2020 FALL REAL ANALYSIS (I) @ NCTU APPL. MATH.
HOMEWORK 9

- Please answer the following questions in details, which means you need to state all theorems and all reasons you have been using.
- Please mark your name, student ID, and question numbers clearly on your answer sheet. The deadline to hand in the exercise is on December 10, 2020.

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(1) Let $f \in L^1(\mathbb{R}^n)$ and $\{K_\delta\}_{\delta>0}$ be an approximation to the identity. Show that
$$\lim_{\delta\to 0}(f * K_\delta)(x) = f(x),$$
for all $x \in \mathcal{L}(f)$, and the limit holds for a.e. $x \in \mathbb{R}^n$.

(2) Suppose that $\varphi \geq 0$ is an integrable function on $\mathbb{R}^n$, with $\int_{\mathbb{R}^n} \varphi(y)\,dx = 1$. Consider $K_\delta(y) := \delta^{-n}\varphi(\delta^{-1}y)$. Show that
  (a) $\{K_\delta\}_{\delta>0}$ is a family of good kernels.
  (b) Assume in addition that φ is bounded and supported in a bounded set. Prove that $\{K_\delta\}_{\delta>0}$ is an approximation to the identity.

(3) Let $E \subset \mathbb{R}^n$ be a measurable set, and $p \in (0,\infty)$. Let $\{f_k\}_{k=1}^\infty$ be a sequence of functions in $L^p(E)$ such that $f_k(x)$ converges to $f(x)$ for a.e.$x \in E$ as $k \to \infty$. Suppose that there exists a constant $C_0 > 0$ such that $\int_E |f_k|^p \leq C_0$, for all $k \in \mathbb{N}$. Prove that
$$\lim_{k\to\infty}\int_E \big||f_k|^p - |f_k - f|^p - |f|^p\big|\,dx = 0.$$
(*Hint: This identity gives the missing term in the Fatou's lemma.*)