2020 FALL REAL ANALYSIS (I) @ NCTU APPL. MATH.
HOMEWORK 10

- Please answer the following questions in details, which means you need to state all theorems and all reasons you have been using.
- Please mark your name, student ID, and question numbers clearly on your answer sheet. The deadline to hand in the exercise is on December 24, 2020.

---

(1) Let $f : E \to \mathbb{R}$ be a measurable function, $1 \le p \le \infty$ with $\frac{1}{p} + \frac{1}{p'} = 1$. Then

$$\|f\|_p = \sup \left\{ \int_E fg \,\middle|\, \|g\|_{p'} \le 1 \text{ and } \int_E fg \text{ exists} \right\}.$$

(2) In 1-dimension, Show that
   (a) For $1 \le p < \infty$, a function $u \in W^{1,p}((0,1))$ implies that u is equal to an absolutely continuous function, and u' exists a.e. such that $u' \in L^p((0,1))$.
   (b) If $u \in W^{1,p}((0,1))$, for some $1 < p < \infty$. Show that

$$|u(x) - u(y) \le |x-y|^{1-\frac{1}{p}} \left(\int_0^1 |u'|^p \, dt \right)^{1/p},$$

   for a.e. $x, y \in [0,1]$.
   (c) Suppose that f is continuous on $[a,b]$, $f'(x)$ exists for every $x \in (a,b)$ and f' is integrable. Show that f is absolutely continuous and

$$f(b) - f(a) = \int_a^b f'(x) \, dx.$$

(3) Show that there exists a constant $C > 0$ independent u such that

$$\int_{B(0,1)} u^2 \, dx \le C \int_{B(0,1)} |Du|^2 \, dx,$$

where $B(0,1)$ is a unit ball with center 0, provided that

$$m\left(\{x \in B(0,1) | \, u(x) = 0\}\right) \ge \alpha > 0, \text{ for } u \in H^1(B(0,1)).$$

(4) Show that the Minkowski's inequality fails for $p < 1$.

(5) Prove the integral version of Minkowski's inequality for $1 \le p \le \infty$, and measurable function $f(x,y)$:

$$\left[\int \left(\int |f(x,y)| \, dx \right)^p dy \right]^{1/p} \le \int \left(\int |f(x,y)|^p \, dy \right)^{1/p} dx.$$

(6) Let

$$f(x) = \sum_{n=0}^{\infty} 2^{-n} e^{2\pi i 2^n x}.$$

Show that f satisfies $|f(x) - f(y)| \leq A_\alpha |x - y|^\alpha$, for each $0 < \alpha < 1$, where $A_\alpha > 0$ is a constant independent of f. Moreover, f is *nowhere differentiable*[1].

[1] If you think nowhere differentiable function is too hard for you, you can ignore the second part of this question.