2020 SPRING CALCULUS 0412: LAST MIDTERM (JUNE 18, 2020)

- Please answer the following questions in details, which means you need to state all theorems and all reasons you used. If you only write the the answer of the question, you will not get grade. The definitions of terminology were taught in the lectures, so you cannot ask instructor or TA about mathematical definitions while taking the midterm.
- Please mark your name, student ID, and question numbers clearly on your answer sheet.

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1. (20 points) Evaluate the integral $\int_0^\infty \frac{\sin x}{x}\, dx$.
2. (20 points) Let $f(x)$ be an $(n+1)$-differentiable function in an open interval I containing 0, then for any $x \in I$, we have
$$f(x) = f(0) + f'(0)x + \cdots + \frac{f^n(0)}{n!}x^n + R_n(x),$$
where $R_n(x)$ is the remainder term, and has the form
$$R_n(x) = \frac{1}{n!}\int_0^x f^{(n+1)}(t)(x-t)^n\, dt.$$
Prove that there exists $c \in (0, x)$ such that
$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}.$$
3. Let $\{a_n\}_{n=1}^\infty$ be a positive sequence. Then
   (a) (5 points) Show that $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = r > 0$ implies that $\lim\limits_{n\to\infty} (a_n)^{\frac{1}{n}} = r$, for some $r > 0$.
   (b) (5 points) Find an example to explain that $\lim\limits_{n\to\infty} (a_n)^{\frac{1}{n}} = r$ cannot imply $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = r$, for some $r > 0$.
4. Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, and $a, b, c \in \mathbb{R}$. Show that
   (a) (6 points) Use the Lagrange multiplier to show that
$$|ax + by + cz| \le \left(|a|^q + |b|^q + |c|^q\right)^{\frac{1}{q}} \left(|x|^p + |y|^p + |z|^p\right)^{\frac{1}{p}}, \tag{0.1}$$
   for any $x, y, z \in \mathbb{R}$.
   (b) (4 points) Find $x, y, z \in \mathbb{R}$ such that the equality holds for the inequality (0.1).
   (c) (5 points) Find the smallest value α such that the inequality
$$(x + 2y + 3z)^4 \le \alpha(x^4 + 2y^4 + 3z^4)$$
   holds for all $x, y, z \in \mathbb{R}$.

5. (15 points) Evaluate the integral $\int_0^1 \frac{x^b - x^a}{\ln x}\, dx$, for $a, b > 0$.

6. (10 points) Find all solutions of the equation

$$1 + \frac{x}{2!} + \frac{x^2}{4!} + \frac{x^3}{6!} + \frac{x^4}{8!} + \ldots + \frac{x^n}{(2n)!} + \ldots = 0.$$

   (You need to think of different cases: Either $x \geq 0$ or $x < 0$.)

7. (10 points) What is the smallest viewing rectangle that contains every number of the family of polar curves $r = 1 + \beta \sin \theta$, where $0 \leq \beta \leq 1$?

8. (10 points) For $p > 1$, evaluate

$$\frac{1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \ldots}{1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \ldots}.$$

9. (10 points) Suppose that f is differentiable in $\mathbb{R}$. Prove that all tangent planes to the surface $z = x f(\frac{y}{x})$ intersect in a common point.