REMAINDER TERM OF THE TAYLOR SERIES

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Taylor series: Use polynomials to approximate functions. If $f(x)$ is a smooth function, which means f is differentiable infinitely many times, and $f(x) = \sum_{k=0}^{\infty} a_k (x-c)^k$, then the coefficients $a_k = \frac{f^{(k)}(c)}{k!}$, where $f^{(k)}(c)$ is the k-th derivative of f evaluated at $x = c$. When $c = 0$, the Taylor series is called the Maclaurin series.

Theorem 0.1. *If $f(x)$ has $(n+1)$-derivatives in an open interval I containing 0, then for any $x \in I$, we have*

$$\begin{aligned} f(x) =& f(0) + f'(0)x + \cdots + \frac{f^n(0)}{n!}x^n + R_n(x) \\ =& P_n(x) + R_n(x), \end{aligned}$$

where $P_n(x)$ is the Taylor expansion with n-finite sums $R_n(x)$ is called the remainder such that

$$R_n(x) = \frac{1}{n!}\int_0^x f^{(n+1)}(t)(x-t)^n \, dt.$$

The proof is based on the integration by parts and change of variables, which I have introduced in the lecture.

We want to give another representation formula for the remainder as follow.

Lemma 0.1. *Under the same hypothesis of the previous theorem, there exists $c \in (0, x)$ such that*

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1},$$

where $R_n(x)$ is the remainder given by the previous theorem.

Proof. When $n = 0$, the lemma holds by the mean value theorem

$$f(x) - f(0) = f'(c)(x-0).$$

We prove by the mathematical induction as follows.

Suppose that $n = k-1$ is okay, i.e., there exists $c \in (0, x)$ such that $R_{k-1}(x) = \frac{f^{(k)}(c)}{k!}x^k$, then we want to show that $n = k$ has the same formula. From the Taylor expansion formula, one can easily see that

$$R_{k-1}(x) = \frac{f^{(k)}(c)}{k!} + R_k(x),$$

which implies that

$$R_k(x) = -\frac{f^{(k)}(0)}{k!}x^k + R_{k-1}(x) = -\frac{f^{(k)}(0)}{k!}x^k + -\frac{f^{(k)}(c)}{k!}x^k$$
$$= \frac{x^k}{k!}\left(f^{(k)}(c) - f^{(k)}(0)\right) = \frac{x^k}{k!}f^{(k+1)}(c')(c-0),$$

where we used the mean value theorem in the last identity, for some $c' \in (0, c)$.

It suffices to show that

$$f^{(k+1)}(c')(c-0) = \frac{f^{(k+1)}(\widetilde{c})}{k+1}x, \tag{0.1}$$

for some $\widetilde{c} \in (0, x)$, then we finish the proof. In order to prove (0.1), let us consider the function

$$F(x) := f(x) - P_n(x), \qquad G(x) = x^{n+1},$$

then by the *Cauchy mean value theorem* for $x \neq 0$, one has

$$\frac{F(x)}{G(x)} = \frac{F(x) - F(0)}{G(x) - G(0)} = \frac{F'(c_1)}{G'(c_1)}, \tag{0.2}$$

for some $c_0 \in (0, x)$. Meanwhile, we also have

$$\frac{F'(c_1)}{G'(c_1)} = \frac{F'(c_1) - F(0)}{G'(c_1) - G(0)} = \frac{F''(c_2)}{G''(c_2)},$$

and by using $F^{(k)}(0) = G^{(k)}(0) = 0$ for all $k = 0, 1, 2, \cdots, n$, then we must have

$$\frac{F'(c_1)}{G'(c_1)} = \frac{F''(c_2)}{G''(c_2)} = \cdots = \frac{F^{(n+1)}(c_{n+1})}{G^{(n+1)}(c_{n+1})} = \frac{f^{(n+1)}(c_{n+1})}{(n+1)!}, \tag{0.3}$$

for $0 < c_{n+1} < c_n < \cdots < c_2 < c_1 < x$. The last identity in the above equation comes from the definitions of F and G.

Finally, by using (0.2) and (0.3), since $R_n(x) = f(x) - P_n(x) = F(x)$, we conclude

$$R_n(x) = F(x) = \frac{f^{(n+1)}(c_{n+1})}{(n+1)!}G(x) = \frac{f^{(n+1)}(c_{n+1})}{(n+1)!}x^{n+1}.$$

This completes the proof. □

Corollary 0.2 (Error estimate)**.** *Let M be a positive number such that $|f^{(n+1)}(t)| \leq M$, for $t \in (0, x)$. Then*

$$|f(x) - P_n(x)| = |R_n(x)| \leq \frac{M}{(n+1)!}|x|^{n+1}.$$

Proof. The proof is simply applied the formula of the remainder R_n and take absolute value. □

Example 0.3. *$f(x) = \sin x$, then $P_4 = x - \frac{x^3}{6}$, $|f^{(5)}(x)| = |\cos x| \leq 1$. Then for any $x \in \mathbb{R}$, $|\sin x - P_4(x)| \leq \frac{|x|^5}{5!}$.*

For instance, take $x = 0, 1$, then $P_r(0.1) = 0.1 - \frac{(0.1)^3}{6} = 0.098333\cdots$, and

$$|\sin 0.1 - P_t(0.1)| \leq \frac{(0.1)^5}{120} < 0.0000001.$$