2020 FALL REAL ANALYSIS (I) @ NCTU APPL. MATH.
HOMEWORK 2

- Please answer the following questions in details, which means you need to state all theorems and all reasons you have been using.
- Please mark your name, student ID, and question numbers clearly on your answer sheet. The deadline to hand in the exercise is on October 8, 2020.

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(1) Show that the Borel σ-algebra $\mathcal{B}$ in $\mathbb{R}^n$ is the smallest σ-algebra containing the closed sets in $\mathbb{R}^n$.

(2) Let $E \subset \mathbb{R}^n$ be any subset, recalling that the outer measure is defined via

$$m_*(E) = \inf \left\{ m(\mathcal{O}) : \ E \subset \mathcal{O}, \text{ and } \mathcal{O} \text{ is an open set in } \mathbb{R}^n \right\}.$$

One can also define an inner measure $m^*(E)$ by

$$m^*(E) = \sup \left\{ m(F) : \ F \subset E, \text{ and } F \text{ is a closed set in } \mathbb{R}^n \right\}.$$

Show that

(a) $m^*(E) \leq m_*(E)$.

(b) E is measurable if and only if that $m^*(E) = m_*(E)$, provided that $m_*(E) < \infty$.

(3) We have introduced F_σ and G_δ sets in $\mathbb{R}^n$.

(a) Show that a closed set is G_δ but an open set is F_σ.

(b) Construct a set, which is F_σ but not G_δ.

(c) Construct a Borel set, which is neither G_δ nor F_σ.

(4) Construct an example which is a Lebesgue measurable set but not a Borel measurable set.

(5) Suppose E is a measure zero set in $\mathbb{R}$. Show that there exists a sequence of open sets $\{\mathcal{O}_n\}$ such that $E \subset \bigcap_{n=1}^{\infty} \mathcal{O}_n$ and $\lim_{n\to\infty} m(\mathcal{O}_n) = 0$.

(6) Try to understand what is a non-measurable set (in the Lebesgue sense)[1].

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[1]Please do not hand in this problem to your TA.