2020 SPRING CALCULUS 0412: SECOND MIDTERM (APRIL 30, 2020)

- Please answer the following questions in details, which means you need to state all theorems and all reasons you used. If you only write the the answer of the question, you will not get grade. The definitions of terminology were taught in the lectures, so you cannot ask instructor or TA about mathematical definitions while taking the midterm.
- Please mark your name, student ID, and question numbers clearly on your answer sheet.

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1. (10 points) Let f be a differentiable function in one-variable. Let $z = y + f(x^2 - y^2)$. Find the value of $y\dfrac{\partial z}{\partial x} + x\dfrac{\partial z}{\partial y}$ when $x = a$ and $y = b$.

2. (10 points) Find the tangent plane to the surface $x^2 + y^2 + z^2 = 6xyz - 3$ at the point $(-1, 1, -1)$.

3. (10 points) Find the limit, if it exists, or show it does not exist.
$$\lim_{(x,y)\to(1,1)} \frac{xy - x - y + 1}{x^2 + y^2 - 2x - 2y + 2}.$$

4. Let $f = f(x, y)$ be the function of the form
$$f(x,y) = \begin{cases} \dfrac{x^3}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$
   (a) (5 points) Find $\lim_{(x,y)\to(0,0)} f(x, y)$.
   (b) (5 points) Is f continuous at $(0, 0)$?
   (c) (5 points) Find partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.
   (d) (5 points) Is $\frac{\partial f}{\partial y}$ continuous at $(0, 0)$?
   (e) (10 points) Is $\frac{\partial^2 f}{\partial x \partial y}(0, 0) = \frac{\partial^2 f}{\partial y \partial x}(0, 0)$?

5. (10 points) Under what kind of conditions of $f = f(x, y)$, we can conclude that $\frac{\partial^2 f}{\partial x \partial y}(p) = \frac{\partial^2 f}{\partial y \partial x}(p)$, for some $p \in \mathbb{R}^2$.?

6. (10 points) Find the exact length of the polar curve $r = \theta^2$, $0 \leq \theta \leq 2\pi$.

7. (10 points) Find a formula for the area of the surface generated by rotating the polar curve $r = f(\theta)$, $a \leq \theta \leq b$ (where f' is continuous and $0 \leq a < b \leq \pi$), about the line $\theta = \frac{\pi}{2}$.

8. Let $z = f(x, y)$ be a differentiable function. Assume that $x = r\cos\theta$, $y = r\sin\theta$.

(a) (5 points) Find $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial \theta}$.

(b) (5 points) Show that $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$

9. (20 points) The gas law for a fixed mass m of an ideal gas at absolute temperature T, pressure P, and volume V is $PV = mRT$, where R is the gas constant. Show that

$$\frac{\partial P}{\partial V}\frac{\partial V}{\partial T}\frac{\partial T}{\partial P} = -1.$$