# 2020 FALL REAL ANALYSIS (I) @ NCTU APPL. MATH.
# HOMEWORK 8

- Please answer the following questions in details, which means you need to state all theorems and all reasons you have been using.
- Please mark your name, student ID, and question numbers clearly on your answer sheet. The deadline to hand in the exercise is on December 3, 2020.

---

(1) Let $\mathcal{R}$ denote the set of all rectangles in $\mathbb{R}^2$ that contain the origin, and with sides parallel to the coordinate axis. Consider the maximal operator associated to the family, i.e.,

$$f^*_{\mathcal{R}}(x) = \sup_{R \in \mathcal{R}} \fint_R |f(x-y)| dy.$$

Then show that

(a) $f \mapsto f^*_{\mathcal{R}}$ does not satisfy the weak type $(1,1)$ inequality

$$m\left(\{x : f^*_{\mathcal{R}}(x) > \alpha\}\right) \leq \frac{A}{\alpha} \|f\|_{L^1(\mathbb{R}^n)},$$

for all $\alpha > 0$, for all $f \in L^1(\mathbb{R}^2)$ and for some constant $A > 0$.

(b) Using this, one can show that there exists $f \in L^1(\mathbb{R})$ so that for $R \in \mathcal{R}$

$$\limsup_{\operatorname{diam}(R) \to 0} \fint_R f(x-y)\, dy = \infty,$$

for a.e. $x$.

(2) Suppose that $\varphi \in L^1(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \varphi(x)\, dx = 1$. Set $K_\delta(x) := \delta^{-n} \varphi(x/\delta)$, for $\delta > 0$.

(a) Prove that $\{K_\delta\}_{\delta > 0}$ is a family of good kernel.

(b) Assume in addition that $\varphi$ is bounded and supported in a bounded set. Verify that $\{K_\alpha\}_{\alpha > 0}$ is an approximation to the identity.