2021 FALL CALCULUS 0417: FIRST MIDTERM (OCTOBER 25, 2021)

- Please answer the following questions in details, which means you need to state all theorems or results you used. L'Hospital rule is **prohibited** to use in this midterm if you cannot prove it (no grade will be given if you really used). The definitions of terminology were taught in the lectures, so you cannot ask instructor or TA about mathematical definitions while taking the midterm.
- Please mark your name, student ID, and question numbers clearly on your answer sheet.
- The exam has a total of 120 points.

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1. (10 points) Use the $\epsilon - \delta$ definition of limit to prove

$$\lim_{x \to 4} \frac{x^2 - 2x - 8}{x - 4} = 6.$$

2. (10 points) Use the ϵ-definition to show that

$$\lim_{x \to -\infty} \frac{1}{x} = 0.$$

3. (10 points) Use the Squeeze Theorem to evaluate

$$\lim_{x \to \infty} \frac{\sin x}{x}.$$

4. (a) (10 points) Prove that

$$\lim_{x \to \infty} f(x) = \lim_{t \to 0^+} f(1/t)$$

and

$$\lim_{x \to -\infty} f(x) = \lim_{t \to 0^-} f(1/t)$$

assuming that these limit exit.

(b) (10 points) Use the question 3, to find

$$\lim_{x \to 0^+} x \sin \frac{1}{x}.$$

5. Determine whether $f'(0)$ exists (need to explain the reasons).

(a) (10 points)

$$f(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

(b) (10 points)

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

6. (10 points) Find the derivative of
$$f(t) = \frac{1}{\sqrt{1+t}},$$
using the definition of a derivative. What is the domain of f'?

7. (10 points) Differentiate
$$f(\theta) = \frac{\sin\theta}{1+\cos\theta}.$$

8. (10 points) Find the derivative of $y = \sqrt{x + \sqrt{x + \sqrt{x}}}$.

9. (Bonus question) Suppose f is a function that satisfies the equation
$$f(x+y) = f(x) + f(y) + x^2y + xy^2$$
for all real numbers x and y. Suppose also that
$$\lim_{x\to 0} \frac{f(x)}{x} = 1.$$
Find
(a) (5 points) $f(0)$.
(b) (5 points) $f'(0)$.
(c) (10 points) $f'(x)$.