2020 FALL REAL ANALYSIS (I): MIDTERM (OCTOBER 22, 2020)

- Please mark your name, student ID, and question numbers clearly on your answer sheet.
- State all reasons, lemmas, theorems clearly, while you are using during answering the questions.
- You can use all results coming from *advanced calculus* without any proofs.
- The notations are the same as in the lectures: $|\cdot|$, $m_*(\cdot)$ and $m(\cdot)$ stand for the volume of rectangles, outer measure, and (Lebesgue) measure, respectively.
- The total score is 120 points. 20 points are your bonus.

1. A subset $E \subset \mathbb{R}$ has measure zero if given any $\epsilon > 0$, there exist countable intervals I_n that cover E, such that $A \subset \bigcup_{n=1}^{\infty} I_n$ and $\sum_{n=1}^{\infty} |I_n| < \epsilon$. Please show that
   (a) (10 points) Every countable set in $\mathbb{R}$ has measure zero.
   (b) (10 points) The Cantor set $\mathcal{C}$ in $[0,1]$ has measure zero.
2. (20 points) Given $0 < \epsilon < 1$, construct a *dense* measurable subset $E \subset [0,1]$ such that $m(E) = \epsilon$.
3. (20 points) Let $\mathcal{L}$ be the collection of Lebesgue measurable sets of $\mathbb{R}$ and m the Lebesgue measure on $\mathcal{L}$. Show that $\mathcal{L}$ is invariant under translation and dilation, i.e., if $E \in \mathcal{L}$, then $E + s \in \mathcal{L}$ and $rE \in \mathcal{L}$ for any $s, r \in \mathbb{R}$. Also show that $m(E+s) = m(E)$ and $m(rE) = |r|m(E)$.
4. (10 points) Let $E \subset \mathbb{R}^n$ be a measurable subset. Let f_n be a sequence of measurable functions defined on E. Show that $g(x) := \limsup_{n\to\infty} f_n(x)$ is a measurable function defined on E.
5. (10 points) Prove or disprove that $|f|$ is measurable $\implies$ f is measurable.
6. (20 points) Suppose that $\{f_n\}$ is a sequence of measurable functions that are all bounded by a number $M > 0$, which are supported in a set $E \subset \mathbb{R}$ of finite measure. Suppose that $f_n(x) \to f(x)$ a.e. in E as $n \to \infty$. Show that f is measurable, bounded, supported on E for a.e. x and
$$\int |f_n - f| \to 0, \text{ as } n \to \infty.$$
7. (20 points) Let $\mathcal{C} \subset [0,1]$ be a Cantor set. Define
$$f(x) = \begin{cases} 0, & x \in \mathcal{C}, \\ n, & x \text{ in the complementary interval of length } 3^{-n}. \end{cases}$$
Show that f is measurable and evaluate the Lebesgue integration of $\int_0^1 f(x)\,dx$.