2020 FALL REAL ANALYSIS (I) @ NCTU APPL. MATH.
HOMEWORK 3

- Please answer the following questions in details, which means you need to state all theorems and all reasons you have been using.
- Please mark your name, student ID, and question numbers clearly on your answer sheet. The deadline to hand in the exercise is on October 15, 2020.

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(1) Let $f(x)$ be a measurable function, and $c \in \mathbb{R}$ be any constant. Show that $f(x) + c$ and $cf(x)$ are measurable functions.

(2) Let $a_k \in \mathbb{R}$ be constants and $E_k \subset \mathbb{R}^n$ be subsets, for $k = 1, 2, \ldots, N$. A simple function $f(x) = \sum_{j=1}^{N} a_k \chi_{E_k}(x)$ is measurable if and only if E_k are measurable sets for all $k = 1, 2, \ldots, N$.

(3) (**The Borel-Cantelli lemma**) Suppose $\{E_k\}_{k=1}^{\infty}$ is a collection of countably many measurable subsets of $\mathbb{R}^n$, and

$$\sum_{k=1}^{\infty} m(E_k) < \infty.$$

Let

$$\begin{aligned} E &= \{x \in \mathbb{R}^n : x \in E_k, \text{ for infinitely many } k\} \\ &= \limsup_{k \to \infty} (E_k). \end{aligned}$$

Prove that

(a) E is measurable,

(b) $m(E) = 0$.