, y \leq 1$ satisfy the following conditions: For each x, $f(x, y)$ is an integrable function of y, and $\frac{\partial f(x,y)}{\partial x}$ is a bounded function of (x, y). Show that $\frac{\partial f(x,y)}{\partial x}$ is a measurable function of y for each x and

$$\frac{d}{dx}\int_0^1 f(x,y)\, dy = \int_0^1 \frac{\partial}{\partial x} f(x,y)\, dy.$$

4. (20 points) Let $\Omega \subset \mathbb{R}^n$ be a connected open set, and $u \in W^{1,p}(\Omega)$ for some $1 \leq p \leq \infty$. Suppose

$$Du = 0 \quad \text{a.e. in } \Omega.$$

   Prove that u is constant a.e. in Ω.

5. (15 points) Suppose $1 < p < n$ and $u \in C_c^1(\mathbb{R}^n)$. Prove that[1]

$$\int_{\mathbb{R}^n} \frac{|u|^p}{|x|^p}\, dx \leq \left(\frac{p}{n-p}\right)^p \int_{\mathbb{R}^n} |\nabla u|^p\, dx.$$

6. (15 points) For $1 \leq p, q \leq \infty$ and $\frac{1}{p} + \frac{1}{q} \geq 1$. Suppose r satisfies $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$. Prove that

$$\|f * g\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q},$$

   where $*$ denotes the convolution of two functions.

[1] Hint: Scaling in the space variable, fundamental theorem of calculus and Hölder's inequality.

7. (20 points bonus) Let $f \in L^1(\mathbb{R}^n)$ and $\{K_\delta\}_{\delta>0}$ be an approximation to the identity. Show that

$$\lim_{\delta\to 0}(f * K_\delta)(x) = f(x),$$

for all x in the Lebesgue set of f, and the limit holds for a.e. $x \in \mathbb{R}^n$.