2020 FALL REAL ANALYSIS (I) @ NCTU APPL. MATH.
HOMEWORK 4

- Please answer the following questions in details, which means you need to state all theorems and all reasons you have been using.
- Please mark your name, student ID, and question numbers clearly on your answer sheet. The deadline to hand in the exercise is on November 5, 2020.

---

(1) If f is integrable in $(0,1)$, show that $x^k f(x)$ is also integrable in $(0,1)$, for all $k \in \mathbb{N}$. Moreover, $\int_0^1 x^k f(x)\,dx \to 0$ as $k \to \infty$.

(2) Let $f(x,y)$, $0 \le x, y \le 1$ satisfy the following conditions: For each x, $f(x,y)$ is an integrable function of y, and $\frac{\partial f(x,y)}{\partial x}$ is a bounded function of (x,y). Show that $\frac{\partial f(x,y)}{\partial x}$ is a measurable function of y for each x and

$$\frac{d}{dx}\int_0^1 f(x,y)\,dy = \int_0^1 \frac{\partial}{\partial x} f(x,y)\,dy.$$

(3) Let $E \subset \mathbb{R}^n$ be any measurable subset, and f be a nonnegative measurable function defined on E. Let $(f_m)(E) := \int_E f(x)\,dx$. Show that
   (a) f_m is a (Lebesgue) measure.[1]
   (b) If T is a measurable and one-to-one map from $\mathbb{R}^n$ to $\mathbb{R}^m$ with a measurable inverse T^{-1}. Show that $(f_m) \circ (T^{-1}(K)) = (f \circ T^{-1})(K)$, where $K \subset \mathbb{R}^m$ is any measurable subset of $\mathbb{R}^m$.[2]

(4) Define f by

$$f(x) = \begin{cases} 0, & x \in \mathbb{Q} \cap (0,1), \\ [1/x]^{-1}, & x \in \mathbb{Q}^c \cap (0,1), \end{cases}$$

where $[x]$ denotes the integer part of x. Evaluate $\int_0^1 f(x)\,dx$.

(5) Let $\{f_k\}$ be a sequence of measurable functions on E. Show that $\sum_{k=1}^{\infty} f_k$ converges absolutely a.e. in E if $\sum_{k=1}^{\infty} \int_E |f_k| < \infty$.

---

[1]For example, when $f = 1$, the integration stands for the usual Lebesgue measure.
[2]Consider the change of variables in your calculus course.