2020 SPRING CALCULUS 0412: FIRST MIDTERM (MARCH 30, 2020)

- Please answer the following questions in details, which means you need to state all theorems and all reasons you used. If you only write the the answer of the question, you will not get grade. The definitions of terminology were taught in the lectures, so you cannot ask instructor or TA about mathematical definitions while taking the midterm.
- Please mark your name, student ID, and question numbers clearly on your answer sheet.

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1. State the following definitions:
   (1) (5 points) What does $\sum_{n=1}^{\infty} a_n = 5$ mean?
   (2) (5 points) What is a monotonic increasing sequence?

2. Determine whether the statement is true or false. If it is true, explain or prove it. If it is false, explain or give an example to disprove the statement.
   (1) (10 points) If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges.
   (2) (10 points) A convergent series is absolutely convergent.

3. (15 points) Find the value of the series $\displaystyle\sum_{n=2}^{\infty} \frac{n^2+1}{n!}$.

4. (10 points) Determine whether the series $\displaystyle\sum_{n=1}^{\infty} (-1)^n \sin\left(\frac{1}{\sqrt{n}}\right) \ln\left(1+\frac{1}{\sqrt{n}}\right)$ is divergent, conditionally convergent, or absolutely convergent.

5. Let $\{f_n\}$ be the Fibonacci sequence defined by $f_1 = f_2 = 1$, $f_{n+1} = f_n + f_{n-1}$, for $n \geq 2$. Define $a_n = \dfrac{f_{n+1}}{f_n}$.
   (1) (4 points) Show that $\{a_{2n}\}_{n=1}^{\infty}$ is decreasing and $\{a_{2n+1}\}_{n=1}^{\infty}$ is increasing.
   (2) (8 points) Show that both limits $\lim_{n\to\infty} a_{2n}$ and $\lim_{n\to\infty} a_{2n+1}$ exist, and find their limits.
   (*Hint: $\{a_n\}$ satisfies the relation $a_{n+1} = 1 + \frac{1}{a_n}$, and try to express a_{n+2} in terms of a_n.*)
   (3) (8 points) Find the radius of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} a_n x^n$.

6. (10 points) Expand the function $f(x) = (8+x)^{1/3}$ as a power series centered at $x = 0$ (you need to write out general terms). Find the radius of convergence.

7. Let $S := \displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n(n-1)} \frac{1}{2^n}$ be a series.
   (1) (5 points) Prove S converges absolutely.
   (2) (10 points) Find the value of S.
   (*Hint: Consider the function* $f(x) = \frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$, $g(x) = \ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots + C$ and $h(x) = \int g(x)\, dx = \cdots$. Try to compute the function $h(x)$.)

8. (20 points) Construct an example that $f(x)$ is differentiable infinitely many times but f cannot be expanded as a power series at $x = 0$ (you need to explain the reason in details).